SOME EXAMPLES OF QUANTUM MARKOVIAN PROCESSES

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Several concrete examples of quantum Markovian processes are considered in detail. The diffusional character of the transition amplitudes found with the use of the Feynman path integral is demonstrated. Two cases of processes with boundaries (absorbing or reflecting) are discussed. Finally, the quantum analogue of the Poisson as well as the Ornstein-Uhlenbeck processes are described.

1. Introduction

In the present paper we collet in a systematical way some simple examples of so-called quantum stochastic processes described in terms of their transition probability amplitudes. In most of the cases considered below we deal with the densities of transition probability amplitudes. Namely, we deal with complex functions $(s, y; t, x)$ each of them giving the density of the probability amplitude for finding a particle at time t at the state x when it is known that at an earlier time s it was at y. The parameters $x, y, z\dots$ vary within the space of states $\mathscr{X}$ relevant to each concrete physical situation. These functions, called for the sake of brevity the densities of transition amplitudes, should satisfy the following postulates (*cf.* [1]):

(*i*) $(s, y; t, x) = (t, x; s, y)^*$

motion reversibility condition,

(*ii*) $\lim\limits_{t \downarrow s} (s, y; t, x) = \delta(y-x)$

time continuity condition,

(*iii*) $\int\limits_{\mathscr{X}} dz(s, y; \tau, z)(s, x; \tau, z)^* = \delta(y-x)$

unitarity condition,

(*iv*) $\int\limits_{\mathscr{X}} dz(s, y; \tau, z)(\tau, z; t, x) = (s, y; t, x) \qquad s < \tau < t$

quantum causality condition,

(*v*) $(s, y; t, x)$

continuous functions in y, x, space continuity condition.

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Any function $(s, y; t, x)$ satisfying the conditions $(i–v)$ is called a quantum Markovian process in the space of states $\mathscr{X}$. The adjective Markovian is justified by the fourth condition, the Smoluchowski–Chapman–Kolmogorov equation (SCH-equation).

The quantum Matkovian processes most interesting from the point of view of physical applications are the quantum diffusional processes, which are characterized besides the postulates $(i–v)$ by the existence of the following limits [5], [6]:

$$\text{A.}\quad \lim_{t \downarrow s} (t-s)^{-1} \int_{\mathscr{X}} dx(s, y; t, x)\,(x_k - y_k) = a_k(s, y),$$

$$\text{B.}\quad \lim_{t \downarrow s} (t-s)^{-1} \int_{\mathscr{X}} dx(s, y; t, x)\,(x_k - y_k)\,(x_j - y_j) = b_{kj}(s, y),$$

$$\text{C.}\quad \lim_{t \downarrow s} (t-s)^{-1} \left[\int_{\mathscr{X}} dx(s, y; t, x) - 1\right] = c(s, y),$$

$$\text{D.}\quad \lim_{t \downarrow s} (t-s)^{-1} \int_{\mathscr{X}} dx(s, y; t, x)\,(x_1 - y_1)^{n_1}(x_2 - y_2)^{n_2}(x_3 - y_3)^{n_3} = 0$$

$$\text{if } n_1 + n_2 + n_3 \geqslant 3;\ k, j = 1, 2, 3.$$

Here, the space of states $\mathscr{X}$ is chosen to be a part (or the whole) of the three dimensional Euclidean space $\mathscr{R}^3$. The functions a_k, b_{kj} and c are called the coefficient functions of quantum diffusion.

It may be shown (*cf. e.g.* [1]) that the densities of transition amplitudes for quantum diffusional processess satisfy the Kolmogorov-Schrödinger equations

$$-\partial_s(s, y; t, x) = [\tfrac{1}{2} b_{kj}(s, y)\partial_k\partial_j + a_k(s, y)\partial_k + c(s, y)](s, y; t, x), \tag{1.1}$$

$$\partial_t(s, y; t, x) = [\tfrac{1}{2}\partial_k\partial_j b_{kj}(t, x) - \partial_k a_k(t, x) + c(t, x)](s, y; t, x). \tag{1.2}$$

These equations, supplemented by the initial contition (ii), provide a method for finding the densities of transition amplitudes if the coefficient functions a_k, b_{kj} and c are given. This is clearly not an easy task and the exact solutions to these equations are known for a very limited number of cases [7]. Since we are interested in examples of processes for which the densities of transition amplitudes are given in the closed form, we shall focus our attention on results obtained with the use of the heuristic Feynman path integral representing solutios of the simple Schrödinger equation [8].

As is know, in the case of systems described by Lagrangians which are quadratic in coordinates and velocities, the "integration" may be crried out to obtain the propagators

$$K(b, a) = \int_a^b e^{\frac{i}{\hbar} S[b,a]}\, \mathscr{D}x(t) = F(t_b, t_a) \exp\left\{\frac{i}{\hbar} S_{cl}[b, a]\right\}. \tag{1.3}$$

Here, $S_{cl}[b, a]$ stands for the action integral calculated with the use of the classical trajectory connecting the points $x_{cl}(t_a) = x_a$ and $x_{cl}(t_b) = x_b$. The function $F(t_b, t_a)$ may be determined from the postulates $(i–v)$ up to a phase factor which was guessed in the several examples of transition amplitudes given by Feynman and Hibbs [9]. We shall discuss them in detail in the next section. In particular, we found it useful at the present stage of development of the quantum stochastic processes theory to check successively all of the conditions (*i–v*) and especially (A–D). Thus, we shall stress the diffusional character of processes. described by the densities of transition amplitudes found by Feynman (examples b–f). Examples a, g, h and i are given by the authors.

2. *Examples of quantum Markovian processes*

We begin with the remark that the notion of quantum Markovian process is not yet fully defined by the postulates (*i–v*) and the conditions (A–D). This is suggested by the existence of the following, rather special, example:

$$(s, y; t, x) = \delta[x-y-v(t-s)], \tag{2.1}$$

where $x, y \in \mathscr{X} = \mathscr{R}^1$ and $v \geqslant 0$.
This function describes a classical particle moving along a real line with constant velocity v. The postulates (*i–iv*) are satisfied, whereas the space continuity condition (*v*) is not. Again the conditions (A–D) are satisfied and

$$a(s, y) = v$$

$$b(s, y) = c(s, y) = 0. \tag{2.2}$$

Clearly, this function has a classical sense as the density of a probability distribution instead of that of a probability amplitude, as it should be in the quantum case. We believe that the condition of a nonvanishing coefficient b would suffice to eliminate such functions permitting a classical interpretation of the quantum diffusional processes.

As the first example of a quantum Markovian process we shall consider an analogue of the classical Poisson process.

2a. Quantum Poisson process

The density of the probability amplitude is given by the formula [3]

$$(t; y, x) \equiv (0, y; t, x) = \sum_{n=0}^{\infty} \frac{(i\lambda t)^n}{n!} \Pi_n(y, x) e^{i\lambda t}, \tag{2.3}$$

where λ is a real number,

$$\Pi_0(y-x) = \delta(y-x), \tag{2.4}$$

$$\Pi_n(y, x) = \int_{\mathscr{X}} dz q(y, z) \Pi_{n-1}(z, x) \qquad n > 0 \tag{2.5}$$

and the function $q(y, z)$, continuous in both variables, satisfies the condition

$$q^*(y, x) = q(x, y) \tag{2.6}$$

and is such that the series is absolutely convergent.

It is quite easy to check all the postulates (*i–v*) if one notices that

$$\int_{\mathscr{X}} dz \Pi_m(y, z) \Pi_n(z, x) = \Pi_{m+n}(y, x). \tag{2.7}$$

Thus, $(t; y, x)$ describes a stationary quantum Markovian process in the space of states $\mathscr{X}$.

2b. Free quantum Brownian motion on a line

The first non-trivial example of a stationary quantum Markovian process of the diffusional type is given by the formula which is well-known in quantum mechanics,

$$(t; y, x) \equiv (0, y; t, x) = \left(\frac{m}{2\pi i \hbar t}\right)^{\frac{1}{2}} \exp i \left[\frac{m}{2\hbar t}(x-y)^2\right] \tag{2.8}$$

where $x, y \in \mathscr{X} = \mathscr{R}^1$, m is the particle mass, and $\hbar$ is the Planck constant devided by 2π.

Such a function is associated with a free particle of mass m undergoing quantum diffusion on a real line.

It is rather obvious that the postulates (*i*) and (*v*) are satisfied. Taking into account that the moments of the process are

$$C_n(t, y) = \int_{-\infty}^{\infty} dx(t; y, x)(x-y)^n = \begin{cases} 0 \text{ for } n \text{ odd} \\ (n-1)!! \left(\frac{i\hbar t}{m}\right)^{\frac{n}{2}} \text{ for } n \text{ even} \end{cases} \tag{2.9}$$

we have for a test function f

$$\lim_{t \downarrow 0} \int_{-\infty}^{\infty} dx(t; y, x) f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(y)}{n!} \lim_{t \downarrow 0} C_n(t, y) = f(y).$$

Thus, the second postulate is also satisfied.

Using the Fresnel integral

$$\int_{-\infty}^{\infty} dx \exp(i\gamma x^2) = \left(\frac{i\pi}{\gamma}\right)^{\frac{1}{2}} \qquad \gamma > 0 \tag{2.11}$$

we may easily find that (*iii*) is also fulfilled,

$$\int_{-\infty}^{\infty} dz(t; y, x)(t; x, z)^* = \exp[i\gamma(y^2 - x^2)]\delta(x-y) = \delta(x-y) \tag{2.12}$$

where $\gamma = \dfrac{m}{2\hbar t}$.

The quantum causality condition (*iv*) is satisfied, as may be verified with a little patience [10].

It follows from formula (2.9) that the conditions (A–D) are satisfied and we have

$$a(y) = \lim_{t \downarrow 0} \frac{C_1(t, y)}{t} = 0,$$

$$b(y) = \lim_{t \downarrow 0} \frac{C_2(t, y)}{t} = i\frac{\hbar}{m},$$

$$c(y) = \lim_{t \downarrow 0} \frac{1}{t}[C_0(t, y) - 1] = 0,$$

$$\lim_{t \downarrow 0} \frac{1}{t} C_n(t, y) = 0 \quad \text{for } n \geqslant 3. \tag{2.13}$$

Hence, the process is diffusional and in such a case one has the following differential equations for the transition amplitude:

$$i\hbar\partial_t(t; y, x) = -\frac{\hbar^2}{2m}\partial_x^2(t: y, x),$$

$$i\hbar\partial_t(t; y, x) = -\frac{\hbar^2}{2m}\partial_y^2(t; y, x). \tag{2.14}$$

The same equations hold for wave functions defined generally as

$$\varphi(s, y) = \int_{\mathscr{X}} dx(s, y; t, x)v(t, x)$$

$$\psi(t, x) = \int_{\mathscr{X}} dyu(s, y)(s, y; t, x) \tag{2.15}$$

where $v(t, x)$ is a final wave function, while $u(s, y)$ is an initial one.

2c. Quantum Brownian motion in one dimension influenced by a constant force

The amplitude for such a proces is given by

$$(t; y, x) = \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{1}{2}} \exp\frac{i}{\hbar}\left[\frac{m(x-y)^2}{2t} + \frac{ft(x+y)}{2} - \frac{f^2t^3}{24m}\right]. \tag{2.16}$$

$x, y \in \mathscr{X} = \mathscr{R}^1$, and f is the magnitude of the force directed along the real line. The physical interpretation of this function may be easily seen from its Kolmogorov-Schrödinger equations, written below.

All the postulates (*i–v*) are satisfied in this case, as may be easily verified. It should be mentioned that the verification of the conditions (A–D) is simplified to a large extent by the fact that only asymptotic estimations of moments at small t are needed instead of their values at any time.

Let us introduce the convenient abbreviations

$$\alpha = \frac{m}{2\hbar t},$$

$$\beta = \frac{tf}{2h},$$

$$\gamma = \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{1}{2}} \exp\left(-\frac{i}{\hbar}\frac{t^3 f^2}{24m}\right). \tag{2.17}$$

With this notation we have for the moments

$$C_0(t, y) = \int_{-\infty}^{\infty} dx(t; x, y) = \gamma\left(\frac{i\pi}{\alpha}\right)^{\frac{1}{2}} \exp\left(-i\frac{\beta^2}{4\alpha} + 2iy\beta\right),$$

$$C_1(t, y) = \int_{-\infty}^{\infty} dx(t; x, y)\,(x-y) = -\frac{\beta}{2\alpha} C_0(t, y),$$

$$C_{2n}(t, y) = \left(\frac{1}{i}\frac{d}{d\alpha}\right)^n C_0(t, y),$$

$$C_{2n+1}(t, y) = \left(\frac{1}{i}\frac{d}{d\alpha}\right)^n C_1(t, y). \tag{2.18}$$

Using the asymptotic expansion of $C_0(t, y)$, which is

$$C_0(t, y) = 1 + i\frac{yf}{h}t + 0(t) \tag{2.19}$$

we may easily estimate all the moments and calculate the coefficients a, b, c. The results are as follows:

$$a(y) = 0$$

$$b(y) = i\frac{h}{m}$$

$$c(y) = -\frac{fy}{i\hbar} \tag{2.20}$$

Hence, the process is a diffusional one and its Kolmogorov-Schrödinger equations (1.1) and (1.2) may be immediately written with the use of the above coefficients.

2d. Quantum diffusion on a line influenced by the Hooke force

This process, usually called the quantum oscillator, is described by the following transition amplitude:

$$(t; y, x) = \left(\frac{m\omega}{2\pi i\hbar \sin \omega t}\right)^{\frac{1}{2}} \exp \frac{im\omega}{2\hbar \sin \omega t} [(x^2+y^2) \cos \omega t - 2xy], \tag{2.21}$$

where $x, y \in \mathscr{X} = \mathscr{R}^1, \omega^2 = \frac{k}{m}$, k being the elasticity coefficient and m the mass of the particle undergoing oscillations.

The postulates (i–v) are satisfied here, which may be proved by performing some elementary calculations.

Let us use the following convenient abbreviations, which have nothing in common with those used previously,

$$\alpha = \frac{m\omega}{2\hbar} \cot \omega t$$

$$\beta = \frac{m\omega}{2\hbar} \frac{\cos \omega t - 1}{\sin \omega t}$$

$$\gamma = \frac{m\omega}{2\hbar} \frac{1}{\pi i \sin \omega t}. \tag{2.22}$$

We find quite easily the following results by using the Fresnel integral,

$$C_0(t, y) = \left(\frac{i\pi\gamma}{\alpha}\right)^{\frac{1}{2}} \exp iy^2 \left(2\beta - \frac{\beta^2}{\alpha}\right) = 1 + \frac{m\omega^2 y^2}{2i\hbar} t + 0(t),$$

$$C_1(t, y) = -y \frac{\beta}{\alpha} C_0(t, y) = 0(t^2)$$

$$C_{2n}(t, y) = \left(\frac{1}{i} \frac{d}{d\alpha}\right)^n C_1(t, y),$$

$$C_{2n+1}(t, y) = \left(\frac{1}{i} \frac{d}{d\alpha}\right)^n C_1(t, y), \tag{2.23}$$

and, therefore, the quantum diffusion coefficients are

$$a(y) = 0$$

$$b(y) = i\frac{\hbar}{m}$$

$$c(y) = \frac{m\omega^2 y^2}{2i\hbar}. \tag{2.24}$$

The condition **D** is satisfied here and, hence, the process is a diffusional quantum Markovian process. It plays an outstanding role in quantum mechanics.

2e. Quantum diffusion on a line influenced by the Hooke force and an external force f depending on time

In this case $x, y \in \mathscr{X} = \mathscr{R}^1$ and the process is not stationary

$$(s, y; t, x) = \left(\frac{m\omega}{2\pi i\hbar \sin \omega(t-s)}\right)^{\frac{1}{2}} \exp \frac{i}{\hbar} \frac{m\omega}{2 \sin \omega(t-s)} \Bigg[(x^2+y^2) \cos \omega(t-s) -$$

$$-2xy + \frac{2x}{m\omega} \int_s^t d\sigma f(\sigma) \sin \omega(\sigma-s) + \frac{2y}{m\omega} \int_s^t d\sigma f(\sigma) \sin \omega(t-\sigma) -$$

$$- \frac{2}{m^2\omega^2} \int_s^t d\sigma \int_s^\sigma d\sigma' f(\sigma) f(\sigma') \sin \omega(t-\sigma) \sin \omega(\sigma'-s) \Bigg]. \tag{2.25}$$

We shall use here the following abbreviations:

$$\gamma = \left(\frac{m\omega}{2\pi i\hbar \sin \omega(t-s)}\right)^{\frac{1}{2}},$$

$$A = \frac{m\omega}{2\hbar} \cot \omega(t-s),$$

$$B = \frac{m\omega}{2\hbar} \frac{\cos \omega(t-s)-1}{\sin \omega(t-s)},$$

$$a = \frac{1}{\hbar \sin \omega(t-s)} \int_s^t d\sigma f(\sigma) \sin \omega(\sigma-s),$$

$$b = \frac{1}{\hbar \sin \omega(t-s)} \int_s^t d\sigma f(\sigma) \sin \omega(t-\sigma),$$

$$c = \frac{-1}{\hbar m\omega \sin \omega(t-s)} \int_s^t d\sigma \int_s^\sigma d\sigma' f(\sigma) f(\sigma') \sin \omega(t-\sigma) \sin \omega(\sigma'-s). \tag{2.26}$$

All the postulates (i–v) may be checked here, although this is a somewhat tedious task in this case. The following elementary identity simplifies the algebra involved:

$$\sin(\alpha_1-\alpha_2) \sin(\alpha_3-\alpha_4) + \sin(\alpha_1-\alpha_3) \sin(\alpha_4-\alpha_2) +$$
$$+ \sin(\alpha_1-\alpha_4) \sin(\alpha_2-\alpha_3) \equiv 0 \tag{2.27}$$

for all $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

The conditions (A–D) may be verified simply if one notices that the zero moment has an asymptotic expansion,

$$C_0(s, t; y) = \int_{-\infty}^{\infty} dx(s, y; t, x) = \left(\frac{i\pi}{A}\right)^{\frac{1}{2}} \gamma \exp i\left\{Ay^2+by+c-\right. \tag{2.28}$$

$$\left.-\frac{\left[(A-B)y-\frac{a}{2}\right]^2}{A}\right\} = 1+\frac{t-s}{i\hbar}\left[\frac{m\omega^2 y^2}{2} - f(s)y\right] + o(t-s),$$

and also that

$$C_1(s, t; y) = \frac{1}{i}\left(\frac{d}{da} - \frac{d}{db}\right) C_0(s, t; y) = \left\{\frac{(A-B)y-\frac{a}{2}}{A} - y\right\} C_0(s, t; y). \tag{2.29}$$

Higher-order moments may be expressed as derivatives of C_0 (even order moments) and C_1 (odd order) with respect to the parameter A and are easily estimable. The condition D is satisfied, and we have for the quantum diffusion coefficients the expressions

$$a(s, y) = 0$$

$$b(s, y) = i\frac{h}{m}$$

$$c(s, y) = \frac{1}{i\hbar}\left[\frac{m\omega^2 y^2}{2} - f(s)y\right]. \tag{2.30}$$

Therefore, the Kolmogorov-Schrödinger equations in this case are

$$-i\hbar\partial_s(s, y; t, x) = \left[-\frac{h^2}{2m}\partial_y^2 + \frac{m\omega^2 y^2}{2} - yf(s)\right](s, y; t, x), \tag{2.31}$$

$$i\hbar\partial_t(s, y; t, x) = \left[-\frac{\hbar^2}{2m}\partial_x^2 + \frac{m\omega^2 x^2}{2} - xf(t)\right](s, y; t, x). \tag{2.32}$$

2f. Quantum Brownian motion in the three dimensional Euclidean space under the presence of a constant magnetic field

If we choose a coordinate system such that the third axis is parallel to a magnetic field $\vec{B} = B\vec{e}_3$, then the density of transition amplitude reads

$$(t; y, x) = \left(\frac{m}{2\pi i\hbar t}\right)^{3/2} \frac{\omega t}{2\sin\frac{\omega t}{2}} \exp\frac{im}{2\hbar}\left\{\frac{(x_3-y_3)^2}{t} + \right.$$

$$\left. + \frac{\omega}{2}\cot\frac{\omega t}{2}\left[(x_1-y_1)^2+(x_2-y_2)^2\right] + \omega(y_1x_2 - x_1y_2)\right\} \tag{2.33}$$

where $x, y \in \mathscr{X} = \mathscr{R}^3$ and $\omega = \dfrac{eB}{mc}$, e and m are the particles charge and mass, and c is the velocity of light.

Postulates (i–v) are again satisfied here. One should notice that the amplitude has the property

$$(t; y, x)_\omega = (-t; x, y)^*_\omega = (t; x, y)_{-\omega} \tag{2.34}$$

which shows the difference between the time-reversal operation and that prescribed by the first postulate.

This is an agreement with the well-known rule which states that a time-reversal operation should be followed by a change, in the direction of the magnetic field (more generally, by a change of sign of frequency ω).

If one introduces the notation

$$\alpha = \left(\frac{m}{2\pi i \hbar t}\right)^{3/2} \frac{\omega t}{2 \sin \dfrac{\omega t}{2}},$$

$$\beta = \frac{m\omega}{4\hbar} \cot \frac{\omega t}{2},$$

$$\gamma = \frac{m}{2\hbar t}$$

$$\eta = \frac{m\omega}{2\hbar} \tag{2.35}$$

the moments are then expressed by the formulae

$$C_{pqr}(t,y) = \int_{\mathscr{R}^3} dx(t;y,x)\,(x_1-y_1)^p\,(x_2-y_2)^q\,(x_3-y_3)^r = \alpha C'_p C''_q C'''_r, \tag{2.36}$$

where

$$C'_p = \int_{-\infty}^{\infty} dz z^p \exp i(\beta z^2 - \eta y_2 z) = \left(\frac{i}{y_2}\frac{d}{d\eta}\right)^p C'_0,$$

$$C''_p = \int_{-\infty}^{\infty} dz z^q \exp i(\beta z^2 + \eta y_1 z) = \left(-\frac{i}{y_1}\frac{d}{d\eta}\right)^q C''_0,$$

$$C'''_r = \int_{-\infty}^{\infty} dz z^r \exp(i\gamma z^2) = \left(-i\frac{d}{d\gamma}\right)^r C'''_0,$$

and for zero moments

$$C_0' = \left(\frac{i\pi}{\beta}\right)^{\frac{1}{2}} \exp\left(-i\frac{\eta^2 y_2^2}{4\beta}\right),$$

$$C_0'' = \left(\frac{i\pi}{\beta}\right)^{\frac{1}{2}} \exp\left(-i\frac{\eta^2 y_1^2}{4\beta}\right),$$

$$C_0''' = \left(\frac{i\pi}{\gamma}\right)^{\frac{1}{2}}. \tag{2.37}$$

Now all quantum diffusional coefficients may be found and are

$$a_1(y) = \lim_{t\downarrow 0} \frac{1}{t} \int_{\mathscr{R}^3} dx(t;\, y, x)\,(x_1 - y_1) = \tfrac{1}{2}\,\omega y_2,$$

$$a_2(y) = -\tfrac{1}{2}\,\omega y_1,$$

$$a_3(y) = 0$$

$$b_{kj}(y) = i\frac{\hbar}{m}\,\delta_{kj}$$

$$c(y) = \frac{m\omega^2}{8i\hbar}\,(y_1^2 + y_2^2). \tag{2.38}$$

Therefore, in this case we have the following Kolmogorov-Schrödinger equations:

$$i\hbar\partial_t(t;\, y, x) = \left[-\frac{\hbar^2}{2m}\Delta_y + \frac{i\hbar\omega}{2}(y_2\partial_1 - y_1\partial_2) + \frac{m\omega^2}{8}(y_1^2 + y_2^2)\right](t;\, y, x), \tag{2.39}$$

$$i\hbar\partial_t(t;\, y, x) = \left[-\frac{\hbar^2}{2m}\Delta_x - \frac{i\hbar\omega}{2}(x_2\partial_1 - x_1\partial_2) + \frac{m\omega^2}{8}(x_1^2 + x_2^2)\right](t;\, y, x). \tag{2.40}$$

2g. Free quantum Brownian motion on a positive half-line

This process is particularly interesting, for it describes the motion of a particle affected by the presence of a boundary placed at the origin. The boundary may absorb or reflect a particle. In the first case the process ends when the absorbing point $x = 0$ is reached. In the second, the barrier reflects the probability flow and, therefore, it is doubled in the space of states $\mathscr{X} = [0, \infty]$. The density of transition amplitude is defined here by the expression

$$(t;\, y, x) = \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{1}{2}} \left\{\exp\left[i\frac{m}{2\hbar t}(y-x)^2\right] + \right.$$

$$\left. +\alpha \exp\left[i\frac{m}{2\hbar t}(y+x)^2\right]\right\}, \tag{2.41}$$

where the parameter α takes the values:

$\alpha = +1$ for reflection at $x = 0$

$\alpha = -1$ for absorption at the origin.

It is not difficult to check each of the postulates $(i-v)$ in this case in the same way as for the case of free quantum Brownian motion on the whole line.

In order to estimate the moments we note that

$$C_n(t) = \left(\frac{\gamma}{i\pi}\right)^{\frac{1}{2}} [A_n(\gamma)+\alpha B_n(\gamma)], \tag{2.42}$$

where

$$\gamma = \frac{m}{2\hbar t},$$

$$A_n(\gamma) = \int_0^\infty dx(x-y)^n \exp i\gamma(x-y)^2 \tag{2.43}$$

and

$$B_n(\gamma) = \int_0^\infty dx(x-y)^n \exp i\gamma(x+y)^2.$$

The asymptotic expansions of these functions may be easily found, thus obtaining for the lower moments the expansions

$$C_0(t) = 1+(1-\alpha)\left(\frac{\gamma}{i\pi}\right)^{\frac{1}{2}} \frac{\exp i\gamma y^2}{2i\gamma y} + 0(\gamma^{-3/2}), \tag{2.44}$$

$$C_1(t) = \left(\frac{\gamma}{i\pi}\right)^{\frac{1}{2}}\left[-\frac{\exp i\gamma y^2}{i\gamma} + 0(\gamma^{-2})\right], \tag{2.45}$$

$$C_2(t) = \frac{i}{2\gamma} + \left[\left(\frac{\gamma}{i\pi}\right)^{\frac{1}{2}}\left(\frac{1}{2y\gamma^2} + \frac{y}{2i\gamma} + \frac{3y\alpha}{2i\gamma} - \frac{\alpha}{2iy\gamma^2}\right) + 0(\gamma^{-3/2})\right]\exp i\gamma y^2, \tag{2.46}$$

$$C_3(t) = 0(\gamma^{-1/2}) \exp i\gamma y^2. \tag{2.47}$$

Using these expresions the quantum diffusion coefficients are found to be

$$a = \frac{2\hbar}{m} \frac{i}{\sqrt{i\pi}} \lim_{\gamma \to \infty} \sqrt{\gamma} \exp i\gamma y^2, \tag{2.48}$$

$$b = \frac{2\hbar}{m} \left[\frac{i}{2} + \frac{1}{\sqrt{i\pi}} \left(\frac{y}{2i} + \alpha \frac{3y}{2i} \right) \lim_{\gamma \to \infty} \sqrt{\gamma} \exp i\gamma y^2 \right], \tag{2.49}$$

$$c = \frac{2\hbar}{m} \cdot \frac{1-\alpha}{2iy\sqrt{i\pi}} \lim_{\gamma \to \infty} \sqrt{\gamma} \exp i\gamma y^2. \tag{2.50}$$

One notices that the coefficients will become reasonable if the limit

$$\lim_{\gamma \to \infty} \sqrt{\gamma} \exp i\gamma y^2$$

is assumed to be zero for $y \neq 0$. This may be understood as some sort of regularization procedure consisting in a change of the mass m by adding a small imaginary part and taking the limit $\varepsilon \downarrow 0$ at the end of calculations. Hence, in fact, we define the density of transition amplitude as

$$(t; y, x) = \lim_{\varepsilon \downarrow 0} (t; y, x)_\varepsilon, \tag{2.51}$$

where the limit is understood to have the sense as defined in the theory of distributions [11]. In this case we obtain the results for $y > 0$:

$$a = 0,$$

$$b = i\frac{\hbar}{m}, \tag{2.52}$$

$$c = 0.$$

2h. Free quantum Brownian motion within a finite interval

In this example, a particle moves freely inside the interval $[0, x_0]$, while at its ends absorption or reflection may occur. The density of transition amplitude is in this case

$$(t; y, x) = \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \left[\exp\left\{ i \frac{m}{2\hbar t} (x - y + 2nx_0)^2 \right\} + \right.$$

$$\left. + \alpha \exp\left\{ i \frac{m}{2\hbar t} (x + y + 2nx_0)^2 \right\} \right]. \tag{2.53}$$

In the case of absorption at the end we may obtain another formuia solving the following boundary problem

$$\partial_t(t; y, x) = i \frac{\hbar}{2m} \partial_x^2(t; y, x),$$

$$(0; y, x) = \delta(y-x),$$

$$(t; y, 0) = t; y, x_0) = 0. \tag{2.54}$$

Using the Fourier method we get the solution

$$(t; y, x) = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \sin\left(\frac{n\pi y}{x_0}\right) \sin\left(\frac{n\pi x}{x_0}\right) \exp\left\{-\frac{i\hbar}{2m}\left(\frac{n\pi}{x_0}\right)^2 t\right\},$$

$$x, y \in (0, x_0). \tag{2.55}$$

The general case of motion with absorption at any two points may be obtained from that just expounded simply by a shift of variables.

In conclusion, we wish to remark that it is possible to get other examples of quantum stochastic processes from classical ones by taking the analytic continuation in time parameter to the purely imaginary values or by the analytic continuation in the parameters of the process. However, the physical meaning of such processes does not always emerge by itself. For instance, we may obtain a quantum counterpart of the Ornstein-Uhlenbeck process [6] by chosing in a specific way the parameters of the process. Namely, if $(t; v, u)$ is the transition amplitude for finding a particles with the velocity u at the time t, the initial velocity having been v, it is given by the formula

$$2i. \qquad (t; v, u) = \left[\frac{\gamma}{\pi b(1-e^{-2\gamma t})}\right]^{\frac{1}{2}} \exp\left\{-\frac{\gamma}{b} \frac{(u-ve^{-\gamma t})^2}{(1-e^{-2\gamma t})}\right\}, \tag{2.56}$$

where we put

$$b = i\frac{\hbar}{m}, \tag{2.57}$$

and γ is a real number.

The Kolmogorov-Schrödinger equations are in this case

$$\partial_t(t; v, u) = \left(\gamma \partial_u u + \frac{b}{2} \partial_u^2\right)(t; v, u),$$

$$\partial_t(t; v, u) = \left(-\gamma v \partial_v + \frac{b}{2} \partial_v^2\right)(t; v, u),$$

$$(0; v, u) = \delta(v-u). \tag{2.58}$$

This process describes the diffusion of a particle in the velocity-space which is damped by a "friction force" represented by the term with γ.

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