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A NEW METHOD OF REDUCTION OF QUASIRELATIVISTIC EQUATIONS TO THE SUBSPACE OF POSITIVE ENERGY STATES

By H. Janyszek

Institute of Physics, Nicholas Copernicus University, Toruń*

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A new method of reduction of arbitrary quasirelativistic quantum-mechanical equation for two charged Dirac particles to the Pauli formalism is presented. The advantages of this method are:

1) it gives a criterion for choosing the most suitable form of quasirelativistic equation from among apparently equivalent alternative possibilities,

2) it permits considerable simplification of formal calculations and provides some demonstrative interpretation of intermediate results,

3) it makes possible direct generalization of results to arbitrary number of identical or different fermions,

4) it leads to the definition of other particle observables (understood in the sense of Foldy-Wouthuysen "mean operators") which together with the Hamiltonian determine the basis of quantum-mechanical description of such systems.

1. Introduction

The generalization of the quantum-mechanical description of a charged Dirac particle in an external electromagnetic field to the approximate "quasirelativistic" description of a system of interacting particles has been widely discussed in several monographies, *e. g.* by Bethe and Salpeter [1], Bethe [2], mainly owing to the importance of this problem in accurate calculations of theoretical spectroscopy of simple atomic system. The first quasirelativistic two-particle equation has been proposed by Breit [3] on the basis of the assumptions of quantum electrodynamics. This equation has found wide applications in spite of some drawbacks resulting from the neglection in its derivation of the hole theory. In 1951 Bethe and Salpeter [4] derived on the basis of the Feynman formalism a new covariant equation which also describes the bound states of two fermions. Salpeter [5] then brought this equation to the form of approximate three-dimensional wave equation. After further transformations (Hermitization of the term which describes the interaction

* Address: Instytut Fizyki, Uniwersytet Mikołaja Kopernika, Toruń, Grudziądzka 5, Poland.

between the particles) Barker and Glover [6] have obtained a quantum-mechanical equation of a type similar to that of the Breit equation (for absent external field) but differing in the term describing the interaction between the particles. This particular quantum -mechanical equation will be called in the further text the Bethe-Salpeter equation.

The starting point for practical applications is most frequently not the complete quasirelativistic equation (in the product space of Dirac spinors of the particles forming the system) but the reduced equation which describes the system of particles in its positive energy states by means of a suitable effective Hamiltonian. Such an approximate Hamiltonian can be obtained (up to second order terms in the fine structure constant $\alpha=e^2/hc$) both by reduction of the quasirelativistic equation and as a direct consequence of the assumptions of quantum electrodynamics. Hanus and Janyszek [7] have given some arguments in favour of the first method regarding it as more correct for the study of the system of interacting Dirac particles. The problem of reduction of a two-particle equation has undergone the same evolution as the analogous problem of one Dirac particle. Breit has applied to his equation the earliest known method of elimination of small components of the spinor thus obtaining the Hamiltonian with non-Hermitian second-order terms, similarly as in the case of the Dirac equation reduced in such a way. Besides in the reduced Breit equation there appeared a term proportional to the fourth power of the charge which was in disagreement with the spectroscopic data for the helium atom. The rejection of this term, as due to incompletely separated negative energy states, required the use of additional arguments (quoted in Refs [1] and [2]). It is well known that the method proposed by Foldy and Wouthuysen [8] of the reduction of the Dirac Hamiltonian after previous transformation of this Hamiltonian to the so-called even form (in which the states with opposite signs of energy are not mixed) by consecutive unitary transformations ("the FW-transformations") played an important role in the quantum-mechanical interpretation of this equation. Direct generalization of this method to the case of reduction of two--particle equations has been carried out in a series of papers by Chraplyvy [9, 10], Chraplyvy and Glover [11], and Barker and Glover [6]. These considerations pointed out the singularity of the case of the Breit equation for particles of identical masses and the necessity of replacing in this case the postulate of complete reduction of the Hamiltonian by a less strong condition which leads to an infinite class of transformations (some of which being non-unitary). These transformations lead to a form of the Hamiltonian which separates only positive (or only negative) energy solutions although the final form of the reduced Hamiltonian is the same in both cases.

It is the purpose of the present considerations to present in detail a new method of reduction which, though also originating from the idea of Foldy and Wouthuysen [8], shows apart from some similar features also essential differences compared to the method proposed in Refs [9] and [6]. The present method refers to the Case transformation [12] and its further interpretation proposed in the paper of Garszczyński and Hanus [13]. In addition to the properties discussed in detail in Section 4 the method is characterized by the fact that together with the gauge invariance condition it defines unambiguously the generalization of the Bethe-Salpeter equation to the presence of external magnetic field. It also justifies the choice of this equation and not the Breit-Wigner equation as the basic quasirelativistic equation for two Dirac particles. These considerations provide detailed justification and extension of the results presented in an earlier paper (Hanus and Janyszek [14]) in shorter form.

2. Generalization of the Case transformation to the two-particle problem. Transformation to "intermediate scheme"

It has been found by Case [12] that rigourous transformation of the Dirac Hamiltonian to the even form is still possible in the presence of external stationary magnetic field, and that the form of this transformation is a simple generalization of the Foldy--Wouthuysen transformation for a free particle. Then it turned out (Garszczyński and Hanus [13]) that the application of this transformation to the case of electromagnetic field, though not leading to completely even form of the Hamiltonian, results in such a form in which the distinction of even and odd terms has a deeper physical meaning and introduces some orderly arrangement thus facilitating their further physical interpretation.

In accordance with Ref. [13] the Dirac Hamiltonian[1]

$$H = \varrho_3 mc^2 + c\varrho_1 z + e\Phi, \qquad z = \boldsymbol{\sigma}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right) \tag{1}$$

transformed by means of the Case transformation

$$\amalg = \exp(iS), \qquad S = \tfrac{1}{2}\varrho_2 \operatorname{tg}^{-1}\left(\frac{z}{mc}\right) \tag{2}$$

in the approximation accepted has the following form

$$H_{\amalg} = \varrho_3\left\{mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} - \frac{e}{2mc}[\boldsymbol{p}, \boldsymbol{A}]_+ - \frac{e\hbar}{2mc}\boldsymbol{\sigma}\mathscr{H} + \right.$$

$$\left. + \frac{e^2}{2mc^2}(A)^2\right\} - \frac{e\hbar^2}{8m^2c^2}\operatorname{div}\boldsymbol{E} - \frac{e\hbar}{8m^2c^2}\boldsymbol{\sigma}(\boldsymbol{E}\times\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{E}) - \varrho_2\frac{e\hbar}{2mc}\boldsymbol{\sigma}\boldsymbol{E}, \tag{3}$$

and thus contains all expected even terms and an odd term proportional to ϱ_2. The elimination of this redundant term is ensured by an additional unitary transformation

$$\amalg' = \exp(iS'), \qquad S' = \varrho_1\frac{e\hbar}{2m^2c^3}\boldsymbol{\sigma}\boldsymbol{E} \tag{4}$$

which, however does not influence the shape of correct even terms.

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[1] The notation and the symbols used throughout this paper are the same as those introduced in the previous paper of Hanus and Janyszek [7]. The quantities referring to two or more particles are denoted by the subscripts $K, L = \mathrm{I}, \mathrm{II}, \ldots N$. All calculations are restricted to second-order terms in the fine structure cionstant α. According to the commonly accepted convention the order of magnitude of the expressions is estimated by powers of c^{-1} (which corresponds to the introduction of the atomic units system).

The transformation (2) can be generalized to the two-particle problem. It then becomes the form

$$Ш = Ш_I Ш_{II} = Ш_{II} Ш_I \tag{5}$$

$$Ш_K = \tfrac{1}{2}\varrho_{2,K}\,\mathrm{tg}^{-1}\left(\frac{z_K}{m_K c}\right), \quad z_K = \sigma_{K'}\left(p_K - \frac{e_K}{c}A_K^{ex}\right), \quad K = \mathrm{I}, \mathrm{II}. \tag{6}$$

We shall apply (5) to the transformation of the Breit Hamiltonian and to the generalized Bethe-Salpeter Hamiltonian proposed in Refs [7] and [14] on the basis of the properties of the transformation Ш and the gauge invariance postulate (for details see Appendix A)

$$H^B = H_I + H_{II} + V, \qquad H^{BS} = H_I + H_{II} + W, \tag{7}$$

$$H_K = \varrho_{3,K} m_K c^2 + c\varrho_{1,K} z_K + e_K \Phi_K^{ex}, \tag{8}$$

$$V = e_I \cdot e_{II}\left(\frac{1}{r} - \frac{1}{2}\varrho_{1,I}\cdot\varrho_{1,II}J\right), \quad W = \frac{1}{4}[\lambda_I + \lambda_{II}, V]_+, \tag{9}$$

$$J = \frac{\sigma_I \sigma_{II}}{r} + \frac{(\sigma_I\cdot r)(\sigma_{II}\cdot r)}{r^3}, \quad r = r_I - r_{II}, \quad r = |r|, \tag{10}$$

$$\lambda_K = T_K\cdot(T_K^2)^{-\frac{1}{2}}, \qquad T_K = \varrho_{3,K} m_K c^2 + c\varrho_{1,K} z_K. \tag{11}$$

One can expect that this transformation does not perform complete separation of states with different sign of energy. It turns out however, that the "intermediate scheme" obtained in this way will play an important role in the comparison of the properties of the investigated quasirelativistic Hamiltonians. The form $H_K^{Ш} = Ш\, H_K Ш^+$, where H_K is given by Eq. (8), results directly from Eq. (3)

$$\begin{aligned} H_K^{Ш} = \varrho_{3,K}\Big\{ m_K c^2 + \frac{p_K^2}{2m_K} - \frac{p_K^4}{8m_K^3 c^2} - \frac{e_K}{2m_K c}[p_K, A_K^{ex}]_+ - \frac{e_K \hbar}{2m_K c}\sigma_K \mathscr{H}_K^{ex} + \\ + \frac{e_K^2}{2m_K c^2}(A_K^{ex})^2\Big\} - \frac{e_K\hbar^2}{8m_K^2c^2}\mathrm{div}_K E_K^{ex} - \frac{e_K\hbar}{8m_K^2c^2}\sigma_K(E_K^{ex}\times p_K - p_K\times E_K^{ex}) + \\ - \varrho_{2,K}\frac{e_K\hbar}{2m_K c}\sigma_K E_K^{ex}, \end{aligned} \tag{12}$$

where

$$E_K^{ex} = -\,\mathrm{grad}_K\,\Phi_K^{ex}, \qquad \mathscr{H}_K^{ex} = \mathrm{rot}_K\, A_K^{ex}. \tag{13}$$

It is thus necessary to calculate only those terms which represent the interaction between the particles, and first of all $V_Ш$. After lenghty but elementary calculations one obtains

$$V_Ш = \sum_{K,L}{}' \Big\{\frac{1}{2}e_K e_L\left(\frac{1}{r} - \frac{1}{2}\varrho_{1,K}\varrho_{1,L}J\right) + \varrho_{2,K}\frac{ie_K e_L}{2m_K c}\left[z_K, \frac{1}{r}\right]_- +$$

$$-\varrho_{3,K}\cdot\varrho_{1,L}\frac{e_K e_L}{4m_K c}[z_K, J]_+ - \varrho_{2,K}\cdot\varrho_{2,L}\frac{e_K e_L}{8m_K m_L c^2}\left[z_K,\left[z_L,\frac{1}{r}\right]_-\right]_- +$$

$$-\frac{e_K e_L}{8m_K^2 c^2}\left[z_K,\left[z_K,\frac{1}{r}\right]_-\right]_- - \varrho_{3,K}\varrho_{3,L}\frac{e_K e_L}{16 m_K m_L c^2}[z_K,[z_L,J]_+]_+ +$$

$$+\varrho_{1,K}\varrho_{1,L}\frac{e_K\cdot e_L}{16m_K^2c^2}[z_K,[z_K,J]_+]_+, \tag{14}$$

where $\sum'_{K,L}$ denotes summation over $K \neq L$ (both of which for the time being become only the values I and II). The expansion of the commutators and anticommutators appearing in Eq. (14) leads to an extremely complicated formula which, however, can be written in a much simpler form by introducing some suitable notations. One can assign special physical meaning to these notations. We shall still continue to write the particular terms in an order according to the components of ϱ_K and to separate the particular powers of c^{-1} for the sake of clarity in the arrangement of these terms according to consecutive orders of approximation. We accept the following denotations:

$$\frac{1}{c}Y_{K,L} = -\frac{e_K\hbar}{2m_K c}\boldsymbol{\sigma}_K \boldsymbol{E}_K^{\text{ex}}, \tag{15}$$

$$\frac{1}{c^2}Q_{K,L} = -\frac{e_K\hbar^2}{8m_K^2c^2}\operatorname{div}_K \boldsymbol{E}_{K,L}, \tag{16}$$

$$\frac{1}{c^2}F_{K,L} = -\frac{e_K\hbar}{8m_K^2c^2}\boldsymbol{\sigma}_K(\boldsymbol{E}_{K,L}\times\boldsymbol{p}_K - \boldsymbol{p}_K\times\boldsymbol{E}_{K,L}), \tag{17}$$

$$\frac{1}{c^2}G_{K,L} = -\frac{e_K\hbar}{4m_K m_L c^2}\boldsymbol{\sigma}_K\cdot(\boldsymbol{E}_{L,K}\times\boldsymbol{p}_L - \boldsymbol{p}_L\times\boldsymbol{E}_{L,K}), \tag{18}$$

$$\frac{1}{c^2}D_{K,L} = \frac{e_K e_L\hbar^2}{4m_K m_L c^2}\left\{\frac{\boldsymbol{\sigma}_K\boldsymbol{\sigma}_L}{r^3} - 3\frac{(\boldsymbol{\sigma}_K\cdot\boldsymbol{r})(\boldsymbol{\sigma}_L\cdot\boldsymbol{r})}{r^5} + \frac{4\pi}{3}\boldsymbol{\sigma}_K\boldsymbol{\sigma}_L\delta(r)\right\}, \tag{19}$$

$$\frac{1}{c^2}M_{K,L} = \frac{e_K e_L\hbar^2}{4m_K m_L c^2}\left\{\frac{\boldsymbol{\sigma}_K\boldsymbol{\sigma}_L}{r^3} - 3\frac{(\boldsymbol{\sigma}_K\cdot\boldsymbol{r})(\boldsymbol{\sigma}_L\cdot\boldsymbol{r})}{r^5} - \frac{8\pi}{3}\boldsymbol{\sigma}_K\boldsymbol{\sigma}_L\delta(\boldsymbol{r})\right\}, \tag{20}$$

$$\frac{1}{c^2}L_{K,L} = -\frac{e_K e_L}{2m_K m_L c^2}\left\{\frac{1}{r}\boldsymbol{p}_K\boldsymbol{p}_L + \frac{\boldsymbol{r}}{r^3}(\boldsymbol{r}\boldsymbol{p}_K)\boldsymbol{p}_L\right\}, \tag{21}$$

$$\frac{1}{c}\hat{P}_{K,L} = -\frac{e_K}{2m_K c}([\boldsymbol{p}_K, \hat{\boldsymbol{a}}_{K,L}]_+ + \hbar\boldsymbol{\sigma}_K\hat{\boldsymbol{b}}_{K,L}), \tag{22}$$

$$\frac{1}{c^2}\hat{\hat{P}}_{K,L} = -\frac{e_K}{2m_K c}([\boldsymbol{p}_K, \hat{\hat{\boldsymbol{a}}}_{K,L}]_+ + \hbar\boldsymbol{\sigma}_K\hat{\hat{\boldsymbol{b}}}_{K,L}), \tag{23}$$

where

$$\hat{a}_{K,L} = \frac{e_L}{2}\left\{\frac{\sigma_L}{r} + \frac{(\sigma_L \cdot r)r}{r^3}\right\}, \quad \hat{b}_{K,L} = \mathrm{rot}_K\, \hat{a}_{K,L}, \tag{24}$$

$$\hat{\hat{a}}_{K,L} = \frac{e_L}{2m_L c}\left\{\frac{1}{r}\, p_L + \frac{r}{r^3}(r \cdot p_L) \pm \frac{\sigma_L \times r}{r^3}\right\}, \ \pm \ \text{for} \begin{cases} K < L \\ K > L \end{cases} \tag{25}$$

$$\hat{\hat{b}}_{K,L} = \mathrm{rot}_K\, \hat{\hat{a}}_{K,L}. \tag{26}$$

The following relationship hold in addition

$$\tfrac{1}{2}\sum_{K,L}{}' \hat{\hat{P}}_{K,L} = \sum_{K,L}{}' \left(\tfrac{1}{2} L_{K,L} + \tfrac{1}{2} M_{K,L} + G_{K,L}\right). \tag{27}$$

Detailed calculations concerning Eqs (22)–(26) are given in Appendix B, where a demonstrative interpretation of the quantities $\hat{a}_{K,L}$ and $\hat{\hat{a}}_{K,L}$ as some operator generalizations of the vector potential for the case of magnetic interaction between Dirac particles has also been introduced. Such interpretation suggested by the form of Eqs (22) and (23) sheds some light on the structure of the expressions discussed, although it is not necessary to make the formal transformations based on Eqs (22)–(27) whose aim is only to simplify the form of final results calculated in this section. The terms (22) and (23) are equivalent to the term appearing in the Pauli equation, as it can be seen from their explicit form. It has already been mentioned before that the quantities defined by Eqs (15)–(21) have some direct physical interpretation: $\frac{1}{c}\, Y_{K,L}$ represents interaction of the electric dipole moment of the K-th particle $e_K\, \hbar\, \sigma_K/2mKc$ with the electric field due to the L-th particle. This term corresponds to the one-particle Hamiltonian with the field E_K^{ex} (12). Similarly $\frac{1}{c^2}\, Q_{K,L}$ and $\frac{1}{c^2}\, F_{K,L}$ correspond to the corrections of Darwin and Frenkel-Thomas in the Hamiltonian (12). On the other hand $\frac{1}{c^2}\, G_{K,L}$ represents the interaction of the K-th particle spin with the orbital angular momentum of the L-th particle. The expression $\frac{1}{c^2}\, M_{K,L}$ is the well-known expression for the interaction of two magnetic dipoles. The term $\frac{1}{c^2}\, D_{K,L}$ corresponds to interaction between two electric dipoles, while $\frac{1}{c^2}\, L_{K,L}$ represents orbit--orbit magnetic interaction. The symbols with two superscripts K, L denote the result of action of the L-th particle on the K-th particle, *e. g.*, $E_{K,L}$ is the Coulomb field produced by the L-th particle at the point at which there is the K-th particle

$$E_{K,L} = -\,\mathrm{grad}_K\, \Phi_{K,L}, \quad \Phi_{K,L} = \frac{e_L}{r}. \tag{28}$$

The denotations and abbreviations introduced in Eqs (15)–(28) permit V_{II} to be expressed in compact form. Indeed long and elaborate calculations yield:

$$\frac{ie_K e_L}{2m_K c}\left[z_K, \frac{1}{r}\right]_- = \frac{1}{c} Y_{K,L}, \tag{29}$$

$$-\frac{e_K \cdot e_L}{4m_K c}[z_K, J]_+ = \frac{1}{c}\hat{P}_{K,L} + \frac{e_K}{m_K c^2}\hat{a}_{K,L} \cdot A_K^{\text{ex}}, \tag{30}$$

$$-\frac{e_K e_L}{8m_K^2 c^2}\left[z_K, \left[z_K, \frac{1}{r}\right]_-\right]_- = \frac{1}{c^2}(Q_{K,L} + F_{K,L}), \tag{31}$$

$$-\frac{e_K \cdot e_L}{8m_K m_L c^2}\left[z_K, \left[z_L, \frac{1}{r}\right]_-\right]_- = \frac{1}{2c^2} D_{K,L}, \tag{32}$$

$$-\frac{e_K e_L}{16 m_K m_L c^2}[z_K, [z_L, J]_+]_+ = \frac{1}{2c^2}\hat{\hat{P}}_{K,L}, \tag{33}$$

$$\frac{e_K e_L}{16 m_K^2 c^2}[z_K, [z_K, J]_+]_+ = \frac{1}{2c^2}\frac{m_K}{m_L}G_{K,L} + \frac{1}{4c^2}\left(\frac{m_K}{m_L} + \frac{m_L}{m_K}\right)M_{K,L} + $$

$$+ \frac{1}{4m_K c^2}[\sigma_K \cdot p_K, \hat{P}_{K,L}]_+. \tag{34}$$

Hence we obtain the final form of the transformed Breit operator

$$V_{\text{II}} = \sum_{K,L}{}' \Big\{ \frac{1}{2} e_K \Phi_{K,L} + \frac{1}{c^2}(Q_{K,L} + F_{K,L}) + \varrho_{3,K} \cdot \varrho_{3,L} \frac{1}{c^2}\left(\frac{1}{2} L_{K,L} + \frac{1}{2} M_{K,L} + G_{K,L}\right) +$$

$$+ \varrho_{2,K}\frac{1}{c} Y_{K,L} + \varrho_{2,K}\varrho_{2,L}\frac{1}{2c^2} D_{K,L} + \varrho_{3,K}\varrho_{1,L}\left(\frac{1}{c}\hat{P}_{K,L} + \frac{e_K}{m_K c^2}\hat{a}_{K,L} A_K^{\text{ex}}\right) +$$

$$+ \varrho_{1,K}\varrho_{1,L}\Big[\frac{1}{2c^2}\frac{m_K}{m_L} G_{K,L} - \frac{1}{4c^2}\left(\frac{m_K}{m_L} + \frac{m_L}{m_K}\right) M_{K,L} +$$

$$- \frac{1}{4}\left(e_K \cdot e_L J + \frac{1}{m_K c^2}[\sigma_K p_K, {}_{K,L}\hat{P}]_+\right)\Big]\Big\}. \tag{35}$$

In a similar form one can express the operator $W_{\text{II}} = \text{II}\, W \text{II}^+$. Since from the properties of the transformation II it follows that

$$\lambda_K^{\text{II}} = \text{II}\lambda_K \text{II}^+ = \text{II}_K \lambda_K \text{II}_K^+ = \varrho_{3,K}, \tag{36}$$

we obtain

$$W_{\text{II}} = \tfrac{1}{4}[\varrho_{3,\text{I}} + \varrho_{3,\text{II}}, V_{\text{II}}]_+. \tag{37}$$

Thus the calculation of W_{II} is reduced to the substitution of (35) into (37). Straightforward calculation yields

$$W_{\text{II}} = \sum_{K,L}{}' \left\{ \frac{1}{2}(\varrho_{3,K}+\varrho_{3,L}) \left[\frac{1}{2} e_K \Phi_{K,L} + \frac{1}{c^2} \left(Q_{K,L} + F_{K,L} + G_{K,L} + \frac{1}{2} L_{K,L} + \right.\right.\right.$$

$$\left.\left.\left. + \frac{1}{2} M_{K,L} \right) \right] + \varrho_{2,K}\varrho_{3,L} \frac{1}{2c} Y_{K,L} + \varrho_{1,K} \left(\frac{1}{2c} \hat{P}_{L,K} + \frac{e_L}{2m_K c} \hat{a}_{L,K} A_L^{\text{ex}} \right) \right\}. \tag{38}$$

By extending the denotations (15)–(17) and (23) to expressions which are dependent on the external field and appear in Eq. (12), *i. e.*, putting

$$\frac{1}{c} Y_K^{\text{ex}} = - \frac{e_K h}{2m_K c} \sigma_K E_K^{\text{ex}}, \tag{39}$$

$$\frac{1}{c^2} Q_K^{\text{ex}} = - \frac{e_K h^2}{8m_K^2 c^2} \operatorname{div}_K E_K^{\text{ex}}, \tag{40}$$

$$\frac{1}{c^2} F_K^{\text{ex}} = - \frac{e_K h}{8\, m_K^2 c^2} \sigma_K (E_K^{\text{ex}} \times p_K - p_K \times E_K^{\text{ex}}), \tag{41}$$

$$\frac{1}{c} P_K^{\text{ex}} = - \frac{e_K}{2m_K c} ([p_K, A_K^{\text{ex}}]_+ + h \sigma_K \mathscr{H}_K^{\text{ex}}), \tag{42}$$

we obtain from Eq. (12)

$$H_K^{\text{II}} = \varrho_{3,K} \left\{ m_K c^2 + \frac{p_K^2}{2m_K} - \frac{p_K^4}{8m_K^3 c^2} + \frac{1}{c} P_K^{\text{ex}} + \frac{e_K^2}{2m_K c^2} (A_K^{\text{ex}})^2 \right\} +$$

$$+ e_K \Phi_K^{\text{ex}} + \frac{1}{c^2} (Q_K^{\text{ex}} + F_K^{\text{ex}}) + \varrho_{2,K} \frac{1}{c} Y_K^{\text{ex}}. \tag{43}$$

The formulae (43), (35) or (38) give the final form which in the intermediate scheme II have the Breit and the generalized Bethe-Salpeter Hamiltonian, respectively. In accordance with what was expected there are still odd terms in both Hamiltonians (which contain the operators $\varrho_{1,K}$ and $\varrho_{2,K}$). For this reason the above-mentioned scheme is called the intermediate scheme. Nevertheless the transformation II (similarly as the Case transformation for one particle) has already introduced far-going arrangement of terms accoıdng to their physical meaning. It is interesting to compare the even terms of both Hamiltonians. If one puts $\varrho_{3,K} = \pm 1$, then both Hamiltonians become identical and represents the total correct reduced effective Hamiltonian ın the subspace of positive energy states. Hence it can be seen that further transformation of these Hamiltonians would be reduced to finding additional unitary transformation which would remove the still remaining odd terms, but which would not change the correct even part of the Hamiltonian. In this respect the properties of both Hamiltonians are completely different. As it can be seen from Eq. (35) the transformation U turned out to be completely useless in case of the Breit Hamiltonian since it did neither remove the odd-odd term for the magnetic interaction of the particles nor even decrease its order of magnitude. This term, *i. e.*, $-\frac{1}{2} e_{\mathrm{I}} \cdot e_{\mathrm{II}} \varrho_{1,\mathrm{I}} \cdot \varrho_{1,\mathrm{II}} J$ remained unaffected (gaining a second order contribution only). It is also not possible to remove this term by means of the modified Case transformation (discussed in the paper of Janyszek [15]) to which magnetic interaction has been added. As it is well known, the authors of Refs [9–11] succeeded in constructing a transformation of the Breit Hamiltonian which reduces the latter to even form for the case of two particles and which separates only a positive energy state solution in case of identical particles, what changes essentially the physical meaning of the transformation. It should be pointed out, however, that the difficulties connected with the appearance of a controversial term proportional to the fourth power of the charge have not been eliminated. Some suggestions about its origin are given in Appendix B. All this indicates that the main source of difficulties is the structure of the Breit interaction itself (in particular of this part which is proportional to $\varrho_{1,\mathrm{I}} \cdot \varrho_{1,\mathrm{II}}$) as a result of neglecting the assumptions of the hole theory in the derivation of the Breit term. Indeed the replacement of the term for the Breit interaction by the corresponding Bethe-Salpeter term changes essentially the structure and the properties of the Hamiltonian. The Bethe-Salpeter term is

$$H_{\mathrm{II}}^{BS} = \sum_K \Big\{ \varrho_{3,K} \Big[m_K c^2 + \frac{p_K^2}{2m_K} - \frac{p_K^4}{8m_K^3 c^2} + \frac{1}{c} P_K^{\mathrm{ex}} + \frac{e_K^2}{2m_K c^2} (A_K^{\mathrm{ex}})^2 \Big] +$$

$$+ e_K \Phi_K^{\mathrm{ex}} + \frac{1}{c^2} (Q_K^{\mathrm{ex}} + F_K^{\mathrm{ex}}) + \varrho_{2,K} \frac{1}{c} Y_K^{\mathrm{ex}} \Big\} +$$

$$+ \sum_{K,L}{}' \Big\{ \frac{1}{2} (\varrho_{3,K} + \varrho_{3,L}) \Big[\frac{1}{2} e_K \Phi_{K,L} + \frac{1}{c^2} \Big(Q_{K,L} + F_{K,L} + G_{K,L} +$$

$$+ \frac{1}{2} L_{K,L} + \frac{1}{2} M_{K,L} \Big) \Big] + \varrho_{2,K} \cdot \varrho_{3,L} \frac{1}{2c} Y_{K,L} +$$

$$+ \varrho_{1,L} \frac{1}{2c} \Big(\hat{P}_{K,L} + \frac{e_K}{m_K c} \hat{a}_{K,L} \cdot A_K^{\mathrm{ex}} \Big) \Big\} \tag{44}$$

and contains no more odd-odd terms (which are proportional to $\varrho_{1,\mathrm{I}} \cdot \varrho_{1,\mathrm{II}}$ and $\varrho_{2,\mathrm{I}} \cdot \varrho_{2,\mathrm{II}}$, respectively), since the latter vanished owing to the anticommutator appearing in W_{II}. The still remaining odd terms (of the first and second order) can be, however, easily eliminated by means of a single unitary transformation which does not change the form of even terms.

3. Additional transformation of the generalized Bethe-Salpeter Hamiltonian to even form and its reduction. Generalization for the case of N fermions

As it can be seen from the shape of the Hamiltonian H_{II}^{BS} (Eq. (44) the situation is similar to the one Dirac particle case (Eq. (12). Thus it is sufficient to construct a single unitary transformation according to the Foldy-Wouthuysen iteration procedure, namely

$$\text{II}' = \exp(iS'), \tag{45}$$

$$S' = -\frac{1}{2c^3}\left\{\sum_K \frac{1}{m_K}\cdot \varrho_{1,K} Y_K^{\text{ex}} + \frac{1}{2}\sum_{K,L}{}' \frac{1}{m_K}\left[\varrho_{1,K}\cdot\varrho_{3,L} Y_{K,L} - \varrho_{2,K}\left(\hat{P}_{L,K} + \right.\right.\right.$$
$$\left.\left.\left.+\frac{e_L}{m_L c}\hat{a}_{L,K} A_K^{\text{ex}}\right)\right]\right\} \tag{46}$$

and after comparatively simple transformations one obtains

$$H' = \text{II}' H_{\text{II}}^{BS}(\text{II}')^+ = \sum_K \left\{\varrho_{3,K}\left[m_K c^2 + \frac{p_K^2}{2m_K} - \frac{p_K^4}{8m_K^3 c^2} + \frac{e_K^2}{2m_K c^2}(A_K^{\text{ex}})^2 + \right.\right.$$
$$\left.\left. + \frac{1}{c} P_K^{\text{ex}}\right] + e_K \Phi_K^{\text{ex}} + \frac{1}{c^2}(Q_K^{\text{ex}} + F_K^{\text{ex}})\right\} + \sum_{K,L}{}' \left\{\frac{1}{2}(\varrho_{3,K} + \varrho_{3,L})\left[\frac{1}{2} e_K \Phi_{K,L} + \right.\right.$$
$$\left.\left. + \frac{1}{c^2}\left(Q_{K,L} + F_{K,L} + G_{K,L} + \frac{1}{2} L_{K,L} + \frac{1}{2} M_{K,L}\right)\right]\right\}. \tag{47}$$

This is the final even form of the generalized Bethe-Salpeter Hamiltonian (which contains only the terms with $\varrho_{3,K}$, or such which do not contain the ϱ_K operators). It can be seen from the form of this Hamiltonian that complete separation of positive and negative energy states is possible. In the preceding considerations there was no necessity of using the concrete representation of the Dirac operators. Only now we choose $\varrho_{3,K}$ in diagonal form. Hence it can be seen that there exist four possible states which belong to four possible combinations of the eigenvalues of the operators $\varrho_{3,\text{I}} = \pm 1$, $\varrho_{3,\text{II}} = \pm 1$. The choice of the eigenvalues $\varrho_{3,K}$ with opposite signs gives rise to vanishing mutual interactions of particles (in accordance with the assumptions of the hole theory which is valid just here). If we accept in particular $\varrho_{3,\text{I}} = \varrho_{3,\text{II}} = \pm 1$, we obtain the well-known form of the reduced effective Hamiltonian in the subspace of positive energy states:

$$H_{\text{eff}} = \sum_K \left\{m_K c^2 + \frac{p_K^2}{2m_K} - \frac{p_K^4}{8m_K^3 c^2} + \frac{1}{c} P_K^{\text{ex}} + \frac{e_K^2}{2m_K c^2}(A_K^{\text{ex}})^2 + \right.$$
$$\left. + e_K \Phi_K^{\text{ex}} + \frac{1}{c^2}(Q_K^{\text{ex}} + F_K^{\text{ex}})\right\} + \sum_{K,L}{}' \left\{\frac{1}{2} e_K \Phi_{K,L} + \frac{1}{c^2}(Q_{K,L} + \right.$$
$$\left. + F_{K,L} + G_{K,L} + \tfrac{1}{2} L_{K,L} + \tfrac{1}{2} M_{K,L})\right\}. \tag{48}$$

This Hamiltonian is the final aim of all methods of reduction and the basis of the quantum-mechanical description in the Pauli formalism for two charged Dirac particles in an external electromagnetic field. It should be noted that contrary to the single-particle case the additional unitary transformation does not commute with the transformation II in the accepted approximation. The role of the transformation II′ is restricted to introducing into the Hamiltonian (44) those terms which compensate the superfluous odd terms owing to the commutation conditions between S' and the terms $\varrho_{3,K} m_K c^2$.

The generalization of the Bethe-Salpeter equation has so far been done for two particles only. This equation, however, can be directly generalized for the case of an arbitrary number of fermions (both identical or not) by assuming that the indices K, L become I, II, ... N. We thus obtain

$$H^{(N)} = \sum_{K=\mathrm{I}}^{N} H_K + \tfrac{1}{2} \sum_{K,L=\mathrm{I}}^{N} W_{K,L}, \qquad W_{K,L} = \tfrac{1}{4} [\lambda_K + \lambda_L, V_{K,L}]_+, \tag{49}$$

$$V_{K,L} = e_K \cdot e_L \left(\frac{1}{r_{K,L}} - \frac{1}{2} \varrho_{1,K} \cdot \varrho_{1,L} J_{K,L} \right), \qquad \boldsymbol{r}_{K,L} = \boldsymbol{r}_K - \boldsymbol{r}_L, \qquad r_{K,L} = |\boldsymbol{r}_{K,L}|. \tag{50}$$

The generalized Case transformation will in this case be the product of N mutually commuting Case transformations for the particular particles of the system. In the intermediate scheme we obtain Eq. (44) in which the summation should be extended from I to N. The additional unitary transformation and the final even form of the N fermion Hamiltonian is expressed by Eqs (46)–(48) (after extending the summation from I to N). In the case of N fermions there are 2^N possible states with definite sign of energy which belong to 2^N possible combinations of the eigenvalues $\varrho_{3,\mathrm{I}} = \pm 1$, $\varrho_{3,\mathrm{II}} = \pm 1 \ldots \varrho_{3,N} = \pm 1$. In particular, if we select all $\varrho_{3,K} = +1$, we obtain the reduced effective N fermion Hamiltonian which can be accepted as the basis of the quantum-mechanical description in the Pauli formalism of a system of N charged Dirac particles in an external electromagnetic field:

$$H_{\mathrm{eff}}^{(N)} = \sum_{K=\mathrm{I}}^{N} \left\{ m_K c^2 + \frac{p_K^2}{2m_K} - \frac{p_K^4}{8 m_K^3 c^2} + \frac{1}{c} P_K^{\mathrm{ex}} + \frac{e_K^2}{2 m_K c^2} (A_K^{\mathrm{ex}})^2 + \right.$$
$$\left. + e_K \Phi_K^{\mathrm{ex}} + \frac{1}{c^2} (Q_K^{\mathrm{ex}} + F_K^{\mathrm{ex}}) \right\} + \sum_{K,L=\mathrm{I}}^{N}{}' \left\{ \frac{1}{2} e_K \Phi_{K,L} + \frac{1}{c^2} (Q_{K,L} + \right.$$
$$\left. + F_{K,L} + G_{K,L} + \tfrac{1}{2} L_{K,L} + \tfrac{1}{2} M_{K,L}) \right\}. \tag{51}$$

4. *Discussion of the results*

The above considerations which concern the reduction of quasirelativistic two-particle equations have shown that out of the two alternative possibilities, *i.e.*, the Breit and the Bethe-Salpeter equation, the latter can be regarded as the basic quasirelativistic two-particle equation. Its generalization taken together with the gauge invarianec condition is uniquely defined. The reduction of this equation leads to a correct effective Hamiltonian in a simple unambiguous way without any other additional assumptions. This reduction is free from several difficulties which appear in attempts of applying analogous procedure to the Breit equation. The results obtained contain both the case of different and identical particles (after including the obvious postulate of antisymmetrization of the state vectors). The generalization of the considerations for the case of arbitrary number of particles turned out to be simple owing to the fact that in contrast with the earlier methods of reduction we did not make use of the concrete representation of the Dirac operators nor of the matrices constructed from these operators (in the 16×16 product space). The method of reduction applied does not contain any ambiguities[2] which appear in the application of the Foldy-Wouthuysen iteration method in Refs [6, 9, 10 and 11]. The generalized Case transformation brought the quasirelativistic two-particle Hamiltonian to the intermediate scheme which is of considerable importance for illustrative interpretation owing to the simple geometrical meaning of this transformation. In particular, the analysis and the comparison of the Breit and the Bethe-Salpeter Hamiltonian provided additional arguments in favour of the latter as the basis of the quantum-mechanical description of the system of Dirac particles. The supplement of this transformation by a simple additional unitary trnasformation which has only ancillary meaning, has brough this Hamiltonian to even form which permits complete separation of the subspaces belonging to different signs of energy. This additional unitary transformation will be of no importance for the transformation of the remaining particle observables owing to the high order of terms appearing in S' (Eq. (46). These observables are only approximately transformed by the transformation $\amalg$. In analogy to Foldy and Wouthuysen (Ref. [8] quoted before) it is possible to make an attempt of defining mean value operators based on the transformation $\amalg$. This problem, however, requıres further study in respect to its physical interpretation. Novertheless the results of the presents paper indicate how to study those particle observables, which together with the Hamiltonian, define the basis of the quantum-mechanical description of the particle system. The above considerations make clear the essential role of the transformation $\amalg$ in the quantum-mechanical description of a system of Dirac particles.

The range of applicability of the many-fermion Hamiltonian presented in this paper can be considerably extended by taking into account the anomalous magnetic moments of the considered Dirac particles which can be included by means of the well-known phenomenological methods. This problem, however, requires still many additional considerations and complicated calculations. Such calculations, however, are in principle possible on the basis of the results presented in this paper.

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[2] The ambiguities appear in the definitions of the generators of the unitary transformation which eliminate the odd terms in subsequent order of approximation from the Hamiltonian. However, it has been shown by Pursey [16] that these ambiguities have no deeper physical meaning. The Foldy-Wouthuysen iteration procedure has then been made unambiguous by Eriksen [17].

APPENDIX A

As the starting point of many considerations on many-Dirac particle system (in particular two particle system) one can regard the Hamiltonian expressed in the product space of these particles. The Hamiltonian one starts with has the form:

$$H^0 = H_{\mathrm{I}}^0 + H_{\mathrm{II}}^0, \tag{A.1}$$

$$H_K^0 = \varrho_{3,K} m_K c^2 + c\varrho_{1,K} \boldsymbol{\sigma}_K \cdot \boldsymbol{p}_K. \quad \text{(A.}$$

The external magnetic field expressed in terms of the potentials A_K^{ex} and Φ_K^{ex} is introduced by the well-known correspondence method

$$\boldsymbol{p}_K \to \boldsymbol{p}_K - \frac{e_K}{c} A_K^{\mathrm{ex}}, \qquad E_K \to E_K - e_K \Phi_K^{\mathrm{ex}}. \tag{A.3}$$

On the other hand the term which describes the interactions between the particles must be justified by means of arguments from outside the formalism of quantum mechanics, but it is finally presented as effective energy which depends on the dynamic variables of the particle only. Such role is played by the Breit and the Bethe-Salpeter operators which describe the interaction between the particles. In case of absence of electromagnetic field we have

$$H_B^0 = H_{\mathrm{I}}^0 + H_{\mathrm{II}}^0 + V, \qquad H_{BS}^0 = H_{\mathrm{I}}^0 + H_{\mathrm{II}}^0 + W^0 \tag{A.4}$$

i. e. the well-known Breit and Bethe-Salpeter Hamiltonians, respectively, the latter in the form proposed by Barker and Glover [6]. In case of the Breit operator there is no difficulty in taking into account the interaction of the particles and the influence of the magnetic field since V is independent of the momenta of the particles. This is different in case of the Bethe-Salpeter interaction W^0. In contrast to V, W^0 cannot be treated as effective potential energy which depends on the coordinates of the particles only since it also depends on their momenta *via* the operators λ_K^0. Unambiguous generalization requires some additional assumptions which fix the sequence of introducing the external field and the interaction W^0 into the Hamiltonian (A.1). A more consequent procedure seems to be the introduction of the potentials of the external field into the Hamiltonian which already contains the mutual interaction between the particles. The proper way of generalizing the shapes of the operators appearing in W^0 is the analysis of the generalized Case transformation which brings the operator

$$\lambda_K = (\varrho_{3,K} m_K c^2 + c\varrho_{1,K} z_K)\left[(\varrho_{3,K} m_K c^2 + c\varrho_{1,K} z_K)^2\right]^{-\frac{1}{2}} \tag{A.5}$$

into the operator $\varrho_{3,K}$, analogously as was the case with the Foldy-Wouthuysen transformation in case of the operator λ_L^0. The operator λ_K depends on the external magnetic field. In the presence of electric field it is not a constant of motion but, having the eigenvalues ± 1 and thus representing the sign of the kinetic energy, it can be regarded as a natural generalization of the operator λ_K^0. Hence we obtain immediately the generalized Bethe-Salpeter Hamiltonian given by Eq. (7). The wave equation which contains the Hamiltonian is invariant with respect to the gauge transformation which fact distinguishes the postulated form of the Hamiltonian in comparison with those resulting from eventual others attempts of generalizing the operator W^0 (*e. g.* by putting $W^0 = W$ which would correspond to the reversed order of introducing the external field and mutual interaction).

Since the dynamic variables of the particles are independent, the gauge transformation of the field can be written in the form (*cf.* Kramers [18], p. 2)

$$A_K^{ex} \to A_K^{ex} + \mathrm{grad}_K f_K, \qquad \Phi_K^{ex} \to \Phi_K^{ex} - \frac{1}{c}\frac{\partial f_K}{\partial t}, \tag{A.6}$$

$$\Psi \to \Psi \cdot \exp\left(\frac{i}{\hbar c}\sum_K e_K f_K\right), \tag{A.7}$$

where

$$f_K \overset{\mathrm{def}}{=} f(\boldsymbol{r}_K, t). \tag{A.8}$$

Hence the condition $\Box f = 0$ imposed on arbitrary function f implies analogous conditions for f_K (which are defined by Eq. (A.8))

$$\Box_K f_K = 0. \tag{A.9}$$

It is interesting to note that the Case transformation is gauge invariant. This follows immediately from the fact that S given by Eq. (2) is expressed by the operator z_K (Eq. (6) which is invariant.

APPENDIX B

There is a possibility of presenting the Breit Hamiltonian as the sum of two formally single-particle terms

$$H^B = \tilde{H}_{\mathrm{I}} + \tilde{H}_{\mathrm{II}}, \qquad \tilde{H}_K = \varrho_{3,K} m_K c^2 + c\varrho_{1,K}\tilde{z}_K + e_K\tilde{\Phi}_K \tag{B.1}$$

where

$$\tilde{z}_K = \boldsymbol{\sigma}_K\left(\boldsymbol{p}_K - \frac{e_K}{c}\tilde{A}_K\right), \qquad \tilde{\Phi}_K = \Phi_K^{ex} + \frac{1}{2}\Phi_{K,L}, \tag{B.2}$$

$$\tilde{A} = A_K^{ex} + \tfrac{1}{2}\tilde{A}_{K,L}, \qquad \Phi_{K,L} = \frac{e_L}{r} \tag{B.3}$$

$$\hat{A}_{K,L} = \varrho_{1,L}\frac{e_L}{2}\left\{\frac{\boldsymbol{\sigma}_L}{r} + \frac{(\boldsymbol{\sigma}_L\cdot\boldsymbol{r})\boldsymbol{r}}{r^3}\right\} = \varrho_{1,L}\hat{\boldsymbol{a}}_{K,L}. \tag{B.4}$$

In reality $\tilde{H}_K$ is the operator of the effective energy of the Kth particle. This energy comprises both the influence of the external field and also a half of the interaction with the other Lth particle. Both contributions are expressed in terms of "effective potentials" $\tilde{\Phi}_K$ and $\tilde{A}_K$. $\hat{A}_{K,L}$ can be regarded as the operator generalization of the vector potential. Such interpretation is suggested by the analogy based on the correspondence principle between the magnetic interaction expressed in the form

$$-\frac{1}{2}\sum_{K,L}{}' \frac{e_K \cdot e_L}{2c^2}(c\boldsymbol{a}_K)\left\{\frac{c\boldsymbol{a}_L}{r}+\frac{\boldsymbol{r}(c\boldsymbol{a}_L\cdot\boldsymbol{r})}{r^3}\right\}, \tag{B.5}$$

and the classical expression for the energy of the current $\vec{\gamma}$ in the magnetic field described by the potential A.

$$\varepsilon = -\frac{1}{2c}\int \boldsymbol{j}(\boldsymbol{r}')A(\boldsymbol{r}')d\boldsymbol{r}'. \tag{B.6}$$

As it can be easily checked the expression (B.6) can be formally brought to (B.5) if the classical quantities A and γ are replaced by the operators $\hat{A}_{K,L}$ and

$$\hat{\boldsymbol{j}}_K = e_K c\boldsymbol{a}_K\delta(\boldsymbol{r}_K-\boldsymbol{r}') \tag{B.7}$$

which corresponds to the acceptance of the Dirac velocity operator of the particle $c\alpha_K$, and $\delta(\boldsymbol{r}_K-\boldsymbol{r}')$ is the three-dimensional Dirac δ-function.

An important role in the calculations made in Section 2 is played by the operators $\hat{\hat{\boldsymbol{a}}}_{K,L}$ which are defined by the operator equation

$$\boldsymbol{\sigma}_K\ddot{\boldsymbol{a}}_{K,L} \overset{\text{def}}{=} \frac{e_L}{4m_Lc}[\boldsymbol{\sigma}_L\cdot\boldsymbol{p}_L, J]_+. \tag{B.8}$$

After calculating the anticommutator and separating the factor $\boldsymbol{\sigma}_K$ we obtain

$$\hat{\hat{\boldsymbol{a}}}_{K,L} = \frac{e_L}{4m_Lc}\left\{\frac{1}{r}\boldsymbol{p}_L+\boldsymbol{p}_L\frac{1}{r}+(\boldsymbol{p}_L\cdot\boldsymbol{r})\frac{\boldsymbol{r}}{r^3}+\frac{\boldsymbol{r}}{r^3}(\boldsymbol{r}\cdot\boldsymbol{p}_L)\pm 2\hbar\frac{\boldsymbol{\sigma}_L\times\boldsymbol{r}}{r^3}\right\} \tag{B.9}$$

or, in a less symmetric form

$$\hat{\hat{\boldsymbol{a}}}_{K,L} = \frac{e_L}{2m_Lc}\left\{\frac{1}{r}\boldsymbol{p}_L+\frac{\boldsymbol{r}}{r^3}(\boldsymbol{r}\cdot\boldsymbol{p}_L)\pm\hbar\frac{\boldsymbol{\sigma}_L\times\boldsymbol{r}}{r^3}\right\}. \tag{B.10}$$

It can beseen from these formulas that the operators $\hat{\hat{\boldsymbol{a}}}_{K,L}$ are connected with the classical expressions for the vector potential through the correspondence principle, this time however, for the magnetic field produced by the motion of the particle with the velocity $\boldsymbol{p}_L/m_L$ and the proper magnetic moment $e_L\hbar\boldsymbol{\sigma}_L/2m_Lc$. This is a nonrelativisic relation. As it can be easily checked

$$\mathrm{div}_K\hat{\boldsymbol{a}}_{K,L} = \mathrm{div}_K\hat{\hat{\boldsymbol{a}}}_{K,L} = 0. \tag{B.11}$$

The analysis of the shape of H^B_{II} (Eqs (12) and (35)) suggests still another remark concerning the circumstances which could be responsible for the appearance of the controversial even term $\sim e^4$, which expresses the effect of the admixture of states in the earlier methods of reduction of the Breit equation. A part of the terms in H_{II}^{B} can be presented in the form:

$$\varrho_{3,K}\left\{\frac{p_K^2}{2m_K}+\frac{1}{c}P_K^{\mathrm{ex}}+\varrho_{1,L}\frac{1}{c}\hat{P}_{K,L}+\varrho_{3,L}\frac{1}{2c^2}\hat{\hat{P}}_{K,L}+\frac{e_K^2}{2m_Kc^2}(A_K^{\mathrm{ex}})^2+\right.$$
$$\left.+\varrho_{1,L}\frac{e_K}{m_Lc^2}\hat{a}_{K,L}A_K^{\mathrm{ex}}\right\}. \qquad (\mathrm{B.12})$$

If we add and subtract a term of analogous structure as the controversial term $\sim e^4$, then this expression can be supplemented to the square

$$\varrho_{3,K}\left\{\frac{1}{2m_K}\left[\sigma_K\cdot\left(P_K-\frac{e_K}{c}A_K^{\mathrm{ex}}-\varrho_{1,L}\frac{e_K}{c}\hat{a}_{K,L}-\varrho_{3,L}\frac{e_K}{2c}\hat{\hat{a}}_{K,L}\right)\right]^2+\right.$$
$$\left.-\frac{e_K^2e_L^2}{8m_Kc^2}J^2\right\} \qquad (\mathrm{B.13})$$

$$J^2=2\left\{\frac{3}{r^2}-2\frac{\sigma_{\mathrm{I}}\cdot\sigma_{\mathrm{II}}}{r^2}+\frac{(\sigma_{\mathrm{I}}\cdot r)(\sigma_{\mathrm{II}}\cdot r)}{r^4}\right\}. \qquad (\mathrm{B.14})$$

The above considerations shed some light on the role of the term containing J^2. This is an even term but it is closely connected with the odd operator $\varrho_{1,L}\frac{e_K}{c}\sigma_K\cdot\hat{a}_{K,L}$ appearing in (B.13), which, after being squared has lost its odd character, although it continues to express the correction due to the admixture of negative energy states.

This example shows that there is a possibility of the appearance of such terms in methods of reduction based on the procedure of formal elimination of odd terms whenever these methods do not take into account the origin and the physical meaning of such terms.

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