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SPLIT STRUCTURES IN GENERAL RELATIVITY

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General approach to decomposition of the tangent bundle of pseudo-Riemannian manifolds, and the associated decomposition of geometric objects are constructed on the basis of the invariantly defined split structure. We define the main geometric objects characterizing decomposition. Invariant non-holonomic generalizations of the Gauss–Codazzi–Ricci's relations have been obtained. All the known types of decompositions (used in theory of frames of reference for the general relativity, in the Hamiltonian formulation for gravity, in the Cauchy problem, in the theory of stationary spaces, and so on) follow from the present work as special cases when fixing the basis and dimensions of subbundles, and parametrization of a basis of decomposition. Method of decomposition have been applied here for the relativistic configurations of a perfect fluid. Discussing an invariant form of the equations of motion we have found the invariant equilibrium conditions and their (3+1) decomposed form. The invariant formulation of the conservation law for the curl have been obtained.

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1. Introduction

Most approaches and formalisms in General Relativity are connected with decomposition of spaces into direct sums of subspaces and the associated decomposition of geometrical objects. It means that in addition to usual structures one should introduce **a split structure** which induces the decomposition of manifolds. This extra structure determines decomposition of all objects and structures defined on a manifold. Among varieties of formalism of decomposition are the methods aimed to describe frames of reference and observable quantities in the theory of gravity. Similar methods have gained the wide acceptance in a great number of problems. Some of these problems are the canonical formalism and the Cauchy problem in

General Relativity, the gravitational waves and quantization of the gravitational field, the theory of stationary and axisymmetric gravitational fields and so on.

The formalism of the decomposition of spaces in coordinate form is presented in [1–4]. The invariant split method was considered in [5] but without any connection with the previous works on decomposition. Objects introduced formally in this work have no clear geometrical meaning.

The invariant method of an $n+m$ decomposition for pseudo-Riemannian manifolds was proposed in [6–9]. There were most approaches to decomposition unified in these works, and the objects introduced there have clear physical and geometrical meaning. For special cases of $(1+4), (1+3), (2+2)$, $(n+4)$ decomposition in the coordinate representation these objects reduce to known physical characteristics of a system [1–4].

The general theory of decomposition of the tangent bundle $T(M)$ of pseudo-Riemannian manifold M into direct sum the non-holonomic subbundles Σ' and Σ'' and the associated decomposition of geometric objects has been constructed in the present work. The $(n+m)$ and $(n+1)$ forms of invariant decomposition have been obtained. We define the main geometric objects characterizing decomposition. Choosing the projection operators and gauges of a basis of decomposition we construct various special cases. The invariant non-holonomic generalization of the Gauss–Codazzi–Ricci's relations has been found as various projections of the curvature tensor.

Method of decomposition have been applied here for the relativistic configurations of a perfect fluid. Discussing an invariant form of the equations of motion we have found the invariant equilibrium conditions and their $(3+1)$ decomposed form. The invariant formulation of the conservation law for the curl have been obtained.

Note that we do not refer to problems of global geometry, but use its invariant formulations to construct decompositions of spaces. We use, mostly, notations and definitions of the works [10–12].

2. The basic notations and definitions

Let M be a pseudo-Riemannian manifold, g a metric on M, $T(M)$ and $T^*(M)$ are the tangent and cotangent bundles over M. The objects $X, Y, Z, u \ldots \in T(M)$ and $\alpha, \beta, \omega, df \in T^*(M)$ denote contravariant and covariant vector fields (d is an exterior differential). We shall denote by $\omega(X)$ an inner product of a one-form ω and vector X. The scalar product of two vectors X, Y and two forms α, β is determined by the metric g

$$X \cdot Y \equiv (X, Y) \equiv g(X, Y)\,; \qquad <\alpha, \beta> \equiv g^{-1}(\alpha, \beta)\,, \tag{2.1}$$

where g^{-1} is the inverse of the metric g.

We need to note that for each vector field $u \in T(M)$ a dual one-form ω is determined uniquely by $\omega(X) = g(X, u)$, $\forall X \in T(M)$. From now on we just will write $\omega = g(\ .\ , u)$. Then the inverse of the metric g is given by

$$g^{-1}(\omega, \alpha) = g^{-1}(g(\ .\ , u), \alpha) = \alpha(u)\,, \qquad \forall u \in T(M), \quad \forall \alpha \in T^*(M)$$

so that $u = g^{-1}(\ .\ , w)$.

A linear operator L on $T(M)$ is a tensor of type $(1,1)$ which acts according to the relations $L \cdot X \equiv L(X) \in T(M), \forall X \in T(M)$. Then $(L^T \cdot \omega)(X) = (\omega \cdot L)(X) \equiv \omega(L(X)), \forall X \in T(M)$ where L^T is a transpose of the operator L. The product of two linear operators is defined by $(L \cdot H) \cdot X = L \cdot (H \cdot X) \in T(M), \forall X \in T(M)$. An operator H is called a symmetric one if $(H \cdot X, Y) = (X, H \cdot Y), \forall X, Y \in T(M)$.

We shall say that **a split structure** $\mathcal{H}^r$ is introduced on M if r linear symmetric operators H^a $(a = 1, 2, \ldots r)$ of a constant rank with the properties

$$H^a \cdot H^b = \delta^{ab} H^b; \qquad \sum_{a=1}^{r} H^a = I\,, \tag{2.2}$$

where I is the unit operator ($I \cdot X = I$, $\forall X \in T(M)$) are defined on $T(M)$.

Then bundles $T(M)$ and $T^*(M)$ are decomposed into the $(n_1 + n_2 + \ldots + n_r)$ subbundles Σ^a, Σ^*_a, so that

$$T(M) = \bigoplus_{a=1}^{r} \Sigma^a; \qquad T^*(M) = \bigoplus_{a=1}^{r} \Sigma^*_a\,,$$

where the sign $\oplus$ denotes the direct sum. The arbitrary vectors, covectors, and metrics are decomposed according to the scheme:

$$X = \sum_{a=1}^{r} X^a\,, \quad \alpha = \sum_{a=1}^{r} \alpha_a, \quad g = \sum_{a=1}^{r} g^a\,, \quad g^{-1} = \sum_{a=1}^{r} g_a^{-1}\,, \tag{2.3}$$

where

$$\begin{aligned}
&X^a = H^a \cdot X \in \Sigma^a\,; && X^a \cdot X^b = 0\,; \quad (a \neq b)\\
&\alpha_a = \alpha \cdot H^a \in \Sigma^*_a\,; && \alpha_a(X^b) = 0\,; \quad (a \neq b)\\
&g^a(X^a, Y^a) \equiv g(X^a, Y^a)\,; && g_a^{-1}(\alpha_a, \beta_a) \equiv g^{-1}(\alpha_a, \beta_a)\,.
\end{aligned} \tag{2.4}$$

In these relations $\{g^a\}$ are the metrics induced on the subbundles $\{\Sigma^a\}$ of the tangent bundle $T(M)$. Using this scheme we can obtain the decomposition of more complex tensors. We assume that all objects with indices $a, b, \ldots$ are defined on the corresponding subbundles $\Sigma^a, \Sigma^b, \ldots$.

Let ∇ be an affine (symmetric and compatible with g) connection such that $\nabla_X Y - \nabla_Y X = [X,Y], X(Y\cdot Z) = Z\cdot\nabla_X Y + Y\cdot\nabla_X Z$, where $[X,Y]f = X(Yf) - Y(Xf)$ is the Lie bracket of two vector fields X and Y, $\nabla_X Y$ is the covariant derivative of Y in the direction X. A consequence of this is that

$$\begin{aligned} 2Z\cdot\nabla_X Y &= X(Y\cdot Z) + Y(Z\cdot X) - Z(X\cdot Y) \\ &\quad + Z\cdot[X,Y] + Y\cdot[Z,X] - X\cdot[Y,Z]\,. \end{aligned} \tag{2.5}$$

The covariant derivative $\nabla_X T$ of a tensor T of type (r,s), where $s = 0,1$ with respect to X is defined by

$$\begin{aligned} (\nabla_X T)(Y_1,\dots Y_r) &= \nabla_X T(Y_1,\dots Y_r) \\ &\quad - \sum_{i=1}^{r} T(Y_1,\dots Y_{i-1},\nabla_X Y_i, Y_{i+1},\dots Y_r)\,. \end{aligned} \tag{2.6}$$

The Lie derivative $\mathcal{L}_X T$ of a tensor T with respect to a vector X and the exterior derivative of an r-form Ω are given by:

$$\begin{aligned} &(\mathcal{L}_X T)(Y_1,\dots Y_r) = \mathcal{L}_X(T(Y_1,\dots Y_r)) \\ &-\sum_{i=1}^{r} T(Y_1,\dots Y_{i-1},\mathcal{L}_X Y_i, Y_{i+1},\dots Y_r)\,, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &(d\Omega)(Y_0,Y_1,\dots Y_r) = \sum_{i=0}^{r}(-1)^i Y_i(\Omega(Y_0,\dots,\hat{Y}_i,\dots Y_r)) \\ &+\sum_{0\le i<j\le r}(-1)^{i+j}\Omega([Y_i,Y_j],Y_0,\dots,\hat{Y}_i,\dots,\hat{Y}_j,\dots,Y_r)\,, \end{aligned} \tag{2.8}$$

where $\mathcal{L}_X Y = [X,Y]$. The symbol "ˆ" means that the associated term is omitted. The curvature tensor is defined by the formula

$$R(X,Y)Z = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z\,. \tag{2.9}$$

Using a split structure $\mathcal{H}^r$, the decomposition of a ∇ is easily set up:

$$\nabla_X Y = \sum_{a,b,c=1}^{r} \nabla^a_{X^b} Y^c\,, \qquad \forall X,Y\in T(M). \tag{2.10}$$

In this sum the objects

$$\nabla^a_{X^a} Y^a \equiv H^a\cdot\nabla^a_{X^a} Y^a\,, \qquad \forall X^a, Y^a\in\Sigma^a \quad (a = 1,2,\dots r) \tag{2.11}$$

define connections $\{\nabla^a\}$ induced on the subbundles $\{\Sigma^a\}$. The objects

$$\nabla^a_{X^b} Y^b \equiv H^a \cdot \nabla_{X^b} Y^b \equiv -B^a(X^b, Y^b)\,, \qquad \forall X^b, Y^b \in \Sigma^b \tag{2.12}$$

are tensors of extrinsic non-holonomicity of subbundles Σ^b $(a \neq b)$. One can think that the objects

$$\nabla^a_{X^b} Y^c \equiv H^a \cdot \nabla_{X^b} Y^c \equiv -Q^a_{bc}(X^b, Y^c), \qquad \forall a \neq b \neq c \tag{2.13}$$

define the generalization of the Ricci coefficients of rotation γ^a_{bc} [13]. In general case they give the objects of rotation Q^a_{bc} of the subbundles Σ^b, Σ^c in the n_a-dimensional direction Σ^a. The other components can be expressed in terms of the introduced objects. Thus, the components $\nabla^a_{X^a} Y^b \equiv H^a \cdot \nabla_{X^a} Y^b$ and $\nabla^a_{X^b} Y^b \equiv H^a \cdot \nabla_{X^b} Y^b$ satisfy the relations

$$\begin{aligned}
&Z^a \cdot \nabla^a_{X^a} Y^b = Y^b \cdot B^b(X^a, Z^a) \quad (a \neq b)\,;\\
&Z^a \cdot \nabla^a_{Y^a} Z^b = Z^b \cdot \Lambda^a(X^b, Y^a) + X^b \cdot B^b(Y^a, Z^a)\,;\\
&\Lambda^a(X^b, Y^c) \equiv H^a \cdot [X^b, Y^c] \qquad (a \neq b \neq c)\,.
\end{aligned} \tag{2.14}$$

The tensor of extrinsic non-holonomicity B^a can be expressed as the sum of symmetric and antisymmetric components

$$B^a(X^b, Y^b) = S^a(X^b, Y^b) + A^a(X^b, Y^b)\,, \tag{2.15}$$

where $S^a(X^b, Y^b)$ and $A^a(X^b, Y^b)$ define the tensors of extrinsic curvature and extrinsic torsion of subbundle Σ^b in the direction of the subbundle Σ^a. For these objects we have

$$2Z^a \cdot S^a(X^b, Y^b) \;=\; (\mathcal{L}_{Z^a} g^b)(X^b, Y^b)\,; \tag{2.16}$$

$$2A^a(X^b, Y^b) \;=\; -H^a \cdot [X^b, Y^b]\,. \tag{2.17}$$

It can be shown that the connection ∇^a induced on the subbundle Σ^a will be symmetric and compatible with the metric g^a. The projecting of the curvature tensor into the subbundles $\Sigma^a, \Sigma^b, \ldots$ gives us nonholonomic generalizations of the Gauss–Codazzi–Ricci equations.

3. An invariant $(n+m)$ split structure on a pseudo-Riemannian manifold

If $r = 2$, then there are only two subbundles Σ' and Σ'' of the tangent bundle $T(M)$ and the previous formulae become much simpler. Owing to importance of this case it was deemed worthwhile to consider the split structure with more details and independently from Sec. 2 [8, 9].

Let H' be a linear idempotent symmetric operator of a constant rank with the property

$$H' \cdot H' = H' \,. \tag{3.1}$$

We shall say that H' defines **a $(n+m)$ split structure** on M if

$$\dim \mathrm{Im} H' = n \,; \qquad \dim \mathrm{Ker} H' = m \,; \qquad \dim M = n + m \,, \tag{3.2}$$

where $\mathrm{Ker} H'$ is the kernel of the operator H'. Since H' is defined, thereby we define the operator H'' such that

$$H'' \cdot H'' = H'' \,; \qquad H'' \cdot H' = H' \cdot H'' = 0 \,; \qquad H' + H'' = I \,. \tag{3.3}$$

Therefore H' and H'' are the projection operators which determine the split structure $\mathcal{H}^2$ on M due to the definition (2.2). We have $T(M) = \Sigma' \oplus \Sigma''$:

$$\begin{aligned} &X = X' + X'' \,; \qquad \alpha = \alpha' + \alpha'' \,; \qquad g = g' + g'' \,; \\ &X' = H' \cdot X \,; \qquad X'' = H'' \cdot X \,; \qquad X' \cdot X'' = 0 \,; \\ &g'(X', Y') = g(X', Y') \,; \qquad g''(X'', Y'') = g(X'', Y'') \,. \end{aligned} \tag{3.4}$$

A connection ∇ is decomposed into the following components: a connection on Σ', and the tensor of extrinsic non-holonomicity of the subbundle Σ', respectively

$$\nabla'_{X'} Y' = H' \cdot \nabla_{X'} Y' \,, \tag{3.5}$$

$$B''(X', Y') = -\nabla''_{X'} Y' = -H'' \cdot \nabla_{X'} Y' \,. \tag{3.6}$$

Other components of ∇ can be expressed in terms of the components (3.5), (3.6) and the Lie derivatives of two vector fields

$$X' \cdot \nabla'_{Y'} Z'' = Z'' \cdot B''(Y', X') \,, \tag{3.7}$$

$$X' \cdot \nabla'_{Y''} Z' = X' \cdot \mathcal{L}_{Y''} Z' + Y'' \cdot B''(Z', X') \,. \tag{3.8}$$

The rest of the components of ∇ $\{\nabla''_{X''} Y'', \ \nabla'_{X''} Y'', \ \nabla''_{X''} Y', \ \nabla''_{X'} Y''\}$ may be written out by substituting $X', Y', B', H', \ldots$ for $X'', Y'', B'', H'', \ldots$ and vice versa in formulae (3.5)–(3.8). This completes the set of all the eight possible projections of the connection.

The tensor B'' may be expressed as the sum of its symmetric and anti-symmetric parts:

$$B''(X', Y') = S''(X', Y') + A''(X', Y') \,; \tag{3.9}$$

$$2Z'' \cdot S''(X', Y') = (\mathcal{L}_{Z''} g')(X', Y') \,, \tag{3.10}$$

$$2A''(X', Y') = -H'' \cdot [X', Y'] \,, \tag{3.11}$$

where S'' and A'' are the tensors of extrinsic curvature and torsion respectively. If $A'' = 0$, the subbundle Σ' will be holonomic (one of the variants of Frobenius theorem).

Using the definition of the curvature tensor (2.9) one can find every possible projection of the curvature tensor

$$
\begin{aligned}
&R(X',Y')Z' \cdot V' = R'(X',Y')Z' \cdot V' + B''(X',Z') \cdot B''(Y',V') \\
&-B''(Y',Z') \cdot B''(X',V') + 2A''(X',Y') \cdot B''(Z',V')\,; && (3.12)\\
&R(X',Y')Z' \cdot V'' = V'' \cdot \{(\nabla''_{Y'}B'')(X',Z') - (\nabla''_{X'}B'')(Y',Z')\} \\
&+2Z' \cdot B'(A''(X',Y'),V'')\,; && (3.13)\\
&R(X',Y'')Z' \cdot V'' = (Z' \cdot (\nabla'_{X'}B') + < X' \cdot B', Z' \cdot B' >)(Y'',V'') \\
&+(V'' \cdot (\nabla''_{Y''}B'') + < Y'' \cdot B'', V'' \cdot B'' >)(X',Z')\,; && (3.14)
\end{aligned}
$$

R' is the curvature tensor of the subbundle Σ'

$$R'(X',Y')Z' \equiv \{\nabla'_{X'}\nabla'_{Y'} - \nabla'_{Y'}\nabla'_{X'} - \nabla'_{[X',Y']'} + 2\mathcal{L}'_{A''(X',Y')}\}Z'\,, \quad (3.15)$$

where $\mathcal{L}'$ is the Lie derivative projected into the subbundle $\Sigma'(\mathcal{L}'_X Y \equiv H' \cdot \mathcal{L}_X Y)$. This definition of the curvature tensor, introduced in the works [7–9], is the invariant generalization of that introduced in coordinate form in [1]. Note that the latter term in (3.15) is necessary in order that the differential curvature operator $R'(X',Y')$ on Σ' be a linear multiplicative one, or, in other words, R' be a tensor of type (1,3) on non-holonomic subbundle Σ'.

The following expression in (3.14), with the fixed vectors X', Z', Y'', V'',

$$(\langle Y'' \cdot B'', V'' \cdot B'' \rangle)(X',Z') \equiv \langle Y'' \cdot B''(X',\ .\ ), V'' \cdot B''(\ .\ ,Z') \rangle$$

defines the scalar product of the two one-forms $\alpha \equiv Y'' \cdot B''(X',\ .\ )$ and $\beta \equiv V'' \cdot B''(\ .\ ,Z')$ according to (2.1) by the metric $(g')^{-1}$. The covariant derivatives of the tensor B' are given by

$$(\nabla'_{X'}B')(Y'',Z'') = \nabla'_{X'}(B'(Y'',Z'')) - B'(\nabla''_{X'}Y'',Z'') - B'(Y'',\nabla''_{X'}Z'')\,,$$

$$(\nabla'_{X''}B')(Y'',Z'') = \nabla'_{X''}(B'(Y'',Z'')) - B'(\nabla''_{X''}Y'',Z'') - B'(Y'',\nabla''_{X''}Z'')\,.$$

The relations (3.12)–(3.14) are nonholonomic analogies of the well–known Gauss–Codazzi–Ricci equations. Other nontrivial projections of the curvature tensor may be written out using the substitution “ $'$ ” for “ $''$ ” and vice versa. In the special case of coordinate representation of $(3+1)$ and $(2+2)$ decomposition, the objects introduced above give us the known tensors [1–4], which have clear physical and geometrical meaning.

Let us note that the objects, presented in the work [5] may be expressed in terms of these tensors. For example, the torsion tensor introduced there as the Nijenhuis tensor [11] proved to be equal

$$\begin{aligned} S_{H'}(X,Y) &= [X,Y']' + [X',Y]' - [X',Y'] - [X,Y]' \\ &= 2A'(X'',Y'') + 2A''(X',Y') \,. \end{aligned}$$

This tensor does not have a simple interpretation even in the classical case of hypersurfaces in M.

4. An invariant (n+1) split structure on a pseudo-Riemannian manifold

In this section we give the invariant generalization of $(n+1)$ decomposition of spaces (the monad method [1,2]) as a special case of $(n+m)$ decomposition when $m=1$.

Let u be a vector field (field of a monad) on M such that $u \cdot u = \varepsilon = \pm 1$. It gives a one-form ω and projection operators uniquely by the formulae

$$\omega(X) = \varepsilon u \cdot X \,, \qquad \forall X \in T(M) \,, \tag{4.1}$$

$$H'' = u \otimes \omega \,; \qquad H' = I - H'' \,. \tag{4.2}$$

The operators H'' and H' satisfy all the necessary relations (3.1)–(3.3), it being known that Σ'' is a one-dimensional subbundle $(m=1)$. The tensor product is denoted by "$\otimes$".

Thus, defining vector or covector fields, u or ω respectively, we, thereby, induce an $(n+1)$ split structure on M. For any vector field X and metric g, this implies

$$X = X' + \omega(X) u \,, \qquad g = g' + \varepsilon \omega \otimes \omega \,, \qquad g^{-1} = (g')^{-1} + \varepsilon u \otimes u \,, \tag{4.3}$$

where $\omega(X)u = X''$. The metrics $g'' = \varepsilon \omega \otimes \omega$ and g' are the metrics on the subbundles Σ'', and Σ' correspondingly. A connection ∇ has the following components:

$$\nabla_{X'} Y' = \nabla'_{X'} Y' - B(X',Y') u \,; \qquad \nabla_u u = \nabla'_u u = -B'(u,u) \equiv F \,, \tag{4.4}$$

where $B(X',Y') = \omega(B''(X',Y'))$. If we consider a congruence of curves for which the vector u is the tangent vector, then F is the first curvature of this congruence. The tensor B of type (0,2) is the tensor of extrinsic non-holonomicity of the subbundle Σ' and can be written as the sum of its symmetric and antisymmetric parts:

$$B(X',Y') = -\omega(\nabla''_{X'} Y') = \varepsilon S(X',Y') + A(X',Y') \,, \tag{4.5}$$

where

$$S(X',Y') = \varepsilon\omega(S''(X',Y'))\,, \quad A(X',Y') = \omega(A''(X',Y'))$$

and

$$2S(X',Y') = (\mathcal{L}_u g')(X',Y')\,, \quad 2A(X',Y') = (d\omega)(X',Y') \tag{4.6}$$

are the tensors of extrinsic curvature and extrinsic torsion of the subbundle Σ'.

The components of the curvature tensor in an $(n+1)$ decomposed form lead to the generalized Gauss–Codazzi–Ricci's equations:

$$\begin{aligned} R(X',Y')Z'\cdot V' &= R'(X',Y')Z'\cdot V' + \varepsilon[2A(X',Y')B(Z',V') \\ &\quad +B(X',Z')B(Y',V') - B(Y',Z')(X',V')]\,, \quad (4.7)\\ R(X',Y')Z'\cdot u &= -2A(X',Y')F\cdot Z' \\ &\quad +\varepsilon[(\nabla_{Y'}B)(X',Z') - (\nabla_{X'}B)(Y',Z')]\,, \quad (4.8)\\ R(X',u)Y'\cdot u &= -Y'\cdot\nabla'_{X'}F + \varepsilon(F\cdot X')(F\cdot Y') \\ &\quad +(\varepsilon\mathcal{L}_u B - < B, B^T >)(X',Y')\,, \quad (4.9) \end{aligned}$$

where the curvature tensor of the subbundle Σ' (see [6]) is given by

$$\begin{aligned} R'(X',Y')Z' &= \{\nabla'_{X'}\nabla'_{Y'} - \nabla'_{Y'}\nabla'_{X'} \\ &\quad -\nabla'_{[X',Y']'} + 2A(X',Y')\mathcal{L}'_u\}Z'\,. \end{aligned} \tag{4.10}$$

5. $(n+m)$ decomposition with respect to an adopted basis

To find all the relations considered above in an $(n+m)$ decomposition form for some fixed basis is a question of great significance for applications. One's choice of one basis or another is dictated by a physical situation, requirements of an interpretation of results, or just by the necessity to use the most comfortable way of calculation. We shall present here the invariant relations of Sec. 3 with respect to an adopted basis of decomposition. Note that all the known types of decomposition can be obtained as special cases of the present formalism by choosing the corresponding concrete bases.

We shall now consider two dual bases of decomposition: a vector one $\{E_\mu\} = \{E_a, E_i\}$ on $T(M)$, and a covector basis $\{\theta^\mu\} = \{\theta^a, \theta^i\}$ on $T^*(M)$, where $E_b \in \Sigma' \equiv \Sigma^n$, $E_i \in \Sigma'' \equiv \Sigma^m$; $\theta^a \in \Sigma^{*\prime} \equiv \Sigma^{*n}$; $\theta^i \in \Sigma^{*\prime\prime} \equiv \Sigma^{*m}$ $(a,b = 1,2,...,n;\ i,k = n+1, n+2, ..., n+m)$. According to (3.4) one has for an adopted basis

$$\theta^a(E_b) = \delta^a_b\,, \qquad \theta^a(E_j) = 0\,; \qquad \theta^i(E_b) = 0\,; \qquad \theta^i(E_k) = \delta^i_k\,; \tag{5.1}$$

$$(E_b \cdot E_k) = 0\,; \qquad \langle \theta^a, \theta^i \rangle = 0\,. \tag{5.2}$$

It should be emphasized that the indices $a, b, c, \ldots$ and $i, j, k, \ldots$ indicate the subbundles Σ^n, Σ^{*n} and Σ^m, Σ^{*m} respectively. With respect to the basis $\{E_\mu\}, \{\theta^\mu\}$ one has

$$H' = E_a \otimes \theta^a\,; \quad H'' = E_i \otimes \theta^i\,; \quad g = g' + g'' = \gamma_{ab}\theta^a \otimes \theta^b + h_{ik}\theta^i \otimes \theta^k\,, \tag{5.3}$$

where $\gamma_{ab} = E_a \cdot E_b$ and $h_{ik} = E_i \cdot E_k$ are the components of the metrics g', g'' induced on the subbundles Σ^n and Σ^m.

Then we introduced the definitions

$$\begin{aligned} \nabla'_{E_a} E_b &= L^c_{ab} E_c\,; \qquad \nabla''_{E_i} E_j = L^k_{ij} E_k\,; \\ B'(E_i, E_k) &= B^a_{ik} E_a\,; \quad B''(E_a, E_b) = B^i_{ab} E_i\,; \end{aligned} \tag{5.4}$$

$$\begin{aligned} [E_a, E_b]' &= \lambda^c_{ab} E_c\,; \qquad [E_i, E_j]'' = \lambda^k_{ij} E_k\,; \\ [E_a, E_i]' &= \lambda^b_{ai} E_b\,; \qquad [E_i, E_a]'' = \lambda^k_{ia} E_k\,, \end{aligned} \tag{5.5}$$

where L^c_{ab} and L^i_{jl} are the coefficients of connections ∇' induced on Σ^n and ∇'' induced on Σ^m. Similarly B^c_{ik} and B^i_{ab} are the coefficients of the tensors of extrinsic non-holonomicity of the subbundles Σ^m and Σ^n respectively. Using the identity (2.5) one can find

$$\begin{aligned} L^c_{ab} &= \triangle^c_{ab} + \gamma^c_{ab}\,; \qquad L^i_{jk} = \triangle^i_{jk} + \gamma^i_{jk}\,; \\ B^a_{ik} &= S^a_{ik} + A^a_{ik}\,; \qquad B^i_{ab} = S^i_{ab} + A^i_{ab}\,, \end{aligned} \tag{5.6}$$

where

$$2\triangle_{cab} = E_a\gamma_{bc} + E_b\gamma_{ac} - E_c\gamma_{ab}\,; \qquad 2\gamma_{cab} = \lambda_{cab} + \lambda_{bca} - \lambda_{abc}\,, \tag{5.7}$$

$$\begin{aligned} 2S_{aik} &= (\mathcal{L}_{E_a} g'')(E_i, E_k) = E_i h_{ik} + \lambda_{ika} + \lambda_{kia}\,; \\ 2A^a_{ik} &= (d\theta^a)(E_i, E_k)\,; \qquad 2A_{aik} = -E^a \cdot [E_i, E_k]\,. \end{aligned} \tag{5.8}$$

The coefficients A_{iab}, S_{iab}, γ_{ijk}, $\triangle_{ijk}$, unwritten here, can be obtained from (5.7), (5.8) by the replacement $(a, b, c, \ldots \leftrightarrow i, j, k, \ldots)$. Adhering to this style here and below we shall write and discuss only those relations which can not be found by the change of indices. We should remember also that the indices $(a, b, c, \ldots)$ are raised and lowered by the metrics γ^{ab} and γ_{ab}. The curvature tensor and its contractions are presented in Appendix A.

In the special case of $(n+1)$ decomposition, *i.e.* when $m=1$ one has adopted bases $\{E_\mu\}=\{E_a,\ E\}$, $\{\theta^\mu\}=\{\theta^a,\ \theta\}$, $(E=E_{n+1},\ \theta=\theta^{n+1};\ a,b=1,2,\dots n)$, so that

$$\begin{aligned}&\theta^a(E_b)=\delta_a{}^b\,; \qquad \theta_a(E)=\theta(E_a)=0\,;\\ &\theta(E)=1\,; \qquad E\cdot E_a=0\,; \qquad E\cdot E\equiv\varepsilon N^2\,,\end{aligned} \tag{5.9}$$

where $\{E_a\}\in\Sigma^n$; $\theta^a\in\Sigma^{*n}$ and $E\in\Sigma^1$; $\theta\in\Sigma^{*1}$. In this case the projectors $H'=E_a\otimes\theta^a$ and $H''=E\otimes\theta$ induce the decomposition of the metric

$$g=g'+g''=\gamma_{ab}\theta^a\otimes\theta^b+\varepsilon N^2\theta\otimes\theta\,. \tag{5.10}$$

Then using the relations (5.4)–(5.8), (A.1)–(A.8) when $i=j=k=1$ or (4.4)–(4.10) when $u=N^{-1}E, \omega=N\theta$ we can find all the necessary relations in the $(n+1)$ decomposition form in an adopted basis. Thus, from (4.4) it follows that

$$F=N^{-2}(G-(E\log N)E)\,; \qquad G=\nabla_E E. \tag{5.11}$$

The tensor of extrinsic non-holonomicity of the subbundle Σ^n can be written in the form

$$B(E_a,E_b)=\varepsilon S_{ab}+A_{ab}\equiv\varepsilon N^{-1}\mathcal{B}_{ab}\,; \qquad \mathcal{B}_{ab}=D_{ab}+\tfrac{1}{2}F_{ab}\,; \tag{5.12}$$

$$\begin{aligned}S_{ab}&=N^{-1}D_{ab}\,; \qquad 2D_{ab}=(\mathcal{L}_E g')(E_a,E_b)\,;\\ 2A_{ab}&=\varepsilon N^{-1}F_{ab}\,; \qquad F_{ab}=\varepsilon N^2 d\theta(E_a,E_b)\,.\end{aligned} \tag{5.13}$$

Acting in the same way as in the previous sections we can find the generalized Gauss–Codazzi–Ricci equations (see Appendix B).

6. Canonical parameterization of an $n+m$ split structure

The relations of Sec. 5 are invariant under the transformation of adopted bases:

$$\theta^a=L^a_b e^b; \quad \theta^l=L^l_k e^k; \quad E_a=(L^{-1})_a{}^b e_b; \quad E_i=(L^{-1})_i{}^k e_k\,, \tag{6.1}$$

where $\{L^a_b\}$ and $\{L^k_i\}$ are $(n\times n)$ and $(m\times m)$ non-singular matrices, and $\{(L^{-1})_a{}^b\}$ and $\{(L^{-1})_i{}^k\}$ are their inverse matrices. Using this property of invariance one can choose, without loss of generality, the simplest basis of decomposition which is useful for applications.

For this purpose we consider the expansion of the covector basis on Σ^{*m} in the domain U of definition of the map $x^\mu(\mu=1,2,\dots n,n+1,\dots n+m)$, *i.e.* $\theta^i = \theta^i_\mu dx^\mu (i,k = n+1, n+2, \ldots, n+m)$. Due to the fact that the rank of the $n \times (n+m)$ matrix $\{\theta^i_\mu\}$ is equal to n, there is an $(m \times m)$ non-singular matrix $\{\theta^i_k\}$ as a box in $\{\theta^i_\mu\}$. Then the covectors θ^i can be written in the form: $\theta^i = \theta^i_k dx^k + \theta^i_a dx^a = L^i_k (dx^k + N^k_a dx^a) \equiv L^i_k e^k$ where $L^i_k = \theta^i_k$, $N^k_a = (L^{-1})^k_i \theta^i_a$. Thus the covector basis θ^i goes over into the new covector basis $e^k \in \Sigma^{*m}$. The vector basis on Σ^n can be written similarly as $E_a = E_a{}^\mu \partial_\mu$. From the condition of duality $e^k(E_a) = 0$ it follows that $E_a = (L^{-1})^b_a (\partial_b - N^k_b \partial_k) \equiv (L^{-1})^b_a e_b$, where $(L^{-1})_a{}^b = E_a{}^b$. Thereby we defined the new vector basis $e_b \in \Sigma^n$. The other vector and covector bases ($e^i \in \Sigma^m$ and $e^a \in \Sigma^{*n}$ respectively) are defined by the condition of duality up to $(n \cdot m)$ functions B^a_i. As a result one obtains the following parameterization of the basis of decomposition:

$$\begin{aligned} &e^a = dx^a + B^a_i e^i \in \Sigma^{*n}; \qquad && e_a = \partial_a - N^i_a \partial_i \in \Sigma^n; \\ &e^i = dx^i + N^i_a dx^a \in \Sigma^{*m}; \qquad && e_i = \partial_i - B^a_i e_a \in \Sigma^m. \end{aligned} \tag{6.2}$$

We shall call this parameterization the canonical one.

If one follows similar procedure beginning with the covector basis $\theta^a \in \Sigma^{*n}$, one will obtain the other canonical parameterization of $(n+m)$ decomposition:

$$\begin{aligned} &e^a = dx^a + A^a_i dx^i \in \Sigma^{*n}; \qquad && e_a = \partial_a - M^k_a e_k \in \Sigma^n; \\ &e^i = dx^i + M^i_a e^a \in \Sigma^{*m}; \qquad && e_k = \partial_k - A^a_k \partial_a \in \Sigma^m. \end{aligned} \tag{6.3}$$

When some metric g is fixed on M, the functions B^a_i (or M^i_a) can be found from the condition of orthogonality (5.2) in terms of $g_{\mu\nu}$ and N^i_a (or A^b_k). If, otherwise, we fix B^a_i (or M^i_a), then we can obtain the metric for both cases according to (5.3):

$$\begin{aligned} g &= \gamma_{ab}(dx^a + B^a_i e^i) \otimes (dx^c + B^c_k e^k) + h_{ik} e^i \otimes e^k \,, \\ g &= \gamma_{ab} e^a \otimes e^b + h_{ik}(e^i + M^i_a e^a) \otimes (e^k + M^k_b e^b) \,. \end{aligned} \tag{6.4}$$

With respect to the canonically parameterized basis (6.2), the objects (5.6)–(5.8) and the Lie bracket of the basic vector fields have the form

$$\begin{aligned} \lambda^c_{ab} &= -2B^c_i A^i_{ab} \,; \qquad \lambda^k_{ij} = (B^a_i e_j - B^a_j e_i) N^k_a \,; \\ \lambda^c_{ai} &= -e_a B^c_i + 2A^k_{ab} B^b_i B^c_k + N^k_{a,i} B^c_k \,; \qquad \lambda^k_{ia} = -2A^k_{ac} B^c_i - N^k_{a,i} \,; \\ 2A^i_{ab} &= e_b N^i_a - e_a N^i_b \,; \qquad 2A^a_{ij} = e_i B^a_j - e_j B^a_i - \lambda^k_{ij} B^a_k \,; \\ 2S_{aik} &= (\mathcal{L}_{e_a} h)(e_i, e_k) \,; \qquad 2S_{iab} = (\mathcal{L}_{e_i} \gamma)(e_a, e_b) \,, \end{aligned} \tag{6.5}$$

where $\gamma = \gamma_{ab} e^a \otimes e^b$ and $h = h_{ik} e^i \otimes e^k$. Here all the geometrical characteristics are expressed in terms of the functions $h_{ij}, \gamma_{ab}, B^a_i, N^k_b$ and their derivatives. Substituting the objects (6.5) for those used in (A.1)–(A.7) we can obtain the Riemann tensor, the Ricci tensor and the scalar curvature in an $(n+m)$ decomposed form with respect to the canonically parameterized basis (6.2). All the relations for the parameterization (6.3) are found from (6.5) by the substitution $(a, b \leftrightarrow i, j;\ \ B^a_i \to M^i_a,\ \ N^i_a \to A^a_i)$.

In the case of $(n+1)$ decomposition both types of parameterizations should be considered independently. Thus for the $(3+1)$ monad method there are two kinds of canonical parameterizations (with respect to local coordinates $\{x^\mu\} = \{t, x^i\}$) determined by

$$\begin{aligned} e_0 &= \partial_t - N^i\partial_i = Nu\,; \qquad e^0 = dt + B_i e^i = N^{-1}\omega\,; \\ e_i &= \partial_i - B_i e_0\,; \qquad\qquad\ \ e^i = dx^i + N^i dt \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} e_0 &= \partial_t - M^i e_i = Vu\,; \qquad e^0 = dt + A_i dx^i = V^{-1}\omega\,; \\ e_i &= \partial_i - A_i\partial_t\,; \qquad\qquad\ \ e^k = dx^k + M^k e^0\,, \end{aligned} \tag{6.7}$$

where u is a monad vector, ω is a one-form of time such that $\omega(u) = 1$.

The first set of bases (6.6) is the generalization of the well-known ADM parameterization [14]. In this case the metric has the form

$$ds^2 = N^2(dt + B_j e^j)^2 - h_{ik}e^i e^k, \qquad (e^i = dx^i + N^i dt)\,. \tag{6.8}$$

The second set of bases (6.7) implies that the metric is given by

$$ds^2 = V^2(e^0)^2 - h_{ik}(dx^i + M^i e^0)(dx^k + M^k e^0), \tag{6.9}$$

where $e^0 = dt + A_j dx^j$.

The latter parameterization is the generalization of those often used when describing stationary spaces. It is worth emphasizing that the redundant "degrees of freedom" of the metrics (6.8)–(6.9) may be used to fix a frame of reference or to simplify the Einstein equations. In the theory of stationary configurations, representation (6.9) is useful for examining of solutions, for which a flux of matter and the timelike Killing vectors are non-collinear (so-called skew solutions [15]).

If B_j vanishes the metric (6.8) goes over into the standard ADM parameterization

$$ds^2 = N^2 dt^2 - h_{ik}(dx^i + N^i dt)(dx^k + N^k dt)\,. \tag{6.10}$$

When M^k vanishes, the metric (6.9) has the form

$$ds^2 = V^2(dt + A_j dx^j)^2 - h_{ik}dx^i dx^k\,. \tag{6.11}$$

This parameterization is often used when describing stationary spaces [15].

7. $(n+m)$ decomposition induced by a family of surfaces

Let $\{M^m \subset M\}$ be an n-parameter family of m-dimensional surfaces. One may think of these surfaces as intersections of the hypersurfaces $x^a=$ const *i.e.* $M^m = \bigcap_a \{x^a = \text{const}\}$. It is obvious that such a family induces $(n+m)$ decomposition of M. Indeed, there exists the vector basis $e_i = \partial_i$ on $T(M)$, $(i = n+1, \ldots, n+m)$, because of holonomicity of the M^m itself. As a consequence of it, the covector basis on the orthogonal to $T(M^m)$ dual subbundles Σ^{*n} is a set of one-forms $\{e^a = dx^a\}$. The corresponding dual bases to the bases $\{e_i\}$ and $\{e^a\}$ are determined up to $(n \cdot m)$ functions N^i_a such that

$$\begin{aligned} e^a &= dx^a \in \Sigma^{*n}\,, & e_a &= \partial_a - N^i_a \partial_i \in \Sigma^n\,; \\ e^i &= dx^i + N^i_a dx^a \in \Sigma^{*m}\,, & e_i &= \partial_i \in \Sigma^m\,. \end{aligned} \tag{7.1}$$

The functions N^i_a are expressed in terms of the components of the metric g by using the condition of orthogonality $e_a \cdot e_i = 0$. Thus the projection operators and the metric have the form:

$$H' = (\partial_a - N^i_a \partial_i) \otimes dx^a\,, \qquad H'' = \partial_k \otimes (dx^k + N^k_a dx^a)\,; \tag{7.2}$$

$$g = \gamma_{ab} dx^a \otimes dx^b + h_{ik}(dx^i + N^i_c dx^c) \otimes (dx^k + N^k_d dx^d)\,. \tag{7.3}$$

From the form of the metric (7.3) it can be seen that here we used the special case of canonical parameterization of $(n+m)$ decomposition (6.2) when B^a_i vanishes. In this case the formulae (6.5) become much simpler. Thus, one finds

$$\lambda^c_{ab} = 0; \quad \lambda^k_{ij} = 0; \quad \lambda^c_{ai} = 0; \quad \lambda^k_{ia} = -N^k_{a,i}; \quad A^a_{ij} = 0; \tag{7.4}$$

$$2S_{iab} = \gamma_{ab,i}\,; \qquad 2A^i_{ab} = e_b N^i_a - e_a N^i_b\,; \tag{7.5}$$

$$2S_{aik} = h_{ik,a} - h_{ik,l} N^l_a - h_{lk} N^l_{a,i} - h_{il} N^l_{a,k}\,; \tag{7.6}$$

$$2L_{cab} = 2\triangle_{cab} = e_a \gamma_{bc} + e_b \gamma_{ca} + e_c \gamma_{ab}\,; \tag{7.7}$$

$$2L_{ijk} = 2\triangle_{ijk} = h_{ij,k} - h_{ik,j} - h_{jk,i}\,. \tag{7.8}$$

The partial derivatives with respect to coordinates x^i and x^a are denoted here by "$_{,i}$" and "$_{,a}$", respectively. Then, according to (A.1)–(A.7), one can find the curvature tensor and its contractions.

8. $(n+m)$ decomposition induced by a group of isometries

Let M admits a non-transitive group of isometries G^n with the n linearly independent Killing vectors $\{\xi_a\}$, which satisfy the relations

$$[\xi_a, \xi_b] = C^d_{ab}\xi_d \qquad (a, b, d = 1, 2, \dots n)\,, \tag{8.1}$$

where C^d_{ab} are the structure constants and obey the Jacobi identity $C^c_{[ab}C^f_{d]c} = 0$ and the condition $C^c_{ab} + C^c_{ba} = 0$. In addition, the metric g satisfies the Killing equations:

$$(\mathcal{L}_{\xi_a} g)(X, Y) = \xi_a(X \cdot Y) - [\xi_a, X] \cdot Y - X \cdot [\xi_a, Y] = 0\,. \tag{8.2}$$

The group G^n decomposes M into a family of $m-$codimensional surfaces $\{M^n\} \subset M$, on which G^n is simply transitive ($\{M^n\}$ are invariant manifolds). Thus, we can say that the group G^n induces $(n+m)$ decomposition of M into the $m-$parameter family of $n-$dimensional surfaces of transitivity. Then the subbundle $\Sigma^n = \bigcup T(M^n)$ is a union of the tangent bundles of the family $\{M^n\}$, and Σ^m is a union of all the $m-$dimensional directions, which are tangent to M and orthogonal to $T(M^n)$.

Now we shall start in the same way as in the previous section. Thus one may think of the surfaces M^n as an intersection of the invariant hypersurfaces $\{x^i = \text{const}\}$, *i.e.* $M^m = \bigcap_i \{x^i = \text{const}\}$, $\ (i = n+1, \dots n+m)$. Moreover, one has $dx^i(\xi_a) = \xi_a x^i = 0$. Thus it is obvious that the invariant differential one-forms dx^i can be chosen as a covector basis on the subbundles Σ^{*m}. Then there exists the vector basis $\{\partial_a\} \in T(M^n)$, so that $dx^i(\partial_a) = 0$ and $\xi_a = \xi^b_{(a)}\partial_b$. Having extended these bases to the "complete ones": $\{dx^i\} \to \{dx^\mu\} = \{dx^a, dx^i\} \in T^*(M)$ and $\{\partial_a\} \to \{\partial_\mu\} = \{\partial_a, \partial_i\} \in T(M)$, where $dx^\mu(\partial_\nu) = \delta^\mu_\nu$ and $[\xi_a, \partial_i] = 0$, we can define one-forms ω^a such that

$$\omega^a(\xi_b) = \delta^a_b; \qquad \omega^a(\partial_i) = 0; \qquad \mathcal{L}_{\partial_i}\omega^a = 0 \tag{8.3}$$

$$\mathcal{L}_{\xi_a}\omega^b = -C^b_{ad}\omega^d\,; \qquad 2d\omega^a = C^a_{bd}\omega^b \wedge \omega^d\,. \tag{8.4}$$

Let us now introduce an auxiliary definition. We shall say that a split structure $\mathcal{H}^2$ is compatible with a group of isometries if the conditions of invariance of $\mathcal{H}^2$ are satisfied, *i.e.* if

$$\mathcal{L}_{\xi_a}H' = 0\,, \qquad \mathcal{L}_{\xi_a}H'' = 0\,, \qquad (a = 1, 2, \dots n)\,. \tag{8.5}$$

Using (5.3) and (8.1) one can easily verify that for the other vector and covector bases $\{E_k\} \in \Sigma^m$ and $\{\theta^a\} \in \Sigma^{*n}$ we have, respectively,

$$\mathcal{L}_{\xi_a}\theta^b = -C^b_{ad}\theta^d\,; \qquad \mathcal{L}_{\xi_a}E_k = 0\,. \tag{8.6}$$

To concretize the basis of decomposition we take $\theta^a = \theta^a_\mu dx^\mu$ and $E_i = E_i^\mu \partial_\mu$. Then the conditions of duality $\theta^a(\xi_b) = \delta^a_b$, $\theta^a(E_i) = 0$, $dx^k(E_i) = \delta^k_i$ determine these bases up to $(n \cdot m)$ functions A^a_i. As a result the basis of $(n+m)$ decomposition has the form:

$$\begin{aligned} &\xi_a \in \Sigma^n\,; \qquad e^a = \omega^a + A^a_i dx^i \in \Sigma^{*n}\,, \\ &e_i = \partial_i - A^a_i \xi_a \in \Sigma^m\,; \qquad dx^k \in \Sigma^{*m}\,, \qquad [\xi_a, e_i] = 0. \end{aligned} \tag{8.7}$$

The projection operators and the metric can be written as

$$H' = \xi_a \otimes (\omega^a + A^a_i dx^i)\,; \qquad H'' = (\partial_i - A^a_i \xi_a) \otimes dx^i\,; \tag{8.8}$$

$$g = g' + g'' = \gamma_{ab}(\omega^a + A^a_i dx^i) \otimes (\omega^b + A^b_j dx^j) + h_{kl} dx^k \otimes dx^l\,. \tag{8.9}$$

This representation of the metric is used when describing spaces of the multidimensional Kaluza–Klein theories [16]. From the Killing equations one finds

$$\xi_a \gamma_{bc} - C^d_{ab}\gamma_{dc} - C^d_{ac}\gamma_{bd} = 0\,; \qquad \xi_a A^b_i - C^b_{ad} A^d_i = 0\,; \qquad \xi_a h_{ik} = 0\,. \tag{8.10}$$

Using these equations we obtain the main geometrical objects

$$A''(\xi_a, \xi_b) = 0\,; \qquad 2A'(e_i, e_k) \equiv F^a_{ik}\xi_a\,; \tag{8.11}$$

$$F^a_{ik} = A^a_{k,i} - A^a_{i,k} + C^a_{bd} A^b_k A^d_i\,; \tag{8.12}$$

$$S'(e_i, e_k) = 0\,; \qquad 2e_i \cdot S''(e_a, e_b) \equiv 2S_{iab} = e_i \gamma_{ab}\,; \tag{8.13}$$

$$2L_{abc} = C_{cab} + C_{bca} + C_{acb}\,; \tag{8.14}$$

$$2L_{ijk} = 2\triangle_{ijk} = e_j h_{ik} + e_k h_{ij} - e_i h_{jk}\,. \tag{8.15}$$

In the end, from (A.1)–(A.7), we can find the curvature tensor, the Ricci tensor and scalar curvature (see Appendix C). When $m = 0$ we come to the case of homogeneous spaces.

9. Relativistic configurations of a perfect fluid

Let us consider space-time M^4 with the metric g in the $(3+1)$ decomposed form

$$g = V^2 e^0 \otimes e^0 - h_{ik} e^i \otimes e^k\,. \tag{9.1}$$

For the time being, we require the basis of decomposition to be an adopted one. Let the source of the gravitational field described by the metric (9.1) be a perfect fluid with the field of 4-velocities $u = V^{-1}e_0 = d/ds$ which is tangent to the flow lines $x^\mu = x^\mu(s)$. Herewith the mass density ρ obeys the conservation law

$$\mathrm{div}(\rho u) \equiv (\nabla_{e_\mu}\rho u)(e_\mu) = V^{-1}h^{-1/2}\mathcal{L}''_{e_0}(\rho h^{1/2}) = 0\,, \tag{9.2}$$

where $\mathcal{L}''_{e_0}$ is the Lie derivative with respect to the basis $\{e_i\}$: $2\mathcal{L}''_{e_0}\sqrt{h} = \sqrt{h}h^{ik}(\mathcal{L}_{e_0}h)(e_i, e_k)$. The equation of motion for the fluid follows from the relation:

$$\mathrm{div}T \equiv (\nabla_{e_\mu}T)(e^\mu,\ .\ ) = 0\,. \tag{9.3}$$

The energy-momentum tensor T is

$$T = \mu V^{-2} e_0 \otimes e_0 + P h^{ik} e_i \otimes e_k\,, \tag{9.4}$$

where μ is the energy density of the fluid, P is the pressure. Using the thermodynamic relations

$$d\mathcal{H} = Tds + \rho^{-1}dP, \qquad \mathcal{H} = (\mu + P)\rho^{-1} \tag{9.5}$$

one finds the equations of motion

$$(\mathrm{div}T)(e_0) \ = \ \rho T V^{-1} u S = -\rho V^{-1} dS/ds = 0\,, \tag{9.6}$$

$$(\mathrm{div}T)(e_i) \ = \ h^{ik}(dP - \rho\mathcal{H}\mathcal{L}_u\omega)(e_k) = 0\,. \tag{9.7}$$

Here we use the following notations: $\mathcal{H}$ is the enthalpy, S is the entropy, T is the temperature, and ω is the covector of the 4-velocity of the fluid ($\omega = Ve^0$, $\omega(u) = 1$). We introduce "the one-form of the enthalpy θ" and "the two-form of the curl Ω" by

$$\theta = \mathcal{H}\omega = \mathcal{H}Ve^0, \qquad \Omega = d\theta\,. \tag{9.8}$$

Then the equations of motion (9.6), (9.7) can be expressed as

$$\mathcal{L}_{e_0}\theta = d(\mathcal{H}V) - VTdS\,. \tag{9.9}$$

Using the formula $\mathcal{L}_{e_0} = i_{e_0}d + di_{e_0}$, where the operator i_{e_0} is defined by the relation $(i_{e_0}\Omega)(Y) = \Omega(e_0, Y)$, $\forall Y \in T(M^4)$, we obtain one more form of the equations of motion

$$i_{e_0}\Omega = -VTdS\,. \tag{9.10}$$

The condition of integrability of these relations leads to the equations of motion for the curl of a perfect fluid

$$\mathcal{L}_{e_0}\Omega = -d(TV) \wedge dS\,. \tag{9.11}$$

In the special case $S = \mathrm{const}$ a perfect fluid is isentropic so that the equations for "the one-form of the enthalpy" (9.9) and "the two-form of the curl" (9.10), (9.11) are reduced to the relations:

$$\mathcal{L}_{e_0}\theta = d(\mathcal{H}V) \tag{9.12}$$

$$i_{e_0}\Omega = 0, \qquad \mathcal{L}_{e_0}\Omega = 0\,. \tag{9.13}$$

It is to be noted that the last equation in (9.13) is the condition of integrability of the equation (9.12). Moreover we may regard this condition as an invariant formulation of the theorem [17], which states that the two-form of the curl Ω is constant along the world lines of particles of an isentropic perfect fluid. From the first relation in (9.13) it follows that Ω is singular, *i.e.* $\Omega(e_0, X) = 0$, $\forall X \in T(M^4)$, and therefore "completely spatial". This implies

$$\Omega = \sum_{i,j} \Omega_{ij} e^i \wedge e^j; \qquad \Omega \wedge \Omega = d\theta \wedge d\theta = 0\,. \tag{9.14}$$

Since in general case $\theta \wedge d\theta \neq 0$, then according to the Darboux theorem (see, for example [10]) it follows that there exist such functions ξ, η, ζ that $\theta = d\xi + \eta d\zeta$. This representation has been used in [18] to construct a number of families of solutions of the Einstein equations for an isentropic perfect fluid.

Now we shall consider the stationary spaces of General Relativity with a timelike Killing vector ∂_t. Then the equations (9.6), (9.7), as well as their consequences (9.9)–(9.13), go over into the equilibrium conditions of a perfect fluid. For an isentropic stationary flow they admit completely $3-$dimensional formulation. Indeed, in this case one has

$$\mathcal{L}_{\partial_t} g = 0\,, \quad \mathcal{L}_{\partial_t} e^{\mu} = 0\,, \quad [\partial_t, e_{\mu}] = 0\,. \tag{9.15}$$

Then using the parameterization of decomposition (6.7) we deduce that the functions V, A_i, M^k, h_{ik} as well as $\rho, \mu, P, \mathcal{H}$ do not depend on time. We define the vector $\vec{M}$ and covector A on the subbundles $\Sigma'' \equiv \Sigma^3$ by

$$\vec{M} = M^i \partial_i\,, \qquad A = A_k dx^k\,. \tag{9.16}$$

In terms of $\vec{M}$ and A the conservation law for mass (9.2) is transformed into the $3-$dimensional equation of continuity of the flow lines

$$\operatorname{div}^{(3)}(\rho\vec{M}) \equiv (\nabla_{e_i}\rho\vec{M})(e^i) = h^{-1/2}\mathcal{L}_{\vec{M}}(\rho h^{1/2}) = 0\,. \tag{9.17}$$

When $S = \text{const}$ the condition (9.9) may be rewritten in the $3-$dimensional form as well

$$i_{\vec{M}} dA = -d\log(\mathcal{H}V); \qquad \vec{M}(\mathcal{H}V) = 0\,. \tag{9.18}$$

From now on the objects and operations are defined on the $3-$dimensional manifold $t = \text{const}$ with respect to the bases $\{\partial_i\}$ and $\{dx^k\}$. For example:

$2dA = \mathcal{F}_{ik} dx^i \wedge dx^k$, where $\mathcal{F}_{ik} = A_{k,i} - A_{i,k}$. The equilibrium condition (9.18) may be expressed in the form

$$\mathcal{L}_{\vec{M}} A = d\{A(\vec{M}) - \log(\mathcal{H}V)\} \tag{9.19}$$

showing that the one-form $\mathcal{L}_{\vec{M}} \vec{A}$ is exact. Hence, as the condition of integrability one obtains the 3−dimensional conservation theorem for the curl dA along the 3−dimensional flow lines, *i.e.*

$$\mathcal{L}_{\vec{M}} dA = 0\,. \tag{9.20}$$

In the case of parameterization (6.6) for the stationary spaces the functions V, B_i, N^k, h_{jk} do not depend on time either. By analogy with (9.19) one has

$$\mathcal{L}_{\vec{N}} B = -d \log(\mathcal{H}V)\,, \tag{9.21}$$

where

$$\vec{N} = N^i \partial_i, \qquad B = B_k dx^k\,. \tag{9.22}$$

The condition of integrability gives the conservation theorem for the curl of B

$$\mathcal{L}_{\vec{N}} dB = 0\,. \tag{9.23}$$

If one of the two objects $\vec{M}$ and A in (9.19) (or $\vec{N}$ and B in (9.21)) vanishes then the equilibrium condition of an isentropic perfect fluid has the simple form

$$\mathcal{H}V = V(\mu + p)/\rho = k\,, \tag{9.24}$$

where k is a constant. Thus the Lagrangian of an isentropic perfect fluid in equilibrium is

$$L_m \equiv -V\sqrt{h}P = (k\rho - \mu V)\sqrt{h} = [k - (1+\varepsilon)V]\rho\sqrt{h}\,, \tag{9.25}$$

where $\varepsilon = \varepsilon(\rho)$ is the internal energy of the fluid and $\mu = \rho(1+\varepsilon)$.

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Appendix A

Components of the curvature tensor with respect to an adopted basis for an $(n+m)$ decomposition

Due to the definitions

$$\{E_\mu\} = \{E_a, E_i\}; \quad R(E_\mu, E_\nu)E_\rho \cdot E_\sigma = R_{\sigma\rho\mu\nu}; \quad R(E_\mu, E_\nu)E_\rho = R^\sigma_{\rho\mu\nu}E_\sigma$$

the generalized Gauss–Codazzi–Ricci equations (3.12)–(3.15) have the form

$$\begin{aligned} R_{abcd} &= R^{(n)}_{abcd} + 2A^i_{.cd}B_{iba} + B^i_{.cb}B_{ida} + B^i_{.db}B_{ica} \qquad (A.1)\\ R_{ibcd} &= B_{icb|d} - B_{idb|c} + 2A^k_{.cd}B_{bki} + B^k_{.db}(B_{cik} - \lambda_{kic})\\ &\quad -B^k_{.cb}(B_{dik} - \lambda_{kid}) \qquad (A.2)\\ R_{ibcj} &= B_{bji|c} - B_{icb|j} - B_{bjk}B_{ci.}{}^k - B_{icd}B_{jb.}{}^d\\ &\quad +B_{bki}\lambda^k_{.jc} + B_{bjk}\lambda^k_{.ic} + B_{idb}\lambda^d_{.cj} + B_{icd}\lambda^d_{.bj}\,, \qquad (A.3)\end{aligned}$$

where the curvature tensor of the subbundle Σ^n is defined by its components $R^{(n)}_{abcd}$ according to

$$R^{(n)a}{}_{bcd} = E_cL^a_{db} - E_dL^a_{cb} + L^f_{db}L^a_{cf} - L^f_{cb}L^a_{df} - \lambda^f_{cd}L^a_{fb} + 2A^i_{.cd}\lambda^a_{bi} \quad (A.4)$$

(and similarly for the replacement $n \to m$ and $a, b, c, \ldots \leftrightarrow i, j, k, \ldots$). Then the components of the Ricci tensor and the curvature scalar have the form

$$\begin{aligned} R_{bd} &= R^{(n)}_{bd} - B^i_{db|i} - S_{b|d} + 2A^i_{ad}A_{ib.}{}^a + 2S_{iad}S^{ia}{}_b\\ &\quad -S_b{}^{ij}S_{dij} + A_{bij}A_d{}^{ij} - S_iB^i_{.db} - B^i_{.da}\lambda^a_{.bi} - B^i_{.ab}\lambda^a_{.di} \qquad (A.5)\\ R_{ia} &= B_{ia.}{}^b{}_{|b} + B_{ai.}{}^k{}_{|k} - S_{i|a} - S_{a|i} - 2S^b_{ik}S^k_{ab} - 6A^b_{ik}A^k_{ab}\\ &\quad +S^k(B_{aik} - \lambda_{kia}) + S^b(B_{iab} - \lambda_{bai}) + B^k_{ab}\lambda_{ki.}{}^b + B^b_{ik}\lambda_{ba.}{}^k \qquad (A.6)\\ R &= R^{(n)} - 2S^i{}_{|i} - S^iS_i - S^i_{ab}S^{ab}_{i..} - A^i_{ab}A^{ab}_{i..}\\ &\quad +R^{(m)} - 2S^a{}_{|a} - S^aS_a - S^a_{ij}S^{ij}_{a.} - A^a_{ij}A_{a..}{}^{ij}\,, \qquad (A.7)\end{aligned}$$

where $S^i = S^i_{ab}\gamma^{ab}$, $S^a = S^a_{ik}h^{ik}$. The signs "${}_{|i}$" and "${}_{|a}$" denote the covariant derivative with respect to the connections L^k_{mn} and L^a_{bc} in the directions of the vectors E_i and E_a, respectively. For example

$$B_{icb|d} = E_dB_{icb} - B_{iab}L^a_{dc} - B_{ica}L^a_{db} \quad (a, b, c \leftrightarrow i, j, k)\,. \qquad (A.8)$$

The other components of the Ricci tensor and the curvature tensor can be found from (A.1)–(A.6) by the formal substitution $a, b, c, \ldots$ for $i, j, k, \ldots$ and otherwise.

Appendix B

Components of the curvature tensor with respect to an adopted basis for (n+1) decomposition

The generalized Gauss–Codazzi–Ricci equations for the metric (5.10) with respect to the basis (5.9) have the form:

$$R_{abcd} = R^{(n)}_{abcd} + \varepsilon N^{-2}(\mathcal{B}_{cb}\mathcal{B}_{da} - \mathcal{B}_{db}\mathcal{B}_{ca} + F_{cd}\mathcal{B}_{ba}) \,, \tag{B.1}$$

$$R_{n+1,bcd} = N\{(N^{-1}\mathcal{B}_{cb})_{|d} - (N^{-1}\mathcal{B}_{db})_{|c}\} - \varepsilon N^{-2}G_b F_{cd} \,, \tag{B.2}$$

$$\begin{aligned} R_{n+1,bc,n+1} &= N\mathcal{L}_E(N^{-1}\mathcal{B}_{cb}) - \mathcal{B}_{ca}\mathcal{B}_{b.}^{\ a} \\ &\quad + \varepsilon N^{-2}G_b G_c - N^2(N^{-2}G_b)_{|c} \,, \end{aligned} \tag{B.3}$$

$$\begin{aligned} R_{bd} = {} & R^{(n)}_{bd} - \varepsilon N^{-2}[N\mathcal{L}_E(N^{-1}\mathcal{B}_{db}) + D\mathcal{B}_{db} + \tfrac{1}{2}F_{ba}F_{d.}^{\ a} - 2D_{ba}D_{d.}^{\ a}] \\ & + \varepsilon(N^{-2}G_b)_{|d} - N^{-4}G_b G_d \,, \end{aligned} \tag{B.4}$$

$$R_{n+1,a} = N[(N^{-1}\mathcal{B}_{a.}^{\ b})_{|b} - E_a(N^{-1}D)] - \varepsilon N^{-1}F_{ab}G^b \,, \tag{B.5}$$

$$\begin{aligned} R_{n+1,n+1} = & - NE(N^{-1}D) - D_{ab}D^{ab} + \tfrac{1}{4}F_{ab}F^{ab} \\ & + N^2(N^{-2}G^a)_{|a} - \varepsilon N^{-2}G_a G^a \,, \end{aligned} \tag{B.6}$$

$$\begin{aligned} R = R^{(n)} & - 2\varepsilon N^{-1}E(N^{-1}D) - \varepsilon N^{-2}(D^2 + D_{ab}D^{ab} + \tfrac{1}{4}F_{ab}F^{ab}) \\ & + 2\varepsilon(N^{-2}G_a)_{|a} - 2N^{-4}G_a G^a . \end{aligned} \tag{B.7}$$

$$R^{(n)a}{}_{bcd} = E_c L^a_{db} - E_d L^a_{cb} + L^f_{db}L^a_{cf} - L^f_{cb}L^a_{df} - \lambda^f_{cd}L^a_{fb} + \varepsilon N^{-2}F_{cd}\lambda^a_b \,, \tag{B.8}$$

where $\lambda^a_b = \theta^a([E_b, E])$ and $R^{(n)} = \gamma^{bd}R^{(n)}_{bd}$; $R^{(n)}_{bd} = R^{(n)a}{}_{bad}$.

Appendix C

Components of the curvature tensor for a $(n+m)$ decomposition induced by a group of isometries

The curvature tensor and its contractions with respect to the basis (8.7) for the metric (8.9) have the form:

$$R^{(m+n)}_{dcab} = R^{(n)}_{dcab} + S_{ic[a}S^{i}_{b]d}\,, \tag{C.1}$$

$$R^{(m+n)}_{icab} = S^{k}_{c[a}F_{b]ki} + S_{icd}C^{d}_{.ba} + 2S_{id[a}\gamma^{d}_{.b]c}\,, \tag{C.2}$$

$$R^{(m+n)}_{ickb} = -S_{ibc;k} + S_{ibd}S_{kc.}{}^{d} + \frac{1}{4}F_{ckj}F_{bi.}{}^{j} - \frac{1}{2}\gamma^{d}_{.bc}F_{dki}\,, \tag{C.3}$$

$$R^{(m+n)}_{ajkl} = F_{aj[l;k]} + F_{bj[k}S_{l]a}{}^{b} + F_{bkl}S_{ja}{}^{b}\,, \tag{C.4}$$

$$R^{(m+n)}_{ijkl} = R^{(m)}_{ijkl} + \frac{1}{2}F_{ai[k}F^{a}_{.l]j} - \frac{1}{2}F_{aij}F^{a}_{kl}\,, \tag{C.5}$$

$$R^{(m)i}_{jlk} = 2e_{[k}\triangle^{i}_{l]j} + 2\triangle^{m}_{j[l}\triangle^{i}_{k]m}, \qquad R^{(n)d}_{.cab} = 2\gamma_{q.[a}{}^{d}\gamma_{b]c.}{}^{q} - C^{qd}_{..c}\gamma_{aqb}\,, \tag{C.6}$$

$$R^{(m+n)}_{ab} = R^{(n)}_{ab} - S^{i}_{ab;i} - S^{i}_{ab}S_{i} + 2S^{i}{}_{ac}S_{i}{}^{c}{}_{b} + \frac{1}{4}F_{aij}F_{b}{}^{ij}\,, \tag{C.7}$$

$$R^{(m+n)}_{ai} = \frac{1}{2}F_{ai}{}^{k}{}_{;k} + \frac{1}{2}F_{ail}S^{l} + C^{d}_{.ba}S^{b}_{id} - C^{b}_{bd}S_{ia}{}^{d}\,, \tag{C.8}$$

$$R^{(m+n)}_{ik} = R^{(m)}_{ik} - S_{(i;k)} - S_{iab}S_{k}{}^{ab} + \frac{1}{2}F_{aij}F^{aj}_{..k}\,, \tag{C.9}$$

$$R^{(m+n)} = R^{(n)} + R^{(m)} - 2S^{i}{}_{;i} - S^{i}S_{i} - S^{iab}S_{iab} - \frac{1}{4}F^{a}_{ij}F_{a}{}^{ij}\,. \tag{C.10}$$

Here $R^{(m)} = h^{ik}R^{(m)}_{ik}$; $R^{(m)}_{ik} = R^{(m)l}{}_{ilk}$ and $R^{(n)} = \gamma^{bd}R^{(n)}_{bd}$; $R^{(n)}_{bd} = R^{(n)a}{}_{bad}$. The covariant derivative in the direction of the vector e_k with respect to the connection $\triangle^{i}_{jk}$ is denoted by "$;k$".

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