STUDYING THERMODYNAMICS IN HEAVY ION COLLISIONS*

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We discuss the possibility of measuring entropy of the system created in heavy ion collisions using the Ma coincidence method. The same method can also be used to test whether the system in question is in a state of equilibrium.

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1. Introduction

The assumption of thermodynamic equilibrium is one of the most commonly used when discussing the system created in central collisions of two relativistic nuclei. It is by no means obvious, however, that the equilibration actually can be achieved, since it is recognized as a process which may take longer time than the life time of the system in question. Be it or not, it is certainly important to verify if the created system is indeed in thermal equilibrium. To test this, one may try to check if the various measured quantities do satisfy relations following from thermodynamics. In the present note we discuss the possibility of testing the relation [1]

$$\left.\frac{\partial S(E,n)}{\partial E}\right|_n = \frac{1}{T}\,, \tag{1}$$

which should be valid in any system at thermal equilibrium. On the other hand the very fact of whether our system is in equilibrium can be tested without checking thermodynamic relations, as we shall argue below.

Testing (1) requires measurement of the temperature T, the energy E, the number of particles n and the entropy S of the system in question. It is

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clear that by measuring the energies of the particles created in the collision we can measure the energy of the system. It is also generally accepted that by measuring the slope in the transverse momentum distribution we can measure the temperature[1]. The real difficulty is the measurement of entropy. In the present note we propose to adapt to this end the coincidence method proposed some time ago by Ma [2]. One of the very attractive features of this method is the possibility of testing whether the system is in equilibrium through a procedure of rescaling of the measured entropy. We also present Monte Carlo estimates of the feasibility of the method, based on a simple model. We conclude that the method has a large potential, as it requires much smaller number of events ($\sim \sqrt{\text{no. of states}}$), than the conventional approach. As a consequence its errors are significantly reduced compared to the simple-minded estimates. It is certainly worth to try it in the present and future high energy experiments.

2. Measurement of entropy by the coincidence method

Below we give a summary of the idea presented in [2].

Ma proposes to count the pairs of configurations of the investigated system. Call N_c the number of pairs of "identical" configurations. Call N_t the total number of pairs of configurations. If all configurations considered are "equivalent" (*i.e.* if they correspond to the same conditions), then entropy is given by the formula (we give the subscript M to the entropy measured through the procedure of Ma)

$$S = S_{\mathrm{M}} = \log\left(\frac{N_t}{N_c}\right). \tag{2}$$

The reason is that $\frac{N_t}{N_c}$ is the volume in the phase-space occupied by the system. This can be seen as follows.

Suppose that the phase-space is divided into cells. Suppose furthermore that our system occupies Γ cells (with a uniform probability). Each cell represents a different state of the system (each cell has as many dimensions as is the number of variables describing the system). Our problem is to calculate Γ : $S = \log(\Gamma)$. Let us select randomly $\mathcal{N}$ configurations of the system (in general $\mathcal{N} \ll \Gamma$, that's the main point). These configurations occupy some cells. The average occupation number of a cell is $\mathcal{N}/\Gamma \ll 1$. Under this condition, the average number of pairs in the same cell is

$$\left(\frac{\mathcal{N}}{\Gamma}\right)^2 \approx \frac{N_t}{\Gamma^2}, \tag{3}$$

[1] This requires correction to the effects of the hydrodynamic flow which seem to be under a reasonable control.

where $N_t \approx \mathcal{N}^2$ is the total number of pairs selected. The total number of coincidences is the sum of (3) over all cells

$$N_c = \Gamma \left(\frac{\mathcal{N}}{\Gamma} \right)^2 , \tag{4}$$

hence

$$\Gamma = \frac{N_t}{N_c} , \tag{5}$$

and thus (2)[2].

If the configurations are not equivalent, one has to divide them into classes: within each class they are now equivalent. If the probability distribution of classes is $P(\lambda)$, then

$$S_{\mathrm{M}} = \sum_{\lambda} P(\lambda) \log \left(\frac{N_t(\lambda)}{N_c(\lambda) P(\lambda)} \right) . \tag{6}$$

The derivation is given in [2] but it can be easily understood as a sum of the "average over classes" $= \sum_\lambda P(\lambda) \log \left(\frac{N_t(\lambda)}{N_c(\lambda)} \right)$ and of the "entropy of the distribution of classes" $= -\sum_\lambda P(\lambda) \log[P(\lambda)]$.

The classes cannot be too small, so that number of configurations in each class is sufficient to obtain a reasonable statistics.

This is what we retain from Ma. In the next section we present a suggestion how to apply this method to measure the entropy of a system of particles produced in high energy interactions, and how, through a rescaling procedure, to test that they come from a system in equilibrium.

3. Application of coincidence method to multiparticle production

A natural possibility to apply the coincidence method to multiparticle production is to identify the configurations of the statistical system produced in a collision, with the events observed in experiment. Once this is accepted, one can proceed as follows.

(a) Select $\mathcal{N}$ events and split them into classes according to the total transverse energy E and multiplicity n recorded. The number of events in each class is denoted by $\mathcal{N}(E, n)$.

(b) Define a "lattice" in momentum space of individual particles [3].

[2] The formula (3) is only approximate. The exact formula is $\frac{\mathcal{N}}{\Gamma} \frac{\mathcal{N}-1}{\Gamma}$ which leads again to (5).

[3] Perhaps a better method is to transform the momenta into variables which give uniform distributions (see *e.g.* [3]).

Within each class:

(i) Call the two configurations "identical" if they have the same occupation numbers within the accuracy of the grid. The number of such pairs is denoted by $N_c(E,n)$.

(ii) Calculate the ratio (6), *i.e.*,

$$S_{\rm M}(E,n) = \log\left(\frac{N_t(E,n)}{N_c(E,n)}\right) = \log\left(\frac{\mathcal{N}(E,n)(\mathcal{N}(E,n)-1)}{N_c(E,n)}\right), \quad (7)$$

where $\mathcal{N}(E,n)$ is the number of events in a given class.

Actually, the condition that the events in one "equivalence class" must have strictly the same multiplicity (otherwise they could never be really identical) could be relaxed, *e.g.* by accepting that in the definition of the "identity" of the two configurations, the occupation numbers may differ by a fixed small amount.

One sees from these arguments that in this way one can measure the entropy only up to an additive constant. Therefore the interesting thing is not to measure the absolute value of entropy but rather its dependence on energy and/or multiplicity.

As we already noted in the Introduction, the measurement described by (7) allows one to perform a simple test of thermalization. When thermodynamics is valid, then the Eqs.(1,20) should be satisfied. Clearly the additive constant is irrelevant. One needs, however, a rather precise measurements because otherwise the numerical estimates of the derivatives $\frac{\partial S_{\rm M}(E,n)}{\partial E}$ and $\frac{\partial S_{\rm M}(E,n)}{\partial n}$ are not reliable.

The very fact of thermalization can also be tested without recourse to the validity of (1) or (20). Indeed, the method of Ma derives the expression (2) for entropy under the assumption of complete randomness of the system. When a system is in a state of equilibrium it has random distributions. The entropy, $S_{\rm M}$, on the other hand, has the scaling property

$$S_{\rm M} = \log(\Gamma) = \log\left(\frac{\Omega}{\sigma}\right) = \log\Omega + \log\left(\frac{1}{\sigma}\right),$$

where Γ is a dimensionless quantity, Ω is a volume in the phase space and σ the volume of the elementary cell. Therefore, we can test the randomness (hence the equilibrium) of the anlyzed system as follows:

— We plot the measured entropy $S_{\mathrm{M}}(\sigma)$ *versus* $\log\left(\frac{1}{\sigma}\right)$.

— When our system is in equilibrium we find this dependence to be linear[4] and its slope equal 1.

Note that testing only the equilibrium is reduced to testing randomness, thus it is not important whether we get an accurate value of the entropy. Measuring entropy is a more demanding task than testing an equilibrium.

In the next section we show the results of a simple Monte Carlo (MC) estimate of the accuracies one can achieve.

4. The classical gas of identical particles

Of course the main problem is for how big systems the coincidence method works in practice. The number of states grows exponentially with the number of particles and the number of subdivisions. Therefore obviously there is a limit to what one can achieve with the finite sample of events. In this section we analyse this question using MC simulations of a classical gas of noninteracting particles. In particular, it is shown that the onset of the thermodynamic behaviour occurs soon enough for the coincidence method to be practically feasible.

We will also determine the *minimal* size of the system for the continuum behaviour to set in as the problem is not really interesting for too small number of subdivisions of the system.

We consider the classical gas of noninteracing, nonrelativistic particles in d dimensions. Since, as mentioned earlier, the method may be applied to the transverse degrees of freedom only we prefer to retain the discussion in arbitrary dimensions. For the same reason we use the number of degrees of freedom N to characterize the size of a system. Of course $N = nd$ in d dimensions [5].

Since for noninteracting particles momentum and space degrees of freedom factorize, we consider for simplicity only the momentum states. The discretized expression for the number of states of N degrees of freedom with the total energy E reads

$$\Gamma(M,N) = \sum_{n_1,\ldots n_N, n_1^2+\ldots n_N^2=M} 1\,, \tag{8}$$

where the momentum $p_i = an_i$ with some discretization scale a. Accordingly $2mE = a^2M$, where the integer M labels the energy of the system and m denotes the mass of a particle.

[4] When σ approaches $1/V$, where V is the volume of the system, the effects of quantum interference between identical particles [4] may disturb this simple behavior.

[5] Provided only the momentum degrees of freedom are considered.

The generating function

$$Z_N(\tau) = \sum_{M=0}^{\infty} \tau^M \Gamma(M,N) = \left(\sum_{n=-\infty}^{\infty} \tau^{n^2} \right)^N = \exp\left(N \log c(\tau)\right), \qquad (9)$$

factorizes and is expressed by a single sum $c(\tau)$. The coefficients Γ can now be simply obtained by calculating recursively expansion of the $f(\tau) = \log c(\tau)$ from that of $c(\tau)$, and subsequently expansion of $Z_N(\tau)$ from that of $f(\tau)$. This procedure provided us with the exact numbers for the density of states $\Gamma(M,N)$, which were used to benchmark the performance of our Monte Carlo.

For large M and N the density of states reaches its continuum limit

$$\Gamma(M,N) \cong \frac{\pi^{N/2} M^{(N/2-1)}}{(N/2-1)!}, \quad M,N \text{ large}. \qquad (10)$$

We will see later that this relation is rather well satisfied even for moderate values of M and N.

On the other hand, *the thermodynamic* limit, $M,N \to \infty, M/N$-fixed, is reached rather slowly. In this limit the entropy density [6] scales depending only on the energy density $\varepsilon = M/N$.

$$\frac{1}{N} \log \Gamma(M,N) \cong \frac{1}{2}[\log(\varepsilon) + \log(2\pi) + 1], \quad M,N \to \infty, \varepsilon = M/N = \text{const}. \qquad (11)$$

The purpose of this exercise is to see whether the coincidence method can detect this behaviour.

4.1. Monte Carlo simulations

In principle one should generate a sample of the $\mathcal{N}$ *configurations* $\{n_1, n_2, \ldots, n_N\}_k$, $k = 1, ..., \mathcal{N}$ of, integer-valued, one-dimensional momenta $n_1, \ldots, n_N$, which satisfy the energy conservation. For our purpose, however, the details of particle kinematics, although practically cumbersome, are not relevant. In order to measure the coincidences, it is enough to label uniquely all multiparticle states and compare the labels. In this way the problem simplifies considerably, yet the essential question of the onset of the thermodynamic behaviour can be addressed.

Consequently each Monte Carlo run consisted of a generation of a sample of $\mathcal{N}$ configurations, represented by integer indices, $(I_1, I_2, \ldots, I_{\mathcal{N}})$, $1 \le I_k \le \Gamma(M,N)$, $k = 1, \ldots, \mathcal{N}$, uniformly distributed in the whole space of available

[6] Here and in the following, we will refer to the entropy per one degree of freedom as the entropy density.

states. Then we counted the number of coincidences $\hat{N}_c$, *i.e.*, the number of pairs (I_j, I_k) such that $I_j = I_k$. The estimate for the number of all states is then

$$\hat{\Gamma} = \mathcal{N}(\mathcal{N}-1)/\hat{N}_c. \tag{12}$$

Moreover assuming the multinomial distribution of $\mathcal{N}$ integers among Γ bins we have calculated also the higher moments of the distribution of the number of coincidencies N_c. In particular, the dispersion of N_c reads [7]

$$\sigma^2[N_c] = 2 < N_c >= \frac{2\mathcal{N}^2}{\Gamma}, \tag{13}$$

which gives for the relative error of the determination of Γ after $\mathcal{N}$ trials

$$\frac{\sqrt{\sigma^2[\Gamma]}}{\Gamma} = \frac{\sqrt{2\Gamma}}{\mathcal{N}}. \tag{14}$$

Therefore the estimate of the error, based on the MC data only, is

$$\hat{\sigma}[\Gamma]/\hat{\Gamma} = \sqrt{2/\hat{N}_c}. \tag{15}$$

Eqs. (12), (14) show directly another advantage of the coincidence method. Namely, it works for much smaller number of trials ($\sim \sqrt{\Gamma}$) than the standard approach which measures average occupation of a single state.

A sample of runs is summarized in Table I. Exact results for $\Gamma(M,N)$ are also quoted. The last column gives the relative deviation of the current estimate (col. 5) from the exact value. It should be compared with the estimate of the error based only on the Monte Carlo data, Eq. (15), given in column 6. The estimated error is steadily decreasing like $1/\mathcal{N}$ and actual deviation follows the suit albeit with some fluctuations. In all runs we have made (about 20 times more than shown in the Table) approximalety 30% of actual deviations were bigger that the MC estimate, as they should. Of course the formula (14) is essential for planning future Monte Carlo simulations. It is interesting to note that the errors decrease as a number of trials and not as $1/\sqrt{\mathcal{N}}$. This is because the true random variable in this problem is the number of pairs, *i.e.* $\mathcal{N}^2$. In particular the computing effort (counting pairs) grows like $\mathcal{N}^2$, and consequently the square root of the computational effort determines decrease of errors as it should. Altogether the Monte Carlo results are well under contol and show that the method is quite reliable. It is however practical only if the total number of states is less than several hundred milions. The last run shown in Table I lasted few hours on a 200 MHz PC. This translates into $N, M \leq\sim 25$. We will discuss now if this is sufficient to see the onset of thermodynamic properties.

[7] After some approximations valid for $1 \ll \mathcal{N} \ll \Gamma$

TABLE I

Monte Carlo results for $\Gamma(\hat{M}, N)$ (col.5) for different N and M. The third and fourth column give the number of generated configurations $\mathcal{N}$, and the number of observed coincidences $\hat{N}_c$. In the last two columns we quote the Monte Carlo estimate of the relative error, *cf.* Eq. (15), and the actual relative deviation $\delta/\Gamma = |\hat{\Gamma} - \Gamma|/\Gamma$ from the exact value Γ also quoted in the Table.

N	M	$\mathcal{N}$	$\hat{N}_c$	$\hat{\Gamma}$	$\hat{\sigma}/\hat{\Gamma}$	δ/Γ
12	6	4 000	218	73 376.	0.096	0.140
		8 000	1000	63 299.	0.045	0.007
		16 000	3866	66 214.	0.023	0.028
		32 000	15884	64 465.	0.011	0.001
	Γ			64 416		
	12	8 000	30	2 133 067.	0.260	0.100
		16 000	124	2 064 387.	0.127	0.065
		32 000	516	1 984 434.	0.062	0.024
		64 000	2 110	1 941 201.	0.031	0.001
		128 000	8 358	1 960 262.	0.015	0.011
	Γ			1 938 336		
	24	20 000	4	99 995 000.	0.707	0.615
		40 000	16	99 997 504.	0.354	0.615
		80 000	106	60 376 604.	0.137	0.025
		160 000	422	60 663 128.	0.069	0.020
		320 000	1596	64 160 200.	0.035	0.036
	Γ			61 903 776		
24	6	8 000	8	7 999 000.	0.500	0.077
		16 000	30	8 532 800.	0.258	0.015
		32 000	132	7 757 334.	0.123	0.105
		64 000	442	9 266 824.	0.067	0.070
		128 000	1842	8 894 610.	0.033	0.027
	Γ			8 667 720		
	8	40 000	16	99 997 504.	0.354	0.487
		80 000	32	199 997 504.	0.250	0.025
		160 000	122	209 834 752.	0.128	0.076
		320 000	510	200 783 680.	0.063	0.029
	Γ			195 082 320		

4.2. Results

Figure 1 shows the entropy density as a function of a scaling variable $\varepsilon = M/N$. Statistical errors of MC results (and the deviation from the exact discrete values given by $\Gamma(M, N)$) are much smaller that the size of symbols. The data follow nicely the curves obtained from the classical formula in the continuum, Eq. (10). Considered as a function of ε and N they obviously show a substantial N-dependence. The N varies from 8 (lowest

curve) to 24 in this plot. On the other hand, the deviation from the ultimate scaling limit, (Eq. (11), the uppermost curve), is around 30% in the worst case (N=8,M=30). With N starting from 12, deviations from the infinite system are smaller than 20%. Note that N denotes the number of degrees of freedom, which in d space dimensions corresponds to N/d particles.

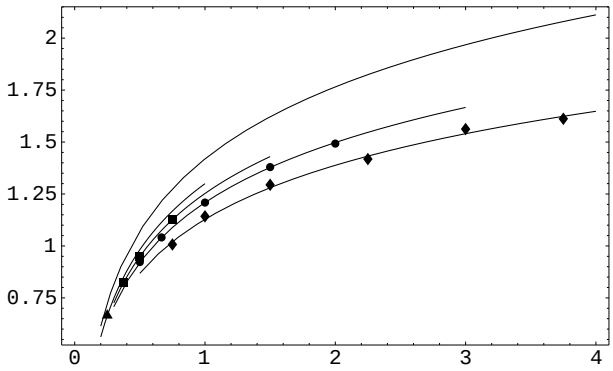

Fig. 1. Entropy density $s = \frac{1}{N}\log\Gamma(M,N)$ vs. the energy density $\varepsilon = M/N$. Black symbols represent our Monte Carlo results for N=8 (diamonds), 12 (circles), 16 (boxes) and 24 (a triangle). Lower solid lines correspond to the continuum approximation, Eq. (10), for each N. The uppermost solid line represents the scaling, thermodynamical limit, Eq. (11).

As a second test we have checked a differential form of Eq. (11)

$$\frac{\partial \log \Gamma}{\partial E} = \frac{N}{2E}, \tag{16}$$

which, together with the equipartition of energy, is the basis of the equilibrium thermodynamics 1. Changing the variable $2mE = a^2M$ gives

$$\frac{\partial \log \Gamma}{\partial E} = \frac{\partial \log \Gamma}{\partial M}\frac{dM}{dE} = \frac{\partial \log \Gamma}{\partial M}\frac{M}{E} = \frac{N}{2E}, \tag{17}$$

or

$$\frac{\partial \log \Gamma}{\partial M} = \frac{N}{2M}, \tag{18}$$

Finally after discretization of the derivative we obtain

$$\log\left(\frac{\Gamma(M+1,N)}{\Gamma(M,N)}\right) = \frac{N}{2M+1}. \tag{19}$$

This equation is tested in Fig. 2, where a half of the inverse of the left hand side, as obtained from simulations, is plotted as a function of ε. Solid line

represents the right hand side [8]. Similarly to the previous case agreement is very good for $N \geq 12$. It was necessary to reduce MC errors to the level of 1%-3% in order to achieve this agreement. Of couse this test is much more sensitive than the previous one since it requires precise measurement of the derivatives.

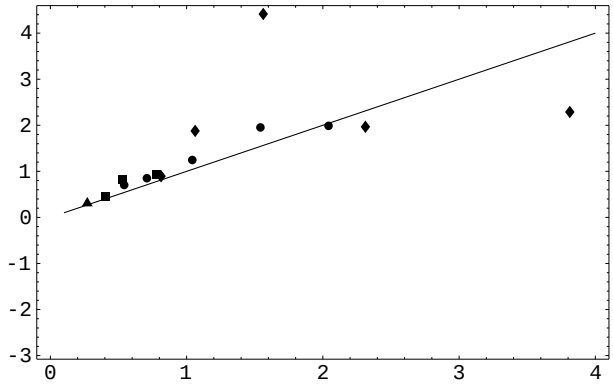

Fig. 2. Testing the relation (19). Half of the inverse of the finite difference (with respect to M) of the entropy $\log \Gamma(M,N)$, as a function of $\varepsilon = M/N$ for $N = 8$ (diamonds), 12 (circles), 16(boxes) and 24 (a triangle). Solid line corresponds to the thermodynamical lmit.

To conclude, the coincidence method is satisfactory in practice for the number of degrees of freedom below ~ 25. This turns out to be sufficient to see the signatures of the thermal equilibrium. For more than 12 degrees of fredom the scaling of the entropy density is confirmed with the accuracy better than 20% . The saddle point relation $\partial S/\partial E = 1/T$ is also very well reproduced.

5. Concluding remarks

Let us end with the following comments.

(a) The relation (1) is but a simplest example of many thermodynamical identities which can be tested once the mesurement of entropy is available. For example, if the particle ratios are measured one may test the relation

$$\left.\frac{\partial S(E,n)}{\partial n}\right|_E = -\frac{\mu}{T}, \tag{20}$$

where μ is the chemical potential. With some additional assumptions about the system in question one may calculate other thermodynamical quantities (*e.g.* free energy) and a host of relations can be tested in principle.

[8] Of course $\varepsilon = (M + 1/2)/N$ in this case.

(b) Employement of the entropy S_{M} brings a bonus: plotting the measured entropy $S_{\mathrm{M}}(\sigma)$ *vs* $\log\left(\frac{1}{\sigma}\right)$, where σ is the volume of the elementary cell of the phase space, and finding this dependence to be linear with the slope 1, we will know, without testing validity of the relations (1) or (20), that our system passes the test of an equilibrium. As we have already pointed out: a reliable measurement of entropy is a more difficult task than just testing an equilibrium.

(c) Our argument presented in Section 3 assumes that the only essential parameters defining the equivalent configurations (in the sense of Ma) are energy and multiplicity. This is surely a rather strong assumption. Nevertheless we feel it is a justified first step in the problem we propose. In fact, a selection of proper variables may be a crucial point in this analysis.

(d) The practical evaluation of the number of coincidences apparently requires $(no.\ of\ events)^2$ operations. However, as shown in [5], actually the necessary computer effort scales almost linearly with the number of events due to the effective algorithm of pair counting.

In conclusion, we have proposed a new method of measurement of the entropy of multiparticle systems created in heavy ion collisions at high energies. This opens the possibility to test thermodynamical properties of such systems. It was also shown that the method seems to be of practical use in the coming experiments.

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