WAVE FUNCTIONS OF FAST-MOVING TWO-BODY SYSTEMS

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A hypothesis is proposed which concerns the description of two-body systems in fast motion. The main idea consists in the determination of an absolute three-dimensional space spanned on the absolute coordinates such as those introduced by Kadyshevsky. The proposed framework results implicitly in the distinguished role of the over-all centre of mass system in accounting for the internal dynamics of an isolated system. The example of two spinless particles is presented which, despite its relative simplicity, exhibits the most important aspects of the proposed framework.

1. *Introduction*

The derivation of a reliable procedure of describing two-body systems in the relativistic theories remains up to now an extremely difficult problem. The literature on this question is rather involved, hence we will mention only certain aspects of the problem which are useful in further considerations. Roughly speaking, there are two ways of treating the two-body problem: the first is based on the quasi-potential approach [1], and the second on the Bethe-Salpeter equation [2]. According to the first approach one starts with two free-body equations

$$R_1\psi_1 = R_2\psi_2 = 0,$$

where $R_{1,2}$ are the Klein-Gordon or Dirac operators, and then instead of two single-body functions one introduces the single two-body wave function $\psi(x_{1\mu}, x_{2\mu})$ satisfying two equations $R_1\psi = R_2\psi = 0$. In the next step one modifies them by introducing "quasi-external" potentials $V_{1,2}$ so that

$$(R_1 - V_1)\,\psi = (R_2 - V_2)\,\psi = 0. \tag{1.1}$$

The mutual consistency of equations (1.1) imposes several constraints which make this type of approach rather restricted. Moreover, the very "philosophy" of replacing the single internal interaction between two particles by two quasi-external potentials does not seem convincing. Different aspects of this approach are discussed by several authors in the classical [3, 4] as well as in the quantum frameworks [1, 5, 6].

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The fundamental difficulty of the relativistic two-body problem follows from the unavoidable interference between the total four-momentum of the system as a whole and those variables which parametrize its internal structure. The same concerns the second approach based on the Bethe-Salpeter equation. The latter can be written in the form

$$(R_1R_2 - V)\,\psi(x_{1\mu}, x_{2\mu}) = 0, \tag{1.2}$$

where V is an invariant function (operator) responsible for the interaction. Apart from the fundamental difficulties of the theory of fields, the greatest disease of this approach is that it leads to abnormal solutions which do not possess the proper behaviour in the non--relativistic ($c \to \infty$) and/or the single-particle ($m_1 \to \infty$) limits [7]. On the other hand, in both these limits we have relatively good theories — the Schroedinger equation and the single-body equation like that of Dirac, respectively — which suggest some modification of the description of two-body systems presented below. It should also be remembered that the four-dimensional symmetry leads, in two- or more than two-body systems, to relative times (energies), the physical interpretation of which is quite obscure.

In order to emphasize some properties of the single-body systems interesting for the generalization on two-body systems, let us consider the hydrogen atom regarding proton as the external centre of force [8]. Here the Dirac equation enables us to evaluate the exact solution $\psi(x_\mu)$ with definite transformation properties under the Lorentz transformations. This gives at once the recipe for boosting the hydrogen atom state from the reference frame R to any other R'. Thus

$$[i(\not{p}-e\not{A})-m]\,\psi(x_\mu)=0, \qquad \psi'(x'_\mu) = D\psi[\Lambda^{-1}_{\mu\nu}(x'_\nu - a_\nu)], \tag{1.3}$$

where D is the corresponding representation of the homogeneous Lorentz group, and $x'_\mu = \Lambda_{\mu\nu}x_\nu + a_\mu$ is the ten-parameter inhomogeneous transformation between R and R'. Of course, the external field A_μ makes the covariant Eq. (1.3) not invariant under the Lorentz-Poincaré transformation. In the reference frame R^0 where the centre of force is at rest and the boundary conditions are independent of time, one can factorize the space from the time dependence of ψ, and hence deal with the positive definite (three-dimensional) operator of the energy of electron and the corresponding eigenvalue problem. Thus in the limiting case of the single-body system the question of indefinite Lorentz metric, and hence all implications of this fact disappear [7]. The stationary solutions of Eq. (1.3) take in R^0 the well-known form

$$\psi^0 = \psi^0_{E^0}(\boldsymbol{x}^0)\exp(-iE^0t^0), \tag{1.4}$$

where E^0 is the total energy of electron in R^0. If this state is represented in the reference frame R which moves with respect to R^0 with the velocity $\boldsymbol{v}$ parallel to the z-axis, then we have

$$\psi(x, y, z, t) = D(v)\,\psi^0[x, y, \gamma(z-vt)]\exp[i(Pz-Et)], \tag{1.5}$$

where $P = v\gamma E^0$, $E = \gamma E^0$, and $\gamma = (1-v^2)^{-1/2}$. Let us point out that the wave function ψ from (1.5) determines also the boosted wave function of the hydrogen atom very similarly as in the impulse approximation where one neglects the internal motion of the constituents of the system [9]. This apparent approximation is strictly connected with the single--body problem. The lack of the translation invariance of Eq. (1.3) (external field) together with the indefinite metric of space-time are responsible for the essential difference between the single-body and the two- (or more than two) -body problem in the covariant theories. In principle, the basis for determining the boost of a two-body state can be the Bethe--Salpeter equation, and this is one of the ways in which one introduces the so-called covariant wave functions [10]. Their effective construction for fast moving systems encounters fundamental difficulties. In this situation one applies the expansion into powers of the velocity v of the system as a whole, evaluating only the first-order corrections proportional to v^2/c^2. It turns out that this relativistic distortion of a composite system starts to play an important role in high-energy collisions. For example in the collisions of the electron with the deuteron, the interaction and the rest-structure of the deuteron are relatively well known, hence, in principle, these corrections should become measurable [11].

The tentative hypothesis proposed below avoids some of the aforementioned difficulties by distinguishing implicitly the over-all centre of mass system of the whole isolated system under consideration.

2. *Two-body problem*

Let us consider the system S_2 composed of two spinless particles of masses m_1, m_2. We assume that S_2 has the hamiltonian $\hat{H}_2$ of the following form ($c = 1$)

$$\hat{H}_2 = (\hat{h}_2^2+\hat{\boldsymbol{P}}^2)^{1/2};\ \hat{h}_2 = (m_1^2+\hat{\boldsymbol{q}}^2)^{1/2}+(m_2^2+\hat{\boldsymbol{q}}^2)^{1/2}+V(\hat{\boldsymbol{y}})$$
$$\hat{h}_2 \xrightarrow[c\to\infty]{} (m_1+m_2)c^2+\hat{\boldsymbol{q}}^2/2\mu+V(\hat{\boldsymbol{y}}), \qquad \mu = m_1m_2/(m_1+m_2), \tag{2.1}$$

which determines the corresponding Schroedinger equation

$$i\partial\Psi/\partial t = \hat{H}_2\Psi(\boldsymbol{X}, \boldsymbol{y}, t). \tag{2.2}$$

Here $\hat{\boldsymbol{X}}, \hat{\boldsymbol{P}}$ is the pair of the canonical coordinates[1], where $\hat{\boldsymbol{P}}$ is the operator of the total momentum of S_2, and $\hat{\boldsymbol{X}}$ is the over-all "position" of S_2. The relation batween $\boldsymbol{X}$ and the centre of gravity of S_2 will be discussed in the next section. The second independent pair of the canonical variables $\boldsymbol{y}$, $\boldsymbol{q}$ parametrizes the internal structure of S_2 in the y-continuum which will be clarified as our account proceeds. Note that both coordinates $\hat{\boldsymbol{X}}$ and $\hat{\boldsymbol{y}}$ are, as seen from (2.2), opposed to the same time t, which fact exhibits the non-covariant character of the proposed framework. In the same way one avoids the known difficulties connected with the relative time (energy) variable of the constituents of S_2 being inherent in any covariant theory. The function $V(\boldsymbol{y})$ spanned on the y-continuum accounts for the interaction between m_1 and m_2, and will be called the absolute potential.

We postulate, and this is the fundamental property of the y-continuum, that the canonical variables $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{q}}$ (as q-numbers) commute with ten generators $\hat{J}_{\mu\nu}$, $\hat{P}_\mu$ of the Lorentz--Poincaré group of the isolated system S_2 [12]

$$[\hat{\boldsymbol{y}}, \hat{J}_{\mu\nu}] = [\hat{\boldsymbol{q}}, \hat{J}_{\mu\nu}] = [\hat{\boldsymbol{y}}, \hat{P}_\mu] = [\hat{\boldsymbol{q}}, \hat{P}_\mu] = 0. \tag{2.3}$$

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[1] Generally $\hat{\boldsymbol{X}}, \hat{\boldsymbol{P}}$ are q-numbers, but in the Schroedinger representation $\hat{\boldsymbol{X}} = \boldsymbol{X}$ and $\hat{\boldsymbol{P}} = -i\nabla_{\boldsymbol{X}}$.

These equalities ensure the absolute nature of the y-continuum and make $\hat{y}$, $\hat{q}$ essentially different from any internal coordinates discussed by several authors, *e. g.* [3] and [13]. The latter are connected *a priori* with the space-time coordinates of the constituents, hence they cannot commute with the generators $\hat{J}_{\mu\nu}$ and $\hat{P}_{\mu}$. In our case, the relations between $\hat{y}$, $\hat{q}$ and the space-time coordinates are established *a posteriori*, as will be explained further on. The commutation of $\hat{y}$, $\hat{q}$ with $\hat{J}_{\mu\nu}$, $\hat{P}_{\mu}$ is similar to the commutation between two pairs of the canonical coordinates $(\hat{x}_1, \hat{p}_1)$ and $(\hat{x}_2, \hat{p}_2)$ of two particles. Both reflect that the corresponding degrees of freedom are entirely independent. However, the commutation relations (2.3) require the following hierarchy of the description of the system: first one must determine the internal state of S_2 in the y-continuum and next, on getting from the internal laws of motion the c-number (absolute) characteristics of S_2 (such as its spin and mass), one determines in the conventional way the generators $\hat{J}_{\mu\nu}$, $\hat{P}_{\mu}$ of S_2. In consequence $\hat{J}_{\mu\nu}$, $\hat{P}_{\mu}$ are independent of $\hat{y}$ and $\hat{q}$, and thus they commute with $\hat{y}$, $\hat{q}$ as required by (2.3).

The structure of the hamiltonian $\hat{H}_2$ (no other reason!) implies that the eigenvalue y of $\hat{y}$ is isomorphic with the relative coordinated of the particles m_1, m_2 represented in the centre of mass R^c where $P = P^c$ vanishes. Consequently, q means the relative momentum of those particles represented also in R^c. We shall say that this isomorphy takes place *a posteriori, i. e.* after realizing some internal state of S_2 in the y-continuum, whereas any quantity parametrized in the y-continuum is *a priori* absolute. *A posteriori*, the c-number eigenvalues of this quantity coincide with the corresponding quantities in space-time represented in R^c. In other words, *a posteriori* this quantity characterized by c-numbers can be relativized, *i. e.* replaced by the suitable covariant quantities represented in R^c. The y and q variables provide an example of this projection onto space-time which will be discussed in detail in Section 3. The same projection operation, which in fact conflicts with the orthodox covariance, is largely practised without clearly indicating its not necessarily covariant background. For example, one evaluates the dipole, quadrupole, or higher multipoles of a composite particle (*e. g.* nucleus) in the rest-frame of this particle where its internal state ψ is known. On getting these absolute (c-number) characteristics one identifies them (*a posteriori*) with the corresponding covariantly defined multipoles represented in the rest-frame R_c of the composite particle [14]. Remember that for loosely bound systems — like nuclei or atoms — when the non-relativistic framework works well, the y-continuum coordinates which parametrize the internal state ψ of those systems mean simply the relative coordinates in the Galilean space. The same projection also takes place in the simplest case when a particle is given of invariant mass W (absolute characteristic) and then one attaches to it the four-momentum P_{μ} (covariant characteristic) which in the centre of mass R^c of this particle takes the form $P_{\mu} = (0, 0, 0 : W)$. In our scheme W is the eigenvalue of the internal hamiltonian $\hat{h}_2$ which is parametrized in the y-continuum — *cf.* Eq. (2.5).

The factorization of Eq. (2.2) in the X and y variables reflects the separation of the internal from the external degrees of freedom of S_2. Remember that the non-relativistic Schroedinger equation (parametrized in the relative and the centre of mass coordinates) has this property without explicitly resorting to the hypothesis of the y-continuum. Notice also that Eq. (2.2) remains Lorentz covariant in the external (space-time) variables X, t, but not in the internal y-variables. So long as S_2 does not interact with any external entity, all internal (absolute) characteristics of S_2 can be evaluated in the y-space, hence the projection onto space-time is of pure kinematic character. Another situation takes place for interacting systems, and this will be discussed in the next section.

According to the factorization of Ψ, let us take the particular solution of (2.2) of the form

$$\Psi = \psi_W(y) \exp [i(\boldsymbol{P} \boldsymbol{X} - Et)],$$

where $\psi_W(y)$ is the eigenstate of $\hat{h}_2$ to the eigenvalue W

$$\hat{h}_2 \psi_W(y) = W \psi_W. \tag{2.5}$$

Equation (2.2), as the Klein-Gordon equation in the $\boldsymbol{X}$, t variables, leads to the relation $W^2 = E^2 - \boldsymbol{P}^2$ which shows that W is the invariant (absolute) mass of S_2. In the centre of mass system R^c where $\boldsymbol{P} = 0$ the wave function (2.4) takes the form

$$\Psi^c = \psi_W(y) \exp(-iWt^c),$$

and Ψ^c fulfils the equation

$$i\partial\Psi^c/\partial t^c = [(\hat{q}^2 + m_1^2)^{\frac{1}{2}} + (\hat{q}^2 + m_2^2)^{\frac{1}{2}} + V(y)]\Psi^c(y, t^c) = W\Psi^c \tag{2.6}$$

known as the semi-relativistic equation of two-body system [15]. Since *a posteriori* the argument y of Ψ^c can be identified with the relative space coordinate $\boldsymbol{x}^c = \boldsymbol{x}_2^c - \boldsymbol{x}_1^c$ in R^c, the wave function $\boldsymbol{\Psi}^c$, which means Ψ^c projected onto space-time continuum (represented in R^c), takes the form identical with Ψ^c except that y is to be replaced by $\boldsymbol{x}^c$. Thus

$$\boldsymbol{\Psi}^c = \psi_W(\boldsymbol{x}^c) \exp(-iWt^c). \tag{2.7}$$

From the covariant formulation point of view, the four arguments $\boldsymbol{x}^c$, t^c of $\boldsymbol{\Psi}^c$ do not parametrize a four-point, much as the four quantities $\boldsymbol{q}$, W do not form a four-vector either. Within the covariant approach we deal namely with two four-vectors $(\boldsymbol{X}, t)$ and $(\boldsymbol{x}, \Delta t)$ and the arguments of $\boldsymbol{\Psi}^c$ are: the time coordinate t of the first four-vector, and the space $\boldsymbol{x}$ of the second — both represented in R^c. The absence of $\boldsymbol{X}^c$ in $\boldsymbol{\Psi}^c$ is obvious as $\boldsymbol{P}^c = 0$, while the absence of Δt^c is due to the single-time formalism implied by Eq. (2.2).

Let us suppose that the mass m_1 of one of the constituents of S_2 tends to infinity. Inserting $\boldsymbol{\Psi}^c = \psi^c \exp(im_1 t^c)$, Eq. (2.6) takes in the limit $m_1 \to \infty$ the following form

$$i\partial\psi^c/\partial t^c = [(\hat{q}^2 + m_2^2)^{\frac{1}{2}} + V(y)]\psi^c(y, t^c) \tag{2.8}$$

which has the stationary solutions $(y = \boldsymbol{x}^c)$

$$\psi^c = \psi_{E^c}^c(\boldsymbol{x}^c) \exp(-iE^c t^c), \tag{2.9}$$

where $E^c = \lim_{m_1 \to \infty} (W - m_1)$. The limiting equation (2.8) becomes covariant in the variables $(y = \boldsymbol{x}^c, t^c)$, hence now the four coordinates $(\boldsymbol{x}^c, t^c)$ parametrize the four-point x_μ (in R^c). Consequently, in the same limit $m_1 \to \infty$, the four quantities $(\boldsymbol{q} = \boldsymbol{p}^c, E^c)$ also become relativized and coincide with the four-momentum p_μ of m_2 (represented in R^c). The singular character of the limit $m_1 \to \infty$ consist in the fact that the time component t of tha four-vector (X, t) and the space component $\boldsymbol{x}$ of the second our-fvector $(\boldsymbol{x}, \Delta t)$ constitute the four-vector x_μ of the particle m_2. This type of singularity characterizes all processes dealing with an external field which breaks the translation invariance of the theory. Here the infinitely heavy body m_1 becomes the centre of force being at rest in R^c, hence $V(\boldsymbol{y}) = V(\boldsymbol{x}^c)$ means now the external potential of definite transformation properties under the Lorentz transformations. Strictly speaking, $V(\boldsymbol{x}^c)$ becomes (*a posteriori*) the fourth component of the external four-potential whose space components vanish in R^c. In the same limit the wave function ψ^c from (2.9) coincides with ψ^0 from (1.4), and R^c is identical with R^0. Thus the y-continuum hypothesis does not modify the single-body problem.

3. *Projection onto space-time*

In the case of S_2 interacting with an external entity one must perform the projection of the internal state of S_2 (obtained in the y-space) onto the equal-time space of some reference frame R. In fact, one deals here with the three-body problem when, according to our picture, one must distinguish between two essentially different cases: a) when in the asymptotic state of the whole three-body system S_3 the subsystem S_2 is in a given bound state, and S_2 as a whole is in the scattering state with the third body, b) when in the asymptotic state there are no two-body bound states. For example, all three particles can create a bound state. In both cases the y-continuum of the whole system S_3 coincides (*a posteriori*) with the over-all centre of mass system R^{cc}. However, in the case a) — which we shall discuss — one must first determine the internal structure of S_2 and then project it onto space-time of R^{cc}, while in the case b) the structure of S_3 is determined from the beginning in the absolute y-continuum of all three particles which requires the suitable absolute hamiltonian $\hat{h}_3$ to be dealt with. In the case a) the Lorentz contraction of the bound state of S_2 — *i. e.* some boundary condition — will afect the very interaction of S_2 with the third body. There is no room for such an effect in the case b).

We restrict ourselves to the case a), and we consider the projection of a given two-body state $\psi(\boldsymbol{y})$ onto space-time represented in an arbitrary reference frame R. The first part of this projection was done in the previous section when the coordinates $\boldsymbol{y}$, $\boldsymbol{q}$ were (*a posteriori*) identified with the relative coordinate and momentum, respectively, of m_1 and m_2 in their centre of mass system R_c. Thus we put

$$\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1, \boldsymbol{p} = a\boldsymbol{p}_2 - b\boldsymbol{p}_1 \quad (a+b = 1) \qquad \text{(a)}$$

where

$$\boldsymbol{y} = \boldsymbol{x}^c, \boldsymbol{q} = \boldsymbol{p}^c, \qquad \text{(b)} \qquad (3.1)$$

the parameter a is so far arbitrary ($\boldsymbol{p}^c = \boldsymbol{q}$ for any a). Since the projection onto space-time introduces as such the Lorentz coordinates of the constituents of S_2, the relations (3.1) can be completed to the four-dimensional language by putting

$$\Delta t = t_2 - t_1, p_0 = ap_{20} - bp_{10}, \qquad (3.2)$$

where $p_{10} = E_1$, $p_{20} = E_2$ are the energies of m_1, m_2, respectively. We then obtain the four-vector $x_\mu = (\boldsymbol{x}, \Delta t)$ and the canonically conjugate relative four-momentum $p_\mu = (\boldsymbol{p}, p_0)$. In fact, the determination of p_0 requires the energies $E_{1,2}$ of the constituents of S_2 to be well defined, which takes place for free particles or in the asymptotic region of the scattering states only.

Together with the relative form-vectors x_μ and p_μ, let us define the second pair of the four-vectors X_μ and P_μ, where

$$X_\mu = ax_{1\mu}+bx_{2\mu}, P_\mu = p_{1\mu}+p_{2\mu}. \tag{3.3}$$

Let us assume that p_0^c vanishes, this being implied by the pure space characteristic of S_2 given by the three-dimensional y-continuum parametrization. In the covariant picture this corresponds to the subsidiary covariant condition $P_\mu p_\mu = 0$ which one imposes onto (covariant) two-body equations of motion [1,6]. The vanishing of p_0^c implies that

$$a = \tilde{a} = \tfrac{1}{2}\,[1+(m_1^2-m_2^2)/W^2] \underset{c\to\infty}{\rightarrow} m_1/(m_1+m_2) = \tilde{a}_{NR}, \tag{3.4}$$

where W always means the invariant mass of S_2. The coordinate

$$\tilde{\boldsymbol{X}} = \tilde{a}\boldsymbol{x}_1+(1-\tilde{a})\,\boldsymbol{x}_2 \tag{3.5}$$

is then called the centre of gravity (mass) of S_2. It is assumed that (3.4) holds also for the bound states of S_2 when $W < m_1+m_2$, which is consisten tbecause (3.4) only requires S_2 to have a well-defined invariant mass W. Since $\tilde{a}$ is an invariant, the centre of gravity (3.5) determines a covariant point which is unlike the usual picture where the "centre of gravity" depends on the reference frame [16]. This important fact follows from the hierarchy of the description of S_2 implied by the proposed framework. The centre of gravity $\boldsymbol{X}$ is determined *a posteriori*, provided that the internal state of S_2 and its invariant mass W are already determined from (2.5) in the absolute y-space — *cf.* (2.3). Note that it is a peculiarity of the non-relativistic framework, due to the lack of energy-mass relation, that $\tilde{a}_{NR}$ is given *a priori* through the masses of the constituents of S_2, independently of the internal state of the system S_2.

Let us consider for the moment the free-particle system S_2 ($V = 0$), and the plane--wave solution of (2.2) equal to

$$\Psi = \exp\,[i(\boldsymbol{PX}-Et)]\exp\,(i\boldsymbol{qy}). \tag{3.6}$$

The external phase $\boldsymbol{PX}-Et$ is manifestly invariant, while the internal phase $\boldsymbol{qy}$ is also absolute because it is represented in the y-continuum of S_2. However, in order to get the corresponding equal-time wave function Ψ in the reference frame R where the external phase is represented, we must perform *(a posteriori)* the projection of the phase $\boldsymbol{qy}$ onto R. From (3.1), and taking into account that p_0^c vanishes, the internal phase can be rewritten in the Lorentz coordinates

$$\boldsymbol{qy} = \boldsymbol{qy}-p_0^c\Delta t^c = p_\mu^c x_\mu^c = p_\mu x_\mu = L-\text{inv}... \tag{3.7}$$

This identity takes place for any four-vector $x_\mu = (\boldsymbol{x}, \Delta t)$ provided only that $\boldsymbol{x}^c = \boldsymbol{y}$ as stated in (3.1b). Thus we define the class of the four-vectors $x_\mu^{(R)} = (\boldsymbol{x}^{(R)}, \Delta t^{(R)})$ such that: 1° $\boldsymbol{x}^{(R)c} = \boldsymbol{y}$, and 2° $x_\mu^{(R)}|_R = (\boldsymbol{x}^{(R)}, 0)$. In consequence, the identity (3.7) expressed in terms of $x_\mu^{(R)}$ takes the form

$$\boldsymbol{q}\boldsymbol{y} = p_\mu x_\mu^{(R)} = \boldsymbol{p}\boldsymbol{x}^{(R)}|_R \equiv \boldsymbol{p}\boldsymbol{x} = L\text{-inv}. \tag{3.8}$$

(we further omit the superscript R understanding by $\boldsymbol{x}$ the space component of $x_\mu^{(R)}$ in R). Thus we deal with the absolute three-dimensional phase $\boldsymbol{p}\boldsymbol{x}$, although its form is not manifestly covariant, as in each reference frame R the four-vector $x_\mu^{(R)}$ depends on the reference frame R. The class of four-vectors $x_\mu^{(R)}$ accounts for the description of the space extension of a "rigid body" which is at rest in a fixed reference frame R^c. We treat the shape of the internal structure of S_2 described by the wave function $\psi(\boldsymbol{y}) = \psi(\boldsymbol{x}^c)$ on exatly the same footing as a realized "rigid body". On the other hand, the covariant wave functions, besides the Lorentz contraction, account also for the internal, space-time motion of the constituents of S_2, which results in additional effects of the relativistic distortion [11] being absent in our picture.

According to the identity (3.8), and taking into account the transformation properties of $\boldsymbol{p}$ and $\boldsymbol{x}$ (as — *a posteriori* — the space components of the corresponding four-vectors), one easily finds the relations between $\boldsymbol{y} = \boldsymbol{x}^c$ and $\boldsymbol{x}$, and $\boldsymbol{q} = \boldsymbol{p}^c$ and $\boldsymbol{p}$ which completes the projection operation. Let us denote by $L(\boldsymbol{v})$ the three-dimensional transformation which reflects the Lorentz contraction in the direction of $\boldsymbol{v}$, where $\boldsymbol{v} = \boldsymbol{P}/(P^2+W^2)^{1/2}$ is the velocity between R^c and R. Then

$$\boldsymbol{x} = L(\boldsymbol{v})\boldsymbol{y}, \boldsymbol{p} = \boldsymbol{q}\, L^{-1}(\boldsymbol{v}), \tag{3.9}$$

and $\boldsymbol{p}\boldsymbol{x} = [\boldsymbol{q}L^{-1}(\boldsymbol{v})]\,[L(\boldsymbol{v})\boldsymbol{y}] = \boldsymbol{q}\boldsymbol{y} = L$-inv., as it should be.

Now let us reconsider the wave function (3.6) written in a general form which does not require the system S_2 to be free

$$\Psi = \exp\,[i(\boldsymbol{P}\boldsymbol{X}-Et)]\,\psi(\boldsymbol{y}) \tag{3.6'}$$

(in particular, $\psi(\boldsymbol{y})$ can be equal to $\exp\,(i\,\boldsymbol{q}\boldsymbol{y})$ as in (3.6)). If $\psi(\boldsymbol{y})$ is projected onto equal-time space represented in R, then

$$\Psi = [\gamma(\boldsymbol{v})]^{1/2} \exp\,[i(\boldsymbol{P}\tilde{\boldsymbol{X}}-Et)]\,D(\boldsymbol{v})\,\psi\,[L^{-1}(\boldsymbol{v})\boldsymbol{x}]. \tag{3.10}$$

Here $D(\boldsymbol{v})$ is the unitary operator of the representation of the Lorentz group which boosts ψ from R^c to R, and the factor $\gamma^{1/2}$ ensures the invariant normalization of ψ: $\int d^3x|\psi|^2 = 1$. Moreover, as the projection requires that $p_0^c = 0$, the coordinate $\tilde{\boldsymbol{X}}$ in (3.10) must denote the centre of gravity of S_2 — *cf.* (3.5).

When S_2 interacts with a third body, then the absolute y-continuum of the whole system S_3 becomes *a posteriori* equivalent with the parametrization in the over-all centre of mass system R^{cc}. Therefore the internal state $\psi(\boldsymbol{y})$ of S_2 must be projected onto the equal-time space of R^{cc}.

Finally let us point out that for a free-body system described by the plane wave we have from (3.1), (3.3) and (3.7) the following identity

$$\Psi = \exp\,[i(\boldsymbol{P}\boldsymbol{X}-Et)]\exp\,(i\boldsymbol{q}\boldsymbol{y}) = \exp\,[i(\boldsymbol{p}_1\boldsymbol{x}_1+\boldsymbol{p}_2\boldsymbol{x}_2)-i(E_1+E_2)t] = \Psi/\gamma^{1/2}.$$

Thus Ψ is identical with the product of two plane waves of each of the constituent taken at equal time t of any given reference frame R. Therefore the modification implied by our hypothesis concerns only bound states, *i. e.* new (composite) particles of finite size. Roughly speaking, free particles behave as if they had their world lines transforming — as such — covariantly, which ceases to be the case for "trajectories" of bound particles. From this point of view our picture resembles that [17] where the notion of the world line as pure classical concept is abandoned.

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