THE EQUATIONS OF MOTION IN EINSTEIN'S UNIFIED FIELD THEORY

BY A. H. KLOTZ AND G. K. RUSSELL

Department of Applied Mathematics, University of Sydney*

(Received April 13, 1972; Revised paper received May 5, 1972)

It is shown that the field equations of the nonsymmetric unified field theory imply equations of motion of a charged test particle with an electromagnetic force term. This result is contingent on the electromagnetic field tensor being suitably expressed in terms of the fundamental tensor. A conjecture of Wyman on the boundary conditions is adopted, and it is shown that to the required order of approximation the electromagnetic force consists of a Lorentz term together with a small correction term, linearly dependent on distance and similar to the Hubble law of cosmic repulsion. It is thus very different from the corrections previously contemplated (Treder 1957). These results remove a major objection to Einstein's unified field theory. The appearance of what is essentially a correction term to a Coulomb field, is a firm prediction of the theory, capable in principle of an empirical verification.

1. *Introduction*

Perhaps the most severe criticism of Einstein's nonsymmetric unified field theory (1954) arises from the claim that its field equations do not seem to imply what is thought to be the correct equations of motion of a charged test particle. On the other hand, Einstein, Infeld and Hoffman (1938, abbreviated by EIH in the sequel) showed that the general relativistic field equations require a test body (an uncharged, gravitating particle) moving on a geodesic of the space-time manifold. This result is regarded as a consequence of the non-linearity of the field equations and should therefore be implied also in every attempt to extend general relativity to include, alongside the gravitational, a geometrical description of the electromagnetic field.

Except for some variations which will be mentioned below, there are two alternative sets of the non-symmetric field equations. We have first the so-called weak field equations

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda} \equiv g_{\mu\nu,\lambda} - g_{\sigma\nu}\Gamma^{\sigma}_{\mu\lambda} - g_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} = 0,$$

$$\Gamma^{\sigma}_{\underset{\vee}{\mu\sigma}} = 0,$$

* Address: Department of Applied Mathematics, University of Sydney, Sydney, N. S. W. 2006, Australia.

$$R_{\mu\nu} = 0, \tag{1.1}$$
$$R_{\underset{\vee}{\mu\nu},\lambda} = 0,$$

where $g_{\mu\nu}$ is a non-symmetric fundamental tensor which replaces the metric tensor of general relativity, $\Gamma^{\lambda}_{\mu\nu}$ denotes the components of an affine connection and $R_{\mu\nu}$ is the Ricci tensor formed from $\Gamma^{\lambda}_{\mu\nu}$ and its first derivatives. As usual, a comma represents partial differentiation while a line and a hook under a pair of indices denote symmetry and skew-symmetry respectively and the dots indicate a cyclic sum. The Greek indices range over 0, 1, 2, 3. Secondly, there are the strong field equations (which, unlike the set (1.1), have not been derived from a variational principle) which are obtained by replacing the last two of (1.1)

$$R_{\mu\nu} = 0. \tag{1.2}$$

The method of EIH of deriving the equations of motion consists of expanding $g_{\mu\nu}$ (which, in their case, was of course the ordinary metric tensor) in a power series of a small parameter ε (say). Using this method and identifying $g_{\underset{\vee}{\mu\nu}}$ with the electromagnetic field tensor $f_{\mu\nu}$, Infeld (1950), Ikeda (1952) and Callaway (1953) showed that both the strong and weak field equations imply equations of motion which are independent of electromagnetic terms up to the fourth order in ε. The same result holds for an alternative interpretation proposed by Hlavaty (1957) where a certain linear combination of the components $g_{\mu\nu}$ is identified with $f_{\mu\nu}$. In other words, it seems to be impossible to derive the Lorentz force on a charged particle moving in an electromagnetic field, directly from the unified field equations. On the other hand, Infeld and Wallace (1940) showed that the general relativistic, Einstein-Maxwell theory does imply the Lorentz equations of motion.

The purpose of the present article is to show that the reason why the equations of motion do not follow from the field equations in Einstein's theory is due to the way in which the electromagnetic field is identified. It will be shown that there exists an alternative expression for the field tensor for which it is possible to derive the equations of motion of a charged test particle. In this expression a Lorentz force term appears in a modified form. The modification (essentially to the Coulomb, inverse square law of electrostatic attraction or repulsion) appears to be related to a cosmological force familiar in a de Sitter universe.

Our correction is different in principle from that of Treder (1957) who showed that an electromagnetic force term follows from the field equations if the current density vector is identified in a special way. He arrived at a potential of the form

$$\varphi = \frac{a}{r} + b + cr + dr^2, \qquad a, b, c, d \text{ const.},$$

but rejected the case $d \neq 0$ on the grounds that it would give an increasing (with r) field. We shall find that this is an unnecessary restriction. In any case, our starting point is quite different since it is the field tensor itself which is re-defined. It should be pointed out that our function Φ is not the electrostatic potential.

The authors wish to express their gratitude to Professor K. E. Bullen, F. R. S., for a critical reading of the manuscript.

2. *The equations of motion*

Let $^*g^{\mu\nu}$ denote the tensorial inverse of $g_{\mu\nu}$. Then, providing the skew-symmetric part of $g_{\mu\nu}$ is calculated before taking covariant derivatives, we can write

$$f_{\mu\nu} = k^* g^{\alpha\beta} g_{\underset{+}{\mu}\underset{-}{\overset{\vee}{\nu}};\alpha\beta}, \tag{2.1}$$

where k is some constant. (The two operations do not commute, since by the first of (1.1)

$$g_{\underset{+}{\mu}\underset{-}{\nu};\alpha\beta} = 0.)$$

The particular form of $f_{\mu\nu}$ has been used by us (Russell and Klotz 1972) to solve the problem raised by a theorem of Tiwari and Pant (1970) that, in the standard interpretation of $f_{\mu\nu}$, there can be no solutions of the weak field equations corresponding to an isolated electrostatic charge. Herein, however, we are interested primarily in the equations of motion.

We shall follow, almost exactly, the method of Infeld (1950), and note for the sake of reference a number of results obtained in this work. In particular, we consider only the approximation to the equations of motion up to the fourth order in ε. We assume, with Infeld, that $\underset{\vee}{g_{\mu\nu}}$ can be expanded in the series

$$\underset{\vee}{g_{ol}} = \dot\varepsilon^3 \underset{\vee}{\overset{3}{g}_{ol}} + (\varepsilon^5 \underset{\vee}{\overset{5}{g}_{ol}} + \dots),$$

$$\underset{\vee}{g_{mn}} = \varepsilon^2 \underset{\vee}{\overset{2}{g}_{mn}} + \varepsilon^4 \underset{\vee}{\overset{4}{g}_{mn}} + \dots,$$

where Latin subscripts range over 1, 2, 3. The weak field equations (1.1) then give

$$\underset{\vee}{\overset{2}{g}_{ms,s}} = 0, \tag{2.2}$$

and

$$(\underset{\vee}{\overset{2}{g}_{ik,l}})_{,ss} = 0. \tag{2.3}$$

According to (2.1), we now have, to the second order,

$$\overset{2}{f}_{mn} = k \underset{\vee}{\overset{2}{g}_{mn,ss}}. \tag{2.4}$$

Suppose now that the field is due to a number of sources. As in general relativity, the equations of motion of the k^{th} source are given by

$$\sum_{i=1}^{\infty} \overset{k}{\underset{i}{c}}_m \varepsilon^i = 0, \tag{2.5}$$

where

$$-2\pi \overset{k}{\underset{i}{c}}_m = \int_{S_k} \Lambda_{mk} n^k dS. \tag{2.6}$$

In (2.6), n^k is the unit outward normal to the surface S_k surrounding the k^{th} source, and

$$\underset{i}{\Lambda}_{mk} = \underset{i-}{R}_{mk} + \tfrac{1}{2}\,\delta_{mk}\underset{i}{R}_{oo} - \tfrac{1}{2}\,\delta_{mk}\underset{i}{R}_{ss}. \tag{2.7}$$

We assume also that $g_{\underline{\mu\nu}}$ gives the metric of the space-time manifold (that is, the gravitational potentials), at least to the second order in ε. Actually, there is no evidence to suggest that this should not be the case exactly, to all orders in ε. Since to the second order, the field equations separate into purely gravitational and purely electromagnetic parts, it follows that we need consider only the electromagnetic correction to the EIH equations

$$-2\pi \overset{k}{\underset{4}{c'_m}} = \int_{S_k} \underset{4}{\Lambda'_{mk}}\, n^k dS, \tag{2.8}$$

where

$$\underset{4}{\Lambda'_{mk}} = \underset{4-}{R}_{mk} - \underset{4}{P}_{mk} + \tfrac{1}{2}\,\delta_{mn}(\underset{4}{R}_{oo} - \underset{4}{P}_{oo}) - \tfrac{1}{2}\,\delta_{mk}(\underset{4}{R}_{ss} - \underset{4}{P}_{ss}); \tag{2.9}$$

$P_{\mu\nu}$ is the Ricci tensor of Riemannian geometry with $g_{\underline{\mu\nu}}$ as the metric tensor, and so is necessarily symmetric.

Infeld has shown that

$$\begin{aligned} \underset{4-}{R}_{mk} - \underset{4}{P}_{mk} = {} & [\overset{2}{\underset{\vee}{g}}_{ms}(\overset{2}{I}_{nps} - \overset{2}{\underset{\vee}{g}}_{np,s}) + \overset{2}{\underset{\vee}{g}}_{sn}(\overset{2}{I}_{pms} - \overset{2}{\underset{\vee}{g}}_{pm,s})]_{,p} - \\ & - \tfrac{1}{2}\,\{[\overset{2}{\underset{\vee}{g}}_{sp}(\overset{2}{I}_{pms} - \overset{2}{\underset{\vee}{g}}_{pm,s})]_{,n} + [\overset{2}{\underset{\vee}{g}}_{sp}(\overset{2}{I}_{pns} - \overset{2}{\underset{\vee}{g}}_{pn,s})]_{,m}\} - \\ & - (\overset{2}{I}_{mts} - \overset{2}{\underset{\vee}{g}}_{mt,s})\,(\overset{2}{I}_{smt} - \overset{2}{\underset{\vee}{g}}_{sn,t}), \end{aligned} \tag{2.10}$$

where

$$I_{abc} = \tfrac{1}{2}\,\underset{\vee}{g}_{ab,c}, \tag{2.11}$$

and that

$$\underset{4}{R}_{oo} - \underset{4}{P}_{oo} = 0. \tag{2.12}$$

We can now apply these results to the problem of determining explicitly the equations of a charged test particle. For the sake of simplicity, we shall discuss a two-body problem only.

3. The two-body problem

Since we are concerned only with a weak (fourth order) approximation, the sources must be regarded as essentially at rest (that is, we cannot expect to get the $\frac{e}{c}\,\underline{v} \wedge \underline{B}$ part of the Lorentz force). Consequently we consider fields far from the sources. Furthermore, the latter may be either singularities in the field (as in Callaway's work), or may be described by singularity free solutions of the field equations, possessing some symmetry about the source and reducing to the classical fields away from it. This has been discussed by Infeld.

Let $\underset{1}{\overset{2}{f}}_{\mu\nu}$ and $\underset{2}{\overset{2}{f}}_{\mu\nu}$ be the electromagnetic fields due to the first and to the second source respectively, so that

$$\overset{2}{f}_{\mu\nu} = \underset{1}{\overset{2}{f}}_{\mu\nu}+\underset{2}{\overset{2}{f}}_{\mu\nu},$$

and let us write

$$m = \varepsilon^2\overset{2}{m}+\varepsilon^4\overset{4}{m}+\ldots$$

for the mass of either source, and

$$e = \varepsilon^2\overset{2}{e}+\varepsilon^4\overset{4}{e},$$

for either charge.

Then, classically, the electrostatic potential far from the two sources is approximately given by

$$\varphi \alpha \frac{\overset{2}{e}_1+\overset{2}{e}_2}{r} \tag{3.1}$$

and

$$\overset{2}{f}_{\mu\nu} \alpha \varepsilon_{\mu\nu\alpha}\varphi_{,\alpha} = \varepsilon_{\mu\nu a}\varphi_{,a},$$

where $\varepsilon_{\mu\nu\lambda}$ is a 3-index permutation symbol.

It follows from the equation (2.5) that

$$\overset{2}{\check{g}}_{mn,ss} = \varepsilon_{mna}\varphi_{,a}. \tag{3.2}$$

Let us define a function Φ, by the equation

$$\Phi_{,ss} = \varphi, \tag{3.3}$$

so that we may take

$$\overset{2}{\check{g}}_{mn} = \varepsilon_{mna}\Phi_{,a}. \tag{3.4}$$

Since φ is a harmonic function, and ε_{mna} is completely skew-symmetric in all its indices, the expression (3.4) satisfies the equations (3.2) as well as (2.2) and (2.3).

The only non-trivial component of I_{abc} (2.11) is now

$$I_{123} = \tfrac{1}{2}(\varepsilon_{12a}\Phi_{,3a}+\varepsilon_{23a}\Phi_{,1a}+\varepsilon_{31a}\Phi_{,2a}) =$$

$$= \tfrac{3}{2}\varphi,$$

and hence, by (2.8),

$$\underset{4}{\Lambda'_{mn}} = -\tfrac{1}{4}\,\delta_{mn}\varphi^2+\tfrac{1}{2}\,\varphi_{,m}\Phi_{,n}+\tfrac{1}{2}\,\varphi_{,n}\Phi_{,m}+\tfrac{1}{2}\,\delta_{mn}\varphi_{,a}\Phi_{,a}-$$

$$-\Phi_{,ma}\Phi_{,na}-\Phi_{,a}\Phi_{,amn}-\varepsilon_{msa}\varepsilon_{npt}\Phi_{,ap}\Phi_{,st}+$$

$$+\delta_{mn}\Phi_{,ab}\Phi_{,ab} =$$

$$= F_{mnp,p}-S_{mn},$$

where

$$F_{mnp} = -F_{mpn} =$$

$$= -\varepsilon_{msa}\varepsilon_{npt}\Phi_{,a}\Phi_{,st}-\delta_{pm}\Phi_{,an}\Phi_{,a}+$$

$$+\delta_{mn}\Phi_{,ap}\Phi_{,a}+\tfrac{1}{2}\,\delta_{mp}\varphi_{,n}\Phi-\tfrac{1}{2}\,\delta_{mn}\varphi_{,p}\Phi,$$

and

$$S_{mn} = \tfrac{1}{4}\,\delta_{mn}\varphi^2-\tfrac{1}{2}\,\varphi_{,m}\Phi_{,n}+\tfrac{1}{2}\,\varphi_{,mn}\Phi.$$

An elementary calculation shows that

$$S_{mn,n} = 0;$$

hence the surface integral (2.7) does not depend on the shape of the surface surrounding the source. Further, unlike in the work of Infeld and Callaway, the extra terms

$$\frac{1}{2\pi}\int S_{mk}n^k dS$$

do not vanish.

The general solution for the bi-harmonic function Φ is

$$\overset{i}{\Phi} = c_1\overset{2}{e_i}r^2+c_2\overset{2}{e_i}r+c_3\overset{2}{e_i}+\frac{c_4\overset{2}{e_i}}{r}, \qquad i = 1, 2, \ldots \tag{3.5}$$

where c_2 is a non-zero constant and c_1, c_3 and c_4 are arbitrary constants.

To simplify the notation, let the two particles have charges e_1 and e_2 and be located at the points ξ^a and η^a respectively. Further, if x^a is a field point, let

$$x^a-\xi^a = R^a,$$

$$x^a-\eta^a = \varrho^a,$$

and let us consider the equations of motion of the first particle. We surround it by a sphere of radius R, and retain only those terms of S_{mk} which are of order R^{-2}, since all the other terms lead to terms in the final equation which depend on the size of the sphere, whilst we know the final result to be independent of it. We then allow the sphere to shrink to zero radius in the usual way, to yield the equations of motion.

Now,

$$\Phi = \overset{1}{\Phi} + \overset{2}{\Phi} =$$

$$= c_1\overset{2}{e_1}R^2 + c_1\overset{2}{e_2}\varrho^2 + c_2\overset{2}{e_1}R + c_2\overset{2}{e_2}\varrho +$$

$$+ c_3\overset{2}{e_1} + c_3\overset{2}{e_2} + c_4\overset{2}{e_1}R^{-1} + c_4\overset{2}{e_2}\varrho^{-1},$$

so that the only surviving terms are

$$2c_1c_2\overset{2}{e_1}\overset{2}{e_2}\varrho^k\frac{R^m}{R^3} + c_2^2\overset{2}{e_1}\overset{2}{e_2}\frac{\varrho^k}{\varrho}\frac{R^m}{R^3} - c_2c_4\overset{2}{e_1}\overset{2}{e_2}\frac{\varrho^k}{\varrho^3}\frac{R^m}{R^3} -$$

$$-c_2c_4\overset{2}{e_1}\overset{2}{e_2}\frac{\varrho^m}{\varrho^3}\frac{R^k}{R^3} + 4c_2^2\overset{2}{e_1^2}\frac{R^mR^k}{R^4}.$$

Integrals of these functions have been considered in EIH, and need not be discussed here. The final result is that the equations of motion take the form

$$\overset{2}{m_1}\overset{2}{\ddot{\xi}^m} + \overset{2}{m_1}\overset{2}{m_2}\frac{\varrho^m}{\varrho^3} = \frac{1}{3}c_1c_2\overset{2}{e_1}\overset{2}{e_2}\varrho^m + \frac{1}{6}c_2^2\overset{2}{e_1}\overset{2}{e_2}\frac{\varrho^m}{\varrho} - \frac{2}{3}c_2c_4\overset{2}{e_1}\overset{2}{e_2}\frac{\varrho^m}{\varrho^3},$$

or, in vector form, if $\boldsymbol{r}$ is the position vector of the first particle relative to an origin which coincides instantaneously with the second,

$$m_1\frac{d^2\boldsymbol{r}}{dt^2} + \frac{m_1m_2}{r^2}\hat{\boldsymbol{r}} = \frac{1}{3}e_1e_2(c_1c_2r + \frac{1}{2}c_2^2)\boldsymbol{r} - \frac{2}{3}e_1e_2c_2c_4\frac{\hat{\boldsymbol{r}}}{r^2} \tag{3.6}$$

(the gravitational constant being taken as unity).

4. *A new law of motion*

So far we have refrained from specifying the units in which e_1 and e_2 are to be measured. However, if we take these as electrostatic then it necessarily follows that

$$c_2c_4 = -\tfrac{3}{2}. \tag{4.1}$$

Further information about the constants c_i can be gathered from dimensional considerations. In fact, let r_0 be a fundamental length constant. Then, equations (3.5), (3.6) and (4.1) enable us to write

$$c_1 = c_5r_0^{-2},$$

$$c_2 = c_6r_0^{-1},$$

and

$$c_4 = -\frac{3}{2}\frac{r_0}{c_6},$$

since c_2 is non-zero.

The radial component of the electrostatic force in (3.6), now takes the form

$$\frac{e_1 e_2}{r^2} + \frac{e_1 e_2 c_5 c_6}{3r_0^3}\left(r + \frac{c_6}{2c_5} r_0\right). \tag{4.2}$$

This form contains a classical Coulomb term together with a new non-Lorentz force term linearly dependent on r. The non-Lorentz term cannot be removed by any choice of the constants since neither r_0 nor c_0 can vanish and the case $c_5 = 0$ leads to an unrealistic physical situation. We must conclude, nevertheless, that the non-Lorentz term is too small to be detected by the resolving power of present day apparatus. We shall identify r_0, tentatively, with the radius of a finite universe. The non-Lorentz force then acquires the character of a cosmological term.

The assumption that at any given time there exists a finite, limiting radius of the universe enables us to re-introduce Wyman's conjecture (1950) that the boundary conditions to be imposed on the solutions of the unified field equations should be the so-called strong boundary conditions. In a spherical polar coordinate system, this takes the form

$$g_{\underset{\vee}{23}} \operatorname{cosec} \theta \to 0 \tag{4.3}$$

as r becomes large.

In our case

$$g_{\underset{\vee}{23}} \operatorname{cosec} \theta = e\left(2c_5 \frac{r^2}{r_0^2} + c_6 \frac{r^2}{r_0} + \frac{3}{2}\frac{r_0}{c_6}\right),$$

whence we conclude that (as $r \to r_0$),

$$2c_5 c_6 + c_6^2 + \tfrac{3}{2} = 0. \tag{4.4}$$

We obtain an "expanding universe" term in (4.2) if we put

$$\frac{c_6}{2c_5} = -\lambda, \quad 0 < \lambda < 1 \tag{4.5}$$

so that (4.2) becomes

$$\frac{e_1 e_2}{r^2} + \frac{e_1 e_2}{4(1-\lambda) r_0^3}(\lambda r_0 - r). \tag{4.6}$$

5. *Discussion*

The arguments of the previous section may well be conjectural but the basic results of this article are not. What we claim to have shown is that Einstein's non-symmetric theory is an unexceptionable approach to the problem of a complete geometrisation of the gravitational and electromagnetic fields. Above all, we have demonstrated that, with suitable geometrical identification of the latter, equations of motion which allow for an electromagnetic interaction can be derived from the field equations of the theory. Secondly, we have found a correction to the Lorentz force, or rather, because of the essentially static character of our model, to the Coulomb law. A result such as this can, in principle, be verified empirically, because it represents a definite prediction.

Original failure to derive the appropriate equations of motion led to several attempts to reformulate radically the unified field theory. Thus, identifying

$$g_{\underset{\vee}{\mu\nu}} = f_{\mu\nu}, \tag{5.1}$$

Bonnor (1954) suggested using a Hamiltonian

$$\mathscr{H}^* = \mathscr{H} + p^2 \sqrt{-g}\, f^{\mu\nu} f_{\mu\nu}, \tag{5.2}$$

where p is an arbitrary constant, $g = \det(g_{\mu\nu})$, and $\mathscr{H}$ is the Hamiltonian of Einstein's theory. This suggestion led to a Lorentz force in the equations of motion. Similarly, Stephenson and Kilmister (1953) and one of us (Klotz 1959 and 1967) effectively took the equations of motion as a fundamental postulate of unified field theory and constructed field equations to fit them. The difficulty of these theories, apart from their arbitrariness, is that they abandon *a priori* all hope of experimental verification, at least on the basis of the classical electromagnetic field laws. The present work shows that such theories are completely unnecessary.

Another point which must be mentioned is an apparent discrepancy between (4.6) and a result previously obtained by the present authors. Using our interpretation of Einstein's geometry, we showed (1972) that the well-known, particular solution of Einstein's field equations, due to Papapetrou (1948), leads to a correction in the Coulomb field of the form

$$E = \frac{e}{r^2}\left(1 - \frac{2m}{r}\right)\hat{r} \tag{5.3}$$

where e and m are the charge and mass of the source. In fact, this form of the electrostatic law gives a (small) constant, radial Lorentz force in the equations of motion. (This is essentially Treder's case). In order to obtain (4.6) a harmonic term (indeed, of the form of a Green's function) must be added. The apparent discrepancy (5.3) only demonstrates the particular nature of Papapetrou's solution.

Finally, we should mention reformulations of the theory discussed in this article due to Einstein and Kaufman (1955) and also studied by one of us (Klotz 1970 and 1972). These theories are perhaps more comprehensive, and philosophically more pleasing than the unification based on the field equations (1.1). Their existence, however, does not invalidate our conclusions. The now resolved objection concerning equations of motion applied to them just as it applied to Einstein's theory. Detailed study of the consequences of the new proposals is now in progress.

It may be of some interest to point out, in conclusion, formal similarity of the non--Lorentz term in (4.6) and the Hubble law of cosmic repulsion as obtained from a de Sitter model of the universe.

REFERENCES

Bonnor, W. B., *Proc. Roy. Soc. London*, Ser. A, **226**, 366 (1954).
Callaway, J., *Phys. Rev.*, **92**, 1567 (1953).
Einstein, A., *The Meaning of Relativity*, Appendix 2, London 1954.
Einstein, A., Infeld, L., Hoffman, B., *Ann. Math.*, **39**, 65 (1938).
Einstein, A., Kaufman, B., *Ann. Math.*, **62**, 1 (1955).
Hlavaty, V., *Geometry of Einstein's Unified Field Theory*, Groningen 1957.
Ikeda, M., *Progr. Theor. Phys.* (Japan), **7**, 127 (1952).
Infeld, L., *Acta Phys. Polon.*, **10**, 284 (1950).
Infeld, L., Wallace, P. R., *Phys. Rev.*, **57**, 797 (1940).
Klotz, A. H., *Nuovo Cimento*, **14**, 135 (1959).
Klotz, A. H., *J. Austral. Math. Soc.*, **7**, 513 (1967).
Klotz, A. H., *Acta Phys. Polon.*, **37**, 261 (1970).
Klotz, A. H., *Acta Phys. Polon.*, **B3**, 341 (1972).
Papapetrou, A., *Proc. Roy. In. Acad.*, **52**, 69 (1948).
Russell, G. K., Klotz, A. H., *Acta Phys. Polon.*, **B3**, 407 (1972).
Stephenson, G., Kilmister, C. W., *Nuovo Cimento*, **10**, 230 (1953).
Tiwari, R., Pant, D. N., *Phys. Letters*, **33A**, 505 (1970).
Treder, H., *Ann. Phys.*, **19**, 369 (1957).
Wyman, M., *Canad. J. Math.*, **2**, 427 (1950).