RELATIVISTIC GRAND ORBITAL MOMENTUM IN CLASSICAL THREE-BODY SYSTEMS

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Using a suitable normalized coordinate system the Hamiltonian for N-free relativistic spinless particles may be written in the following form $H = \sqrt{(\sum_{\alpha=1}^{N} m_\alpha)^2 + \sum_{\alpha=1}^{N} \pi_\alpha^2}$. A generalized Poincaré group is identified as a group of symmetry of the system. The relativistic grand orbital momentum tensor Γ_{ij} $(i, j = 1, 2, ..., 3N-3)$ is defined and its connection with the many-body impact parameter — b has the following simple form $\Gamma^2 = b^2[M^2 - (\sum_{\alpha=1}^{N} m_\alpha)^2]$, where M is the total invariant mass of the system, and $\Gamma^2 = \frac{1}{2}\sum(\Gamma_{ij})^2$ is Poincaré invariant.

The whole scheme is presented for a three particle system, but can be easily generalized to $N > 3$.

1. Introduction

Non-relativistic collisions involving 3 or more particles are often described by means of the so-called grand orbital momentum tensor which was introduced by Smith ([1], [2]). Let us briefly recall the construction. One considers a system consisting of three freely moving particles using six coordinates, three of them (ξ_1, ξ_2, ξ_3) describing the relative position of a chosen pair of particles, and three (ξ_4, ξ_5, ξ_6) the relative position of the third particle and the centre of mass of the pair. Then we build the 6×6 grand orbital momentum tensor Γ, whose components are $\Gamma_{ij} = \xi_i \pi_j - \xi_j \pi_i$, where π_i are the relative momenta canonically conjugate to ξ_i, $(i = 1, 2, ..., 6)$. Clearly, the central three-body collisions can be characterized by the condition $\Gamma^2 = 0$, where $\Gamma^2 = \frac{1}{2} \sum_{i,j=1}^{6} (\Gamma_{ij})^2$. Next one defines the three-body impact distance $b^2 = \min [\sum_{i=1}^{6} \xi_i^2]$ (an analogue of the two--body impact parameter), which is uniquely determined by the kinetic energy of the system T and the value Γ^2, namely [1] $\Gamma^2 = 2\mu T b^2$, where $\mu^2 = \prod_{\alpha=1}^{3} m_\alpha / \sum_{\alpha=1}^{3} m_\alpha$. This is a simple generalization of the relation $l^2 = 2\mu T b^2$ for the two-particle system with l denoting the orbital

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momentum. For not too high energies this relation enables us to use only a few terms of the partial wave expansion with respect to Γ.

Of course we are actually interested in finding the fully relativistic description of the asymptotic free-particle states. Several attempts to construct the relativistic theory of the grand orbital momentum are known in literature (Ref. [3]–[6]). However, either some relativistic features only were taken into account or methods used in the construction were criticized as ambiguous. The aim of this paper is to present a new more satisfactory relativistic version of the grand orbital momentum. This paper deals with classical formalism. The quantum mechanical treatment will be the subject of another paper.

In the first place we give a construction of relative coordinates and momenta for 3 relativistic particles. Further, we introduce a symmetric, normalized coordinate system which exhibits the symmetry of the Hamiltonian.

In Section 4 the symmetry is exploited and a generalized Poincaré group is identified as a group of symmetry of the system. We construct generators and write their Poisson bracket relations.

In the last Section we construct the relativistic grand orbital momentum and study its properties. A connection between grand orbital momentum, invariant mass and impact distance is established.

2. *Relativistic relative variables*

Let us start with the problem of construction of relative momenta. We shall use a method similar to the non-relativistic procedure which leads to the relative momenta *via* the Galilean transformation. Clearly, in our case the Galilean transformation will be replaced by the Lorentz one.

First we concentrate on the particles 1 and 2 in their centre of momentum frame (CM (12)). Their momenta referred to the chosen frame must satisfy the condition $\boldsymbol{p}_{12}^1+\boldsymbol{p}_{12}^2 = 0$. Now we difine $\boldsymbol{P}_1 \equiv \frac{1}{2}(\boldsymbol{p}_{12}^1-\boldsymbol{p}_{12}^2)$, which has the meaning of relative momentum of particles 1 and 2 measured in CM(12). If we observe all three particles from the overall CM(123) then the momenta fulfil the condition $\boldsymbol{p}_{123}^1+\boldsymbol{p}_{123}^2+\boldsymbol{p}_{123}^3 = 0$. The subsystem (12) considered as a particle with the invariant mass $m_{12}^2 = (E^1+E^2)^2-(\boldsymbol{p}^1+\boldsymbol{p}^2)^2$ moves with the velocity

$$\boldsymbol{V} = \frac{\boldsymbol{p}_{123}^1+\boldsymbol{p}_{123}^2}{E_{123}^1+E_{123}^2}.$$

Then the relations (given by a parallel Lorentz transformation [2]) between momenta $\boldsymbol{p}_{12}^\alpha$, energy E_{12}^α, $\alpha = 1, 2$ and quantities $\boldsymbol{p}_{123}^\alpha$, E_{123}^α, $\alpha = 1, 2$ measured in CM(123) have the following form

$$\boldsymbol{p}_{12}^\alpha = \boldsymbol{p}_{123}^\alpha + \frac{\boldsymbol{p}_{123}^1+\boldsymbol{p}_{123}^2}{m_{12}}\left[\frac{(\boldsymbol{p}_{123}^1+\boldsymbol{p}_{123}^2)\cdot\boldsymbol{p}_{123}^\alpha}{m_{12}+E_{123}^1+E_{123}^2} - E_{123}^\alpha\right],$$

$$E_{12}^\alpha = \frac{E_{123}^1+E_{123}^2}{m_{12}}\left(E_{123}^\alpha - \frac{(\boldsymbol{p}_{123}^1+\boldsymbol{p}_{123}^2)\cdot\boldsymbol{p}_{123}^\alpha}{E_{123}^1+E_{123}^2}\right), \tag{1}$$

with $\alpha = 1, 2$. We define

$$P_2 \equiv \tfrac{2}{3}\left[\tfrac{1}{2}(p^1_{123}+p^2_{123})-p^3_{123}\right],$$

which may be considered as the relative momentum of the third particle and the subsystem (12) measured in CM(123).

Let us assume that we observe three particles from an arbitrary inertial frame. Then $p^1+p^2+p^3 = P(\equiv P_3)$. The centre of momentum frame CM(123) has then the following velocity

$$V = \frac{P}{E}$$

where $E = E^1+E^2+E^3$ is the total energy of the system. Now, the relations between linear momenta measured in CM(123) and in our inertial frame can be written in the form

$$p^\alpha_{123} = p^\alpha + \frac{P}{M}\left[\frac{P\cdot p^\alpha}{E+M} - E^\alpha\right],$$

$$E^\alpha_{123} = \frac{E}{M}\left(E^\alpha - \frac{Pp^\alpha}{E}\right), \tag{2}$$

where $\alpha = 1, 2, 3$ and $M^2 \equiv E^2 - P^2$.

By inverting (1) and (2) we obtain, after a straightforward calculation, the relations between the individual momenta p^α, $\alpha = 1, 2, 3$, the relative momenta P_1 and P_2 and the total momentum P_3:

$$\begin{aligned} p^1 &= a^1P_3+b^1P_2+P_1, \\ p^2 &= a^2P_3+b^2P_2-P_1, \\ p^3 &= a^3P_3-P_2, \end{aligned} \tag{3}$$

where

$$b^1 = 1-b^2 = \frac{1}{2}\left[1+\frac{E^1_{123}-E^2_{123}+E^1_{12}-E^2_{12}}{E^1_{123}+E^2_{123}+E^1_{12}+E^2_{12}}\right],$$

$$a^\alpha = \frac{E^\alpha}{M} - \frac{Pp^\alpha}{M(E+M)}.$$

Clearly b^α , a^α, should be considered as functions of P_1, P_2, P_3.

Let us proceed to the relative positions. Having the transformation of momenta (3) we would like to find a transformation law

$$X_\alpha = X_\alpha(x^1, x^2, x^3, p^1, p^2, p^3). \tag{4}$$

for the canonically conjugate positions. Let us assume that formulae (3) and (4) define a canonical transformation. This assumption leads to the conclusion that

$$(X_\alpha)_k = \sum_{i=1}^{3}\sum_{\beta=1}^{3} \frac{\partial(p^\beta)_i}{\partial(P_\alpha)_k}(x^\beta)_i, \tag{5}$$

with $\alpha = 1, 2, 3$, $k = 1, 2, 3$. Strictly speaking we could add an arbitrary function $g^\alpha(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{p}^3)$ but this would lead to no benefit in further studies.

Then inserting (3) to (5) we obtain an explicit expression for the conjugate positions. The position canonically conjugate to the total momentum has the following form

$$X_3 = \frac{\sum_{\alpha=1}^{3} \boldsymbol{x}^\alpha E^\alpha}{H} - \frac{\boldsymbol{P} \times (\boldsymbol{P} \times \boldsymbol{K} - \boldsymbol{J}H)}{H(H^2 - \boldsymbol{P}^2)^{\frac{1}{2}}(H + (H^2 - \boldsymbol{P}^2)^{\frac{1}{2}})}, \tag{6}$$

where

$$H = \sum_{\alpha=1}^{3} E^\alpha,$$

$$\boldsymbol{P} = \sum_{\alpha=1}^{3} \boldsymbol{p}^\alpha,$$

$$\boldsymbol{J} = \sum_{\alpha=1}^{3} \boldsymbol{x}^\alpha \times \boldsymbol{p}^\alpha,$$

$$\boldsymbol{K} = \sum_{\alpha=1}^{3} \{\boldsymbol{x}^\alpha E^\alpha - \boldsymbol{p}^\alpha t\} \tag{7}$$

are ten generators of the Poincaré group [7]. This position vector was studied by Pryce [8], Newton and Wigner [9] and Fleming [10]. The last one called this vector — the centre of spin position vector. The vectors $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ may be respectively considered as the relative position of particles 1 and 2 in their CM and the relative position of subsystem (12) and the third particle in CM(123). Because the $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ referred to an arbitrary inertial frame are given by very involved formulae, we only give the expressions referred to CM(123)

$$\boldsymbol{X}_1 = \boldsymbol{x}^1 - \boldsymbol{x}^2 + \boldsymbol{P}_2 \cdot (\boldsymbol{x}^1 - \boldsymbol{x}^2) \left[\frac{\boldsymbol{P}_2}{m_{12}(m_{12} + E_{12})} - 4\frac{(m_1^2 - m_2^2)E_{12} + m_{12}\boldsymbol{P}_1 \cdot \boldsymbol{P}_2}{E_{12}(m_{12}^4 - (m_1^2 - m_2^2)^2)} \boldsymbol{P}_1 \right],$$

$$\boldsymbol{X}_2 = \boldsymbol{Q}_{12} - \boldsymbol{x}^3, \tag{8}$$

where $E_{12} = E_{123}^1 + E_{123}^2$ and $\boldsymbol{Q}_{12}$ is the centre of spin position vector for the particles 1 and 2 referred to CM(123). It is worth while stressing at this point that in the case $\boldsymbol{P} = 0$ our procedure gives the same expressions for $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ as those obtained by Barsella and Fabri [11].

After the canonical transformation given by (3) and (5) the Hamiltonian takes the following form

$$H = \sqrt{M^2 + \boldsymbol{P}_3^2},$$

where

$$M = \sqrt{m_{12}^2 + \boldsymbol{P}_2^2} + \sqrt{m_3^2 + \boldsymbol{P}_2^2}, \quad m_{12} = \sqrt{m_1^2 + \boldsymbol{P}_1^2} + \sqrt{m_2^2 + \boldsymbol{P}_1^2}. \tag{9}$$

3. *Normalized coordinate system and a symmetric form of the Hamiltonian*

Let us recall that the key notion leading to the concept of the non-relativistic grand orbital momentum was a symmetric, normalized coordinate system [1]. Namely, instead of the conventional non-relativistic (NR) relative variable $(X_\alpha)_{NR}$, $(P_\alpha)_{NR}$, $\alpha = 1, 2, 3$ Smith used

$$\begin{aligned} (\xi_1)_S &= d^{-1}(X_1)_{NR}, & (\pi_1)_S &= d(P_1)_{NR}, \\ (\xi_2)_S &= d(X_2)_{NR}, & (\pi_2)_S &= d^{-1}(P_2)_{NR}, \\ (\xi_3)_S &= (X_3)_{NR}, & (\pi_3)_S &= (P_3)_{NR}, \end{aligned} \tag{10}$$

where $\mu_{12} = m_1 m_2/(m_1+m_2)$, $\mu_3 = m_3(m_1+m_2)/\sum_{\alpha=1}^{3} m_\alpha$, $d^2 = (\mu_3/\mu_{12})^{\frac{1}{2}}$. After the transformation the Hamiltonian (the kinetic energy)

$$H = \frac{(P_3)^2_{NR}}{2(\sum_{\alpha=1}^{3} m_\alpha)} + \frac{(P_1)^2_{NR}}{2\mu_{12}} + \frac{(P_2)^2_{NR}}{2\mu_3}$$

becomes conveniently symmetric, namely

$$H = \frac{(\pi_3)^2_S}{2\sum_{\alpha=1}^{3} m_\alpha} + \frac{1}{2\mu}\left((\pi_1)^2_S + (\pi_2)^2_S\right)$$

where $\mu^2 = \mu_{12}\,\mu_3$.

In the relativistic case before we introduce the normalized coordinate system we must make an additional transformation leading to relativistic reduced masses [12]. We introduce a momentum vector $\boldsymbol{\Pi}_1$ which is collinear with $\boldsymbol{P}_1$ and satisfies the following condition

$$m_{12} \equiv \sqrt{m_1^2+\boldsymbol{P}_1^2} + \sqrt{m_2^2+\boldsymbol{P}_1^2} = \frac{m_1+m_2}{\sqrt{m_1 m_2}}\sqrt{m_1 m_2+\boldsymbol{\Pi}_1^2}\,. \tag{11}$$

Thus the total invariant mass takes the form

$$M = \frac{m_1+m_2}{\sqrt{m_1 m_2}}\sqrt{m_1 m_2+\boldsymbol{\Pi}_1^2+\frac{\mu_{12}}{m_1+m_2}\boldsymbol{P}_2^2} + \sqrt{m_3^2+\boldsymbol{P}_2^2}\,.$$

In the next step we define $\boldsymbol{\Pi}_2$ which is collinear with $\boldsymbol{P}_2$ and satisfies the equation

$$M = \frac{m_1+m_2}{\sqrt{m_1 m_2}}\sqrt{m_1 m_2+\boldsymbol{\Pi}_1^2} + \frac{m_3}{\sqrt{m_3\mu_3}}\sqrt{m_3\mu_3+\boldsymbol{\Pi}_2^2}\,. \tag{12}$$

We see that the quantities $\sqrt{m_1 m_2}$ and $\sqrt{m_3\mu_3}$ have the meaning of relativistic reduced masses [12].

Let us write explicit formulae for the above described transformation

$$\boldsymbol{P}_1 = A_1(\Pi_1)\boldsymbol{\Pi}_1, \quad \boldsymbol{P}_2 = A_2(\Pi_1, \Pi_2)\boldsymbol{\Pi}_2, \quad \boldsymbol{P}_3 = \boldsymbol{\Pi}_3, \tag{13}$$

where

$$A_1 = \left[\frac{(m_{12}^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2}{4m_{12}^2 \Pi_2^2}\right]^{\frac{1}{2}},$$

$$A_2 = \left[\frac{(M^2 - m_{12}^2 - m_3^2)^2 - 4m_{12}^2 m_3^2}{4M^2 \Pi_2^2}\right]^{\frac{1}{2}}.$$

Clearly m_{12}, M should be expressed in terms of $\boldsymbol{\Pi}_1$, $\boldsymbol{\Pi}_2$ (by means of (11) and (12)).

The last step leads us to a symmetric form of the Hamiltonian. In order to achieve it we introduce momenta $\boldsymbol{\pi}_1$, $\boldsymbol{\pi}_2$, which are collinear with $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ respectively, but they have norms changed in the following way

$$\pi_1^2 = \left(\frac{m_1+m_2}{\sqrt{m_1 m_2}}\sqrt{m_1 m_2 + \boldsymbol{\Pi}_1^2} - m_1 - m_2\right)\left(M + \sum_{\alpha=1}^{3} m_\alpha\right),$$

$$\pi_2^2 = \left(\frac{m_3}{\sqrt{m_3 \mu_3}}\sqrt{m_3 \mu_3 + \boldsymbol{\Pi}_2^2} - m_3\right)\left(M + \sum_{\alpha=1}^{3} m_\alpha\right). \tag{14}$$

The explicit formulae for these transformations are given by

$$\boldsymbol{\Pi}_1 = B_1(\pi_1, \pi_2)\boldsymbol{\pi}_1, \ \boldsymbol{\Pi}_2 = B_2(\pi_1, \pi_2)\boldsymbol{\pi}_3 \tag{15}$$

where

$$B_1 = \frac{\sqrt{m_1 m_2}}{m_1 + m_2}\left[\frac{\left(\dfrac{\pi_1^2}{M(\pi_1, \pi_2) + \sum_{\alpha=1}^{3} m_\alpha} + m_1 + m_2\right)^2 - (m_1 + m_2)^2}{\pi_1^2}\right]^{\frac{1}{2}},$$

$$B_2 = \frac{\sqrt{\mu_3 m_3}}{m_3}\left[\frac{\left(\dfrac{\pi_2^2}{M(\pi_1, \pi_2) + \sum_{\alpha=1}^{3} m_\alpha} + m_3\right)^2 - m_3^2}{\pi_2^2}\right]^{\frac{1}{2}}$$

with

$$M = \sqrt{\left(\sum_{\alpha=1}^{3} m_\alpha\right)^2 + \boldsymbol{\pi}_1^2 + \boldsymbol{\pi}_2^2}\,. \tag{16}$$

The mapping

$$\boldsymbol{P}_\alpha \to \boldsymbol{\pi}_\alpha = D_\alpha^{-1} P_\alpha, \ \boldsymbol{P}_3 \to \boldsymbol{\pi}_3 = \boldsymbol{P}_3, \tag{17}$$

where $D_\alpha = A_\alpha B_\alpha$, $\alpha = 1, 2$ induces a transformation $(X_1, X_2, X_3) \to (\xi_1, \xi_2, \xi_3)$. It follows from (5) and (17) that

$$\xi_\alpha = D_\alpha\left(X_\alpha - \frac{P_\alpha}{P_\alpha}\left(X_\alpha \frac{P_\alpha}{P_\alpha}\right)\right) + \frac{P_\alpha}{D_\alpha}((X_1P_1)R_\alpha + (X_2P_2)S_\alpha),$$

$$\xi_3 = X_3 \tag{18}$$

where $\alpha = 1, 2$ and

$$R_1 = \frac{m_{12}^4 - (m_1^2 - m_2^2)^2}{4m_{12}^4 D_1^2 (M + \sum_{\alpha=1}^{3} m_\alpha)^2}\left(\frac{m_{12}+m_1+m_2}{m_{12}-m_1-m_2} + \frac{E_{(3)}}{M}\right),$$

$$R_2 = -\frac{m_{12}^4 - (m_1^2 - m_2^2)^2}{4m_{12}^3 D_1^2 M (M + \sum_{\alpha=1}^{3} m_\alpha)^2},$$

$$S_1 = \frac{M^2 - m_{12}^2 + m_3^2}{4M^4 D_2^2 (M + \sum_{\alpha=1}^{3} m_\alpha)^2}((E_{(3)} + m_3)(M + \sum_{\alpha=1}^{3} m_\alpha) - 2Mm_{12}),$$

$$S_2 = \frac{M^2 - m_{12}^2 + m_3^2}{4M^4 D_2^2 (M + \sum_{\alpha=1}^{3} m_\alpha)^2}\left((E_{(3)} + m_3)(M + \sum_{\alpha=1}^{3} m_\alpha) + 2Mm_{12}\frac{M + m_{12} + m_3}{E_{(3)} - m_3}\right),$$

where

$$E_{(3)} = \frac{m_3}{\sqrt{m_3\mu_3}}\sqrt{m_3\mu_3 + \Pi_2^2(P_1, P_2)}.$$

Taking into account (12) and (14) one obtains the Hamiltonian in the symmetric form

$$H = \sqrt{(\sum_{\alpha=1}^{3} m_\alpha)^2 + \sum_{\alpha=1}^{3} \pi_\alpha^2}. \tag{19}$$

Before turning to the discussion of the above formula we should consider the meaning of the new variables (given in (17) and (18)).

In the first place we should inquire into the problem of correspondence between our relative (one should bear in mind that the following Poisson bracket relations are valid, namely $\{\xi_\alpha, P\} = 0$, $\alpha = 1, 2$) variables ξ_α, π_α, $\alpha = 1, 2$ and those $(\xi_\alpha)_S$, $(\pi_\alpha)_S$ introduced by Smith. It can be checked that in the limit $m_\alpha \to \infty$, $\alpha = 1, 2, 3$ we get

$$\xi_1 \to c_1(X_1)_{NR}, \quad \pi_1 \to c_1^{-1}(P_1)_{NR},$$

$$\xi_2 \to c_2(X_2)_{NR}, \quad \pi_2 \to c_2^{-1}(P_2)_{NR},$$

where $c_1 = (\mu_{12}/\sum_{\alpha=1}^{3} m_\alpha)^{\frac{1}{2}}$, $c_2 = (\mu_3/\sum_{\alpha=1}^{3} m_\alpha)^{\frac{1}{2}}$ and $\xi_3 \to (X_3)_{NR}$, $\boldsymbol{\pi}_3 \to (\boldsymbol{P}_3)_{NR}$. Comparing these asymptotic formulae with (10) one finds full correspondence (except for the common factor $(\mu/\sum_{\alpha=1}^{3} m_\alpha)^{\frac{1}{2}}$).

The transformation (17) of the relative momenta is something like a quasi-dilatation. It can be regarded as a result of a pure Lorentz transformation in the direction of $\boldsymbol{P}_\alpha$, $\alpha = 1$ and 2. As far as the relative coordinates (18) are concerned it is clear that the transformation is a result of the quasi-dilation and a transformation which changes only the longitudinal component (*i. e.*, the component in the direction of the linear momentum $\boldsymbol{P}_\alpha$) of $\boldsymbol{X}_\alpha = (\boldsymbol{X}_\perp)_\alpha + (\boldsymbol{X}_\parallel)_\alpha$. In spite of this modification, the vectors ξ_α, $\alpha = 1, 2$ may be considered as the proper relative positions.

It is worth saying at this point that the relative orbital momentum $\boldsymbol{l}^1$ of the pair (12) and the relative orbital momentum $\boldsymbol{l}^2$ of the third particle with respect to the pair (12) are invariant under all the transformation (17) and (18), *i. e.*,

$$l^1 \equiv \boldsymbol{X}_1 \times \boldsymbol{P}_1 = \xi_1 \times \boldsymbol{\pi}_1,$$

$$l^2 \equiv \boldsymbol{X}_2 \times \boldsymbol{P}_2 = \xi_2 \times \boldsymbol{\pi}_2. \tag{20}$$

4. *A symmetry group of the system*

The form of expression (19) suggests that the Hamiltonian is the generator of time translation in a generalized Poincaré group consisting of 0(9,1) and 10-dimensional translation.

In order to simplify the formalism we use the 9-dimensional notation

$$(\xi_i)_{i=1,\ldots,9} = (\xi_1, \xi_2, \xi_3),$$

$$(\pi_i)_{i=1,\ldots,9} = (\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\pi}_3).$$

The generators of the group can be written in the form

$$\pi_i,\ H,\ K_i = \xi_i H - \pi_i t, \quad \Gamma_{ij} = \xi_i \pi_j - \xi_j \pi_i, \tag{21}$$

with $i, j = 1, \ldots, 9$. Clearly, the above quantities span the Lie algebra of the group in which the product of the elements is defined as their Poisson bracket.

We find the following Poisson relations

$$\{H, \pi_i\} = 0,$$

$$\{H, \Gamma_{ij}\} = 0,$$

$$\{\pi_i, \pi_j\} = 0,$$

$$\{\pi_m, \Gamma_{ij}\} = \pi_i \delta_{mj} - \pi_j \delta_{mi},$$

$$\{\Gamma_{nm}, \Gamma_{ij}\} = \Gamma_{mi}\delta_{nj} + \Gamma_{jm}\delta_{in} + \Gamma_{in}\delta_{jm} + \Gamma_{nj}\delta_{im},$$

$$\{K_i, H\} = \pi_i,$$

$$\{K_m, \Gamma_{ij}\} = K_i\delta_{jm} - K_j\delta_{im},$$

$$\{K_i, \pi_j\} = H\delta_{ij},$$

$$\{K_i, K_j\} = -\Gamma_{ij}, \tag{22}$$

where all the subscripts run from 1 to 9.

Taking into account results of the previous section (especially (20)) one finds that all generators (7) of the proper Poincaré group except $\boldsymbol{K}$ belong to the set of generators (21), *i. e.*,

$$H, \boldsymbol{P} = (\pi_7, \pi_8, \pi_9) \equiv \boldsymbol{\pi}_3,$$

$$J_{kl} \equiv \sum_{m=1}^{3} \varepsilon_{klm} J_m = \sum_{\alpha=1}^{3} (\xi_k^\alpha \pi_l^\alpha - \xi_l^\alpha \pi_k^\alpha), \tag{23}$$

$$(k, l = 1, 2, 3).$$

If we use the following notation $\boldsymbol{K}_3 \equiv \boldsymbol{\xi}_3 H - \boldsymbol{\pi}_3 t = (K_7, K_8, K_9)$, then we get

$$\boldsymbol{K} = \boldsymbol{K}_3 - \frac{(\boldsymbol{J} - \boldsymbol{X}_3 \times \boldsymbol{P}) \times \boldsymbol{P}}{H + (H^2 - \boldsymbol{P}^2)^{\frac{1}{2}}}. \tag{24}$$

Therefore, the generator of the special Lorentz transformation $\boldsymbol{K}$ belongs to our algebra in CM(123) only.

5. *Grand orbital momentum*

Basing on an analysis of the non-relativistic case we know that the subalgebra, spanned by Γ_{ij}, $i, j = 1, \ldots, 6$ plays an important part in the description of free particle systems. They form an antisymmetric 6×6 tensor which will be called the grand orbital momentum tensor (one should remember the connection with the orbital momenta (20)). A basic property of the grand orbital momentum is described by the preposition: $\Gamma_{ij} = 0$ for all $i, j = 1, 2, \ldots, 6$ if and only if we deal with a central 3-body collision (*i. e.*, at some instant t_0 the particle trajectories intersect: $\boldsymbol{x}^1 = \boldsymbol{x}^2 = \boldsymbol{x}^3$).

From the generators Γ_{ij} we construct the positive definite quantity:

$$\Gamma^2 = \frac{1}{2} \sum_{i,j=1}^{6} (\Gamma_{ij})^2 \tag{25}$$

which is a suitable measure of the togetherness with which all particles emerge from or go in towards a common region of interaction. Another important property of Γ^2 is described by the following Poisson bracket relations

$$\{\Gamma^2, H\} = \{\Gamma^2, P_i\} = \{\Gamma^2, J_{ki}\} = \{\Gamma^2, K_i\} = 0 \tag{26}$$

with $k, i = 1, 2, 3$, which can be obtained by means of (22), (23) and (24). It means that Γ^2 is Poincaré invariant.

It follows immediately from the definition of Γ_{ij} that

$$\Gamma^2 = (\sum_{i=1}^{6} \xi_i^2)(\sum_{j=1}^{6} \pi_j^2) - (\sum_{i=1}^{6} \xi_i \pi_i)^2. \tag{27}$$

Now, we introduce a "radial" variable (in our 6-dimensional space of relative motion)

$$\varrho^2 = \sum_{i=1}^{6} \xi_i^2.$$

The time derivative of ϱ is

$$\dot{\varrho} \equiv \{\varrho, H\} = (\sum_{i=1}^{6} \xi_i \pi_i)/\varrho H. \tag{28}$$

Taking account of (16) and (28) we find that

$$\Gamma^2 = \varrho^2[\sum_{i=1}^{6} \pi_i^2 - \dot{\varrho}^2 H^2] =$$

$$= \varrho^2[M^2 - (\sum_{\alpha=1}^{3} m_\alpha)^2 - \dot{\varrho}^2 H^2], \tag{29}$$

where M is the total invariant mass of the system.

The minimum value b of ϱ, for which clearly $\dot{\varrho} = 0$, may be considered as the relativistic analogue of the 3-body impact distance (which was introduced by Smith for the nonrelativistic case). It follows that it is related to the quantity Γ^2 by the very simple and interesting formula

$$\Gamma^2 = b^2[M^2 - (\sum_{\alpha=1}^{3} m_\alpha)^2]. \tag{30}$$

Thus we see that given the total invariant mass of the system in the asymptotic state (when the particles are far-away from each other, moving without interaction) the value of Γ^2 determines uniquely the impact distance of the particles. The above formula is the analogue of non-relativistic expression:

$$\Gamma^2 = 2\mu T b^2.$$

6. Concluding remarks

A group of three, successive transformation has led to the symmetric form of the Hamiltonian. The Hamiltonian is a generator which belongs to the algebra of generalized Poincaré group. This group of symmetry has enabled us to introduce the relativistic grand angular momentum. We have found a Poincaré invariant parameter Γ^2 which together with the total invariant mass of the system describes the impact distance of the many-

-particle system. This relation enables us to answer the question whether the particles come out from the same interaction region of a given diameter.

The whole procedure has been presented for a three particle system, but can be easily generalized to a many-body system.

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