NONLOCALITY AND GENERALIZED ANALYTICITY

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It is shown that for the nonlocal theory of Snyder's type dispersion amplitudes are generalized analytic functions of their arguments.

The true test of the microcausality at high energies are indicated.

1

In the previous papers [1–6] the nonlocal relativistic theory of elementary particles had been suggested. The fundamental point is the assumption that dispersion amplitudes are generalized analytic functions of their arguments. The discussion is the following.

It is known that basing on the fundamental principles of local relativistic theory of quantized fields, dispersion amplitudes of some scattering processes were proved to be analytic functions of the complex energy variable for fixed momentum transfer [7, 8] and of the complex momentum transfer variable for fixed energy [9, 10] and moreover that they are bounded by a certain polynomial for high energies.

Therefore two possibilities of modifying the microcausal principle may be down as follows.

1. Either the dispersion amplitudes are still analytic functions of their arguments, but they increase faster than a certain polynomial for high energy, *i. e.*

$$|f(z)| > Ae^{\alpha|z|^{\beta}} \text{ as } z \to \infty. \tag{1}$$

Here A, α and β are positive constants.

2. Or the dispersion amplitudes are not analytic functions and their real and imaginary parts verify equations more complicated than Cauchy-Riemann's

$$u_x = g(v_y, u, v), \quad u_y = h(v_x, u, v). \tag{2}$$

The first possibility had been examined by [11] and the second considered by [1–6]. It is shown that if we suppose g and h to be two linear functions of their arguments and

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the principle of correspondence is taken into account, then the general case (2) is the combination of the two following ones:

a)
$$u_x = \frac{1}{p} v_y, \quad u_y = -\frac{1}{q} v_x \tag{3}$$

here

$$p(x, y) > 0, \quad q(x, y) > 0.$$

In particular, for $p \equiv q$, we obtain the p-analytic functions. The function, whose real and imaginary parts fulfil equations (3) is said a (p, q)-analytic function [12].

b) The dispersion amplitudes are the generalized analytic function of Vekua [13], hence their real and imaginary parts verify the equations

$$u_x = v_y + au + bv, \quad u_y = -v_x + cu + dv \tag{4}$$

or

$$\partial_{\bar{z}} f(z) + A(z) f(z) + B(z) \bar{f}(z) = 0.$$

The result of paper [5] have shown that generalized analyticity of the dispersion amplitudes leads in fact to the violation of microcausality and in addition it is possible to establish a class of distribution in order to satisfy the microcausal principle given by Efimov [14] at large distances. For the purpose of clarifying the connection between the violation of the microcausality and the generalized analyticity of the dispersion amplitudes, the analytic property will be considered in the Snyder's nonlocal theory. The true test of the microcausality at high energy are indicated in Paragraphs 3 and 4. The discussion and conclusions are given in the last paragraph.

2

In this paragraph the connection between the nonlocality and the generalized analyticity will be explained. Namely, we shall prove that dispersion amplitudes in the nonlocal relativistic theory of Snyder type [15] are generalized analytic functions of their arguments.

For simplicity, it is possible to restrict ourselves to consider the scattering of two scalar particles

$$a_1 + b_1 \rightarrow a_2 + b_2.$$

The 4-momentums of a_i and b_i are denoted respectively by p_i and q_i. As is well known, the dispersion amplitude of the above process $T(s, t)$ in a local relativistic theory is an analytic function of two Mandelstam variables

$$s = (p_1 + q_1)^2 \text{ and } t = (p_1 - p_2)^2.$$

For convenience, the following representations are introduced

$$f(s) \equiv T(s, t) \text{ for fixed } t$$

$$s = x^1 + ix^2,$$

$$f(s) = U_1(x^1, x^2) + iU_2(x^1, x^2).$$

The functions $U_i (i = 1, 2)$ are harmonic in the (x^1, x^2)-plane.

Let us now consider x^1 and x^2 as two moving coordinates of a certain surface which is described by the following vector equation

$$\vec{r} = \vec{r}(x^1, x^2) \tag{5}$$

im a certain 3-dimensional space.

It is easily seen that for an arbitrary nonlocal theory the equation (5) represents a certain surface different from a plane. Indeed, it is known that in the case when the theory is nonlocal, a certain smallest length of coordinates space must exist. Therefore in momentum space the squares of all 4-momentum are bounded by a certain largest quantity. This means that (5) is different from the equation of a plane. The main discussion follows; it is known that the microcausality of a local relativistic theory is equivalent to the assumption that the dispersion amplitude $f(s)$ is an analytic function of s in the s-plane, *i. e.* $U_1(x^1, x^2)$ and $U_2(x^1, x^2)$ verify the well-known equations of Cauchy-Riemann

$$\frac{\partial}{\partial x^1} U_1 = \frac{\partial}{\partial x^2} U_2 \tag{6a}$$

$$\frac{\partial}{\partial x^2} U_1 = -\frac{\partial}{\partial x^1} U_2. \tag{6b}$$

These equations are invariant under linear transformations of coordinates consisting of rotations and translations in the (x^1, x^2)-plane.

For an arbitrary nonlocal relativistic theory (5) represents a Riemann space of two dimensions, the metric of which is as follows

$$ds^2 = g_{\alpha\beta}(x) dx^\alpha dx^\beta \tag{7}$$

here

$$g_{\alpha\beta}(x) = \frac{\partial \vec{r}}{\partial x^\alpha} \cdot \frac{\partial \vec{r}}{\partial x^\beta}.$$

The vector field $U_i(x^1, x^2)$ becomes now the functions given in this space. The Riemann space allows the transformations to be more general than the linear transformations mentioned above

$$'x^\alpha = f^\alpha(x). \tag{8}$$

The equations (6) are clearly not invariant under (8). Therefore, it is necessary to seek other equations which fulfil the following conditions:

1. They are invariant under (8).
2. They contain only the first derivatives of $U_i(x^1, x^2)$ with respect to x^α.
3. They are a generalisation of (6).

It is easily seen that these equations are as follows

$$\nabla_1 U_1 = \nabla_2 U_2, \tag{9a}$$

$$\nabla_2 U_1 = -\nabla_1 U_2, \tag{9b}$$

here ∇_α are covariant derivatives. It is clear that the equations (9) are the unique ones to fulfil all the above imposed requirements.

By using the symbols of Christoffel $\Gamma^\lambda_{\alpha\beta}$ the equations (9) can be written in the following form

$$\frac{\partial}{\partial x^1} U_1 - \frac{\partial}{\partial x^2} U_2 - (\Gamma^\lambda_{11} - \Gamma^\lambda_{22}) U_\lambda = 0, \tag{10a}$$

$$\frac{\partial}{\partial x^2} U_1 + \frac{\partial}{\partial x^1} U_2 - 2\Gamma^\lambda_{12} U_\lambda = 0. \tag{10b}$$

Or in the complex form

$$\frac{\partial}{\partial \bar{s}} f(s) + A(s) f(s) + B(s) \bar{f}(s) = 0 \tag{11}$$

here

$$A(s) = \tfrac{1}{4}\{(\Gamma^1_{22} - \Gamma^1_{11} - 2\Gamma^2_{12}) + i(\Gamma^2_{11} - \Gamma^2_{22} - 2\Gamma^1_{12})\}$$

$$B(s) = \tfrac{1}{4}\{(\Gamma^1_{22} - \Gamma^1_{11} + 2\Gamma^2_{12}) - i(\Gamma^2_{11} - \Gamma^2_{22} + 2\Gamma^1_{12})\}.$$

The equation (11) shows that $f(s)$ is a generalized analytic function of the Vekua type.

Let us now consider the important case where (7) represents a sphere. We obtain then

$$g_{11} = g_{22} = \sqrt{g} = \frac{1}{(1+a^2 s\bar{s})^2}, \qquad g_{12} = 0$$

$$k = \frac{1}{a} > 0$$

and therefore

$$f(s) = (1+a^2 s\bar{s})^{-2}\, \Phi(s) \tag{12}$$

here $\Phi(s)$ is a certain analytic function. The formula (12) shows that $f(s)$ is a generalized analytic function with $e^{\omega(s)}$ given by the simple factor

$$\exp\{\omega(s)\} = (1+a^2 s\bar{s})^{-2}$$

$f(s)$ verifies the following equation

$$\frac{\partial}{\partial \bar{s}} f(s) + \frac{2s}{1+a^2 s\bar{s}} f(s) = 0$$

which is of type

$$\frac{\partial}{\partial \bar{s}} f(s) + A(s) f(s) = 0.$$

This type of equation was considered in the previous papers [1–6]. This special case presented above corresponds to the nonlocal theory of Snyder type.

It is very interesting to remind that by Vekua [13] the coefficient B of equation (11) is identical to zero, $B \equiv 0$, if and only if equation (7) represents a surface of second degree with positive curvature.

In resuming, for an arbitrary nonlocal theory, we have proved in this paragraph that the dispersion amplitude $f(s)$ is a generalized analytic function of the complex energy variable. In particular the nonlocal theory of Snyder type was also considered and the dispersion amplitude in this case is defined by the formula (12).

It is easily seen that analogous results are obtained immediately in the case when the s variable is replaced by the t variable. Hence, it is very possible to assert that in a class of nonlocal theories dispersion amplitudes are generalized analytic functions of their arguments.

This assertion is clearly very important for studying the nonlocal theories of elementary particles.

3

Thus the connection between the nonlocality and the generalized analyticity of dispersion amplitudes was shown. This fundamental result plays, probably, an important role in studying the nonlocal theory of elementary particles for the simple reason that the study of nonlocal theory is reduced to the simpler problem of considering the analytic property of dispersion amplitudes by means of generalized analytic function theory. It is similar to the actual analytic theory of elementary particles.

On the other hand, the results of previous papers [1–5] show that the asymptotic theorems, such as Pameranchuk's theorem, which for a long time were considered as the criterion to check the validity of the analyticity of dispersion amplitudes at high energies, were also deduced from generalized analyticity of dispersion amplitudes. This means that the role of those theorems, considered as the test of the microcausality, is negated. Therefore, the following problem of fundamental value arises: what is the true test of the microcausality at high energy?

This problem will be solved in part in this paragraph. In order to seek the true test of the microcausality, as suggested by [4], it is very necessary to find the relations which are deduced only from the analyticity and cannot be deduced from any generalized analyticity. To do this, let us consider, what is the fundamental feature of the analyticity of an arbitrary function? As is known, the analyticity of $f(z)$ is characterized by the simple connection between its real and imaginary parts $u(x, y)$ and $v(x, y)$, *i. e.* by Cauchy-Riemann equations. It is clear that the theorem of Cauchy characterizes fully this feature. As the direct consequences of this theorem the dispersion relations and the dispersion sum rules may be considered to be true tests for checking the validity of microcausality at high energy.

In this paragraph, it will be shown that, indeed, the dispersion relations of the acausal theory and of the causal theory are different from each other.

To begin, we notice that similar to the causal theory the mathematical basis of the acausal dispersion relations is the following generalized formula of Cauchy:

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \Omega_1(z, z')f(z')dz' - \Omega_2(z, z')\bar{f}(z')dz'.$$

For simplicity, we restrict ourselves to the study of the case when $B \equiv 0$, and we obtain then

$$f(z) = \frac{e^{\omega(z)}}{2\pi i} \int_{\Gamma} \frac{e^{-\omega(z')}f(z')}{z'-z}\, dz'$$

here

$$\omega(z) = \frac{1}{\pi} \iint_G \frac{A(z')}{z'-z}\, dx'dy'.$$

Basing on the results obtained in the previous paragraph, we can confirm that in case when $f(z)$ represents the acausal amplitude, the function $\varphi(z)$ given by

$$\varphi(z) = e^{-\omega(z)}f(z) \tag{13}$$

can be considered to be the causal amplitude and the factor exp $\{\omega(z)\}$ characterizes the nonlocality. Following [16] we shall examine only the so-called m i n i m u m nonlocal theory, the dispersion amplitudes of which fulfil the following requirements:

1. Causal and acausal amplitudes possess the same spectra. This means that their singularities are to coincide to each other for finite value of arguments.

2. Acausal dispersion amplitudes verify the crossing symmetry relations.

3. Macrocausality at large distances is guaranteed. It is clear that the above mentioned conditions imposed severe restrictions on the factor of the nonlocality exp $\{\omega(z)\}$. Indeed, let $T_a(s, t)$ and $T_c(s, t)$ be the acausal and causal amplitudes which are related to each other by

$$T_a(s, t) = e^{\omega(s,t)}T_c(s, t).$$

The requirement 1) means that $e^{\omega(s,t)}$ does not possess any singularity in the s- and t-planes. Therefore, unphysical singularities are guaranteed to disappear.

It is easily seen that $e^{\omega(s,t)}$ has to be a generalized analytic function, as an arbitrary analytic to fulfil the imposed conditions is identically to zero.

The requirement 2) tells that T_a verifies the relation of the form

$$T_a(u, t) = T_a(s, t)^*$$

from which we obtain

$$\omega(u, t) = \omega(s, t)^*.$$

The macrocausal requirement 3) shows that for a certain sufficiently large value of energy, we need to have

$$e^{\omega(s,t)} \approx 1.$$

It will be shown below that by using the generalized formula of Cauchy and the above imposed restrictions the acausal dispersion relations will be established easily.

As an illustrative example, let us consider the πN-scattering which is well known. Let p_i and q_i be respectively the 4-momentums of nucleons and pions at initial and final states. Their masses are denoted by M and m.

The dispersion amplitudes of this process are given by

$$F_a(s, t) = \delta_{\alpha\beta} F_a^+(s, t) + \tfrac{1}{2}[\tau_\alpha, \tau_\beta] F_a^-(s, t)$$

here $F_a^\pm(s, t)$ are expressed in term of the spinor invariant amplitudes $A_a^\pm(s, t)$ and $B_a^\pm(s, t)$ by

$$F_a^\pm(s, t) = A_a^\pm(s, t) + \tfrac{1}{2}\hat{k} B_a^\pm(s, t)$$

in which $k = q_1 + q_2$. The causal dispersion amplitudes $A_c^\pm$ and $B_c^\pm$ are related to $A_a^\pm$ and $B_a^\pm$ by the following relations

$$A_a^\pm(s, t) = e^{\omega(s,t)} A_c^\pm(s, t)$$

$$B_a^\pm(s, t) = e^{\omega(s,t)} B_c^\pm(s, t).$$

Following Chew [17] it is convenient to introduce the new variables ν and χ^2. Functions $A_a^\pm$, $B_a^\pm$ and ω now become functions of these variables. The crossing symmetry relation reduces to the fact that $\omega(\nu)$ is an even function of ν

$$\omega(\nu) = \omega(-\nu).$$

The dispersion relations for $A_a^\pm(\nu; \chi^2)$ and $B_a^\pm(\nu; \chi^2)$ are easily established

$$\mathrm{Re}\, A_a^\pm(\nu; \chi^2) = \frac{e^{\omega(\nu)}}{\pi} P \int_{m-\frac{\chi^2}{M}}^{+\infty} d\nu' e^{-\omega(\nu')}\, \mathrm{Im}\, A_a^\pm(\nu') \left(\frac{1}{\nu'-\nu} \mp \frac{1}{\nu'+\nu}\right)$$

$$\mathrm{Re}\, B_a^\pm(\nu; \chi^2) = \frac{G^2}{2M} e^{\omega(\nu)} \left(\frac{1}{\nu_B - \nu} \mp \frac{1}{\nu_B + \nu}\right) +$$

$$+ \frac{e^{\omega(\nu)}}{\pi} P \int_{m-\frac{\chi^2}{M}}^{+\infty} d\nu' e^{-\omega(\nu')}\, \mathrm{Im}\, B_a^\pm(\nu') \left(\frac{1}{\nu'-\nu} \mp \frac{1}{\nu'+\nu}\right),$$

here

$$\nu_B = -\frac{m^2}{2M} - \frac{\chi^2}{M}.$$

Thereby the dispersion relations for forward scattering have the form

$$D_+(\nu) = \tfrac{1}{2}\, e^{\omega(\nu)-\omega(m)}\left(1+\frac{\nu}{m}\right) D_+(m) + \frac{2e^{\omega(\nu)}}{m^2}\cdot\frac{f^2}{4\pi}\cdot\frac{k^2}{\nu-\dfrac{m^2}{2M}} +$$

$$+\frac{e^{\omega(\nu)}}{4\pi^2}\, k^2 \int_m^{+\infty} \frac{d\nu'}{k'}\, e^{-\omega(\nu')}\left(\frac{\sigma_+(\nu')}{\nu'-\nu} + \frac{\sigma_-(\nu')}{\nu'+\nu}\right),$$

$$D_-(\nu) = \tfrac{1}{2}\, e^{\omega(\nu)-\omega(m)}\left\{\left(1+\frac{\nu}{m}\right) D_-(m) - \left(1-\frac{\nu}{m}\right) D_+(m)\right\} -$$

$$-\frac{2e^{\omega(\nu)}}{m^2}\cdot\frac{f^2}{4\pi}\cdot\frac{k^2}{\nu+\dfrac{m^2}{2M}} + \frac{e^{\omega(\nu)}}{4\pi^2}\, k^2 \int_m^{+\infty} \frac{d\nu'}{k'}\, e^{-\omega(\nu')}\left[\frac{\sigma_+(\nu')}{\nu'+\nu} + \frac{\sigma_-(\nu')}{\nu'-\nu}\right].$$

Here $D_\pm$ and $\sigma_\pm$ possess the sense given by [17].

4

It is known that the dispersion sum rules are considered as an instrument, with the aid of which low energy region is related to the high energy one. In this paragraph the acausal dispersion sum rules will be established in a simple way for illustration.

Suppose $f(\nu)$ and $\varphi(\nu)$ are respectively the dispersion amplitudes of acausal and causal theories. They are related by

$$f(\nu) = e^{\omega(\nu)}\varphi(\nu).$$

For the function $f(\nu)\, e^{-\omega(\nu)}$, which is analytic in the upper half plane, we can apply the ordinary theorem of Cauchy with the contour to be the real axis and the semi circle in the upper plane

$$\int_{-A}^{A} f(\nu')e^{-\omega(\nu')}d\nu' + \int_{C_A} f(\nu')e^{-\omega(\nu')}d\nu' = 0.$$

The integral over C_A can be neglected for $f(\nu)$ decreasing sufficiently rapidly as $\nu \to \infty$ and we obtain the following relation

$$\int_{-\infty}^{+\infty} \operatorname{Im} f(\nu)e^{-\omega(\nu)}d\nu = 0$$

which is called the acausal superconvergent sum rule.

If the amplitude $f(v)$ does not decrease for large v, then the sum rule is obtained by evaluating the integral over C_A. By using the Regge behaviour for high energies the sum rule is obtained easily

$$\int_{-A}^{A} \operatorname{Im} f(v) e^{-\omega(v)} dv = \sum \frac{A^{\alpha_i+1}}{\alpha_i+1} \operatorname{Im} b_i(1+e^{i\pi\alpha_i}).$$

This is acausal Regge sum rule.

5

The main results of this paper are as follows. By paragraph 2 it is proved that dispersion amplitudes of nonlocal theory of Snyder type are generalized analytic functions of their arguments. In considering a particular case, the factor $e^{\omega(v)}$ which characterizes the nonlocality, is found. With the aid of this assertion, the study of nonlocal theory become more effective, therefore it can play an important role in developping nonlocal theory of elementary particles. The latter then can be constructed in a way similar to the actual analytic theory of elementary particles, *i. e.* it is necessary to establish the relations which allow them to be compared directly with the experiment, for instance, dispersion relations, sum rules and the Mandelstam representation. In the paragraphs 3 and 4 it is shown that the dispersion relations and the dispersion sum rules can be considered as the criterions in order to check the microcausality at high energies. This assertion bases on the fact that they are different from each other for nonlocal and local theories.

For the purpose of checking directly with the experiment, the dispersion relations of nonlocal theory are established πN-scattering processes.

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REFERENCES

[1] Trân hu'u Phát, *Scientific Comm. of Hanoi University*, Section of Phys., Vol. III, 1970.
[2] Trân hu'u Phát, *Acta Phys. Polon.*, **B1**, 323 (1970), **B1**, 331 (1970).
[3] Trân hu'u Phát, *Ann. Phys.* (Germany), **26**, 201 (1971).
[4] Trân hu'u Phát, *Ann. Phys.* (Germany), **27**, 33 (1971).
[5] Trân hu'u Phát, *Acta Phys. Polon.*, **B2**, 763 (1971).
[6] Trân hu'u Phát, *Acta Phys. Hungar.*, **30**, 381 (1971).
[7] N. N. Bogolyubov, D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Interscience Publishers Inc., New York 1959.
[8] N. N. Bogolyubov, B. V. Medvedev, M. K. Polivanov, *Problems in Theory of Dispersion Relations*, Fizmatgiz, Moscow 1958.
[9] J. Bros, A. Messiah, R. Stora, *J. Math. Phys.*, **2**, 639 (1961).
[10] J. Bros, H. Epstein, V. Glaser, *Nouvo Cimento*, **31**, 1265 (1964).
[11] A. A. Logunov, N. V. Hieu, *Preprint* P2873, Dubna 1966.

[12] G. N. Poloji, *Generalized Analytic Functions*, Kiev 1965 (in Russian).
[13] I. N. Vekua, *Generalized Analytic Functions*, Fizmatgiz, Moscow 1958 (in Russian).
[14] G. V. Efimov, *Preprint* P2870, Dubna 1966.
[15] H. Snyder, *Phys. Rev.*, **71**, 38 (1947).
[16] D. I. Blokhinsev, *Space and Time in Microworld*, Nauka, Moscow 1968 (in Russian).
[17] G. F. Chew *et al.*, *Phys. Rev.*, **106**, 1345 (1957).