LOCALIZATION TRANSITIONS FROM FREE RANDOM VARIABLES

Romuald A. Janik[a], Maciej A. Nowak[a,b], Gábor Papp[c] and Ismail Zahed[d]

[a]Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Kraków, Poland
[b]GSI, Planckstr. 1, D-64291 Darmstadt, Germany
[c]Institute for Theoretical Physics, Eötvös University
Budapest, Hungary
[d]Department of Physics and Astronomy, SUNY
Stony Brook, New York 11794, USA.

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We motivate and use the concept of free random variables for the study of the de-pinning transition of flux lines in superconductors as recently discussed by Hatano and Nelson in one dimension. Our analysis yields naturally to a generalization of the concept of Coherent Phase Appproximation (CPA) for nonhermitean Hamiltonians, and is exact for Cauchy randomness. We derive analytical conditions for the critical points of the complex eigenvalue distribution, in very good agreement with numerical calculations. We suggest a relation between dimensionally reduced nonhermitean quantum mechanics and weak nonhermiticity.

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1.

Recently Hatano and Nelson [1] have shown that the de-pinning of flux lines from columnar defects in superconductors in D+1-dimensions, may be mapped onto the world lines of bosons in D-dimensions. The pinning and hence localization by the columnar defects is mapped onto an on site real randomness, and the de-pinning by the transverse magnetic field is mapped onto a directed hoping, resulting in nonhermitean quantum mechanics. While it is generally accepted that all eigenstates are localized in one- and two-dimensions in the presence of randomness, it is clear that the flux lines are de-pinned by a strong transverse magnetic field. The de-pinning in one- and two-dimensions was studied numerically in [1], and the

phenomenon of nonhermitean localization generated a lot of investigations [1–8, 18].

In this paper, we would like to show that the nonhermitean tight-binding model discussed by Hatano and Nelson in one-dimension can be analyzed in a straightforward way using the concepts of free random variables. In Section 2 and 3, we introduce the model and motivate the use of the addition law for free random variables. In Section 4, we derive an explicit condition for the end-points of the distribution of (localized) eigenvalues on the real axis for arbitrary transverse magnetic field. In Section 5, we extend our analysis to the complex eigenvalue plane obtaining the *complex* eigenvalue distribution and the critical values of the magnetic field. In Section 6, we use the uncertainty principle to show that the constant mode sector of this and related models may be amenable to nonhermitean random matrix models with weak nonhermiticity [2,3].

2.

Following Hatano and Nelson, we consider the nonhermitean tight--binding Hamiltonian in second quantized form for $D = 1$

$$H = H_0 + \mathcal{V} = \sum_{A=1}^{N} \left(\frac{t}{2}(e^{+h}\, c_{A+1}^{\dagger} c_A + e^{-h}\, c_{A-1}^{\dagger} c_A) + V_A\, c_A^{\dagger} c_A \right) , \qquad (1)$$

where $c_A^{\dagger}$ is a boson creation operator at site A. Throughout, the lattice spacing $a = 1$. The diagonal entries are random with elements distributed uniformly between $(-\Delta, \Delta)$, and the deterministic part H_0 is off-diagonal with hopping strengths $te^{\pm h}/2$ $(t < 0)$. Here h is the typical strength of the 'transverse magnetic field' in units of the flux quantum [1].

The eigenvalues of H are complex valued for $h \neq 0$. Due to reality of the partition function and the symmetry $H(-h) = H^T(h)$, the complex eigenvalues are symmetric under reflection along the x and y axes. Their distribution in the z-plane is shown in Fig. 1, for $h = 0.1, 0.3$ and $\Delta = t = 1$. The results are for an ensemble of 100 matrices of size 100×100. For $h = 0.1$

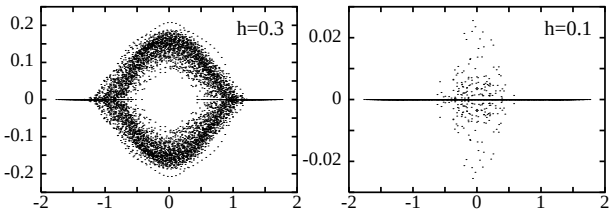


Fig. 1. Re ε – Im ε for $t = \Delta = 1$.

the eigenvalues are mostly real (localized). For $h = 0.3$ the eigenvalues around $z = 0$ are mostly complex (delocalized). For larger size matrices, the width of the rim shrinks to zero in agreement with the results discussed in [1].

An important question regarding the character of the spectrum is the occurrence of a critical $h_c^{(1)}$ for which a gap near $z \sim 0$ sets in for the localized states. For increasing $h > h_c^{(1)}$ the eigenvalues migrate from the real axis to the complex plane as also discussed by Feinberg and Zee [9]. The migration is total for $h = h_c^{(2)} > h_c^{(1)}$. To try to quantify this and the bulk aspects of the spectrum in Fig. 1, we will analyze (1) using the addition law for free random variables [10, 11].

3.

All the information on the eigenvalue distribution of the Hatano–Nelson Hamiltonian is encoded in the Green's function:

$$G(z) = \left\langle \frac{1}{z - H_0 - \mathcal{V}} \right\rangle . \tag{2}$$

The distribution of eigenvalues, which is one-dimensional in this case, can be reconstructed from the discontinuities of $G(z)$. It is convenient to express it through

$$G(z) = \langle \partial_z \log \det(z - H_0 - \mathcal{V}) \rangle . \tag{3}$$

The determinant splits into a sum of two terms:

$$\det(z - H_0 - \mathcal{V}) \sim \left(\frac{t}{2}\right)^N \cdot \left[e^{Nh} + e^{-Nh}\right] + e^{NF_{\text{out}}(z,\mathcal{V})}(1 + \ldots) . \tag{4}$$

We see that when $h + \log(t/2) > F_{\text{out}}$, the latter term can be neglected and the Green's function is $G(z) = 0$. In the opposite case, in particular for large z,

$$G(z) = \partial_z F_{\text{out}} \tag{5}$$

(after averaging over $\mathcal{V}$). We will now proceed to find the Green's function in the latter region.

A standard method used in the treatment of lattice models with site disorder is the Coherent Phase Approximation (CPA). The basic assumption is that one can implement the effects of disorder through a *scalar* (*i.e.* diagonal) self-energy $\Sigma(z)$:

$$G^{\text{CPA}}(z) = \frac{1}{N} \operatorname{tr} \frac{1}{z - H_0 - \Sigma(z)} \equiv G_{H_0}(z - \Sigma(z)) , \tag{6}$$

where Σ satisfies

$$\left\langle \frac{\mathcal{V}_a - \Sigma(z)}{1 - (\mathcal{V}_a - \Sigma(z))G^{\mathrm{CPA}}(z)} \right\rangle = 0\,. \tag{7}$$

This can be recast in the form

$$G^{\mathrm{CPA}}(z) = G_{\mathcal{V}}\left(\frac{1}{G^{\mathrm{CPA}}(z)} + \Sigma(z)\right)\,. \tag{8}$$

We will now show that a solution of the CPA equations is equivalent to the addition formalism of the Blue's functions [13], which are the functional inverses of the Green's functions *i.e.* they satisfy

$$B_{\mathcal{V}}(G_{\mathcal{V}}(z)) \equiv z\,. \tag{9}$$

We will now evaluate the Blue's function for the random part $B_{\mathcal{V}}$ on both sides of (8):

$$B_{\mathcal{V}}(G^{\mathrm{CPA}}(z)) = \frac{1}{G^{\mathrm{CPA}}(z)} + \Sigma(z) \tag{10}$$

and evaluate the Blue's function for the deterministic part B_{H_0} on both sides of (6):

$$B_{H_0}(G^{\mathrm{CPA}}(z)) = z - \Sigma(z)\,. \tag{11}$$

Adding the last two equations together gives the addition law for random matrices [13]

$$B_{\mathcal{V}}(G^{\mathrm{CPA}}(z)) + B_{H_0}(G^{\mathrm{CPA}}(z)) - \frac{1}{G^{\mathrm{CPA}}(z)} = z\,. \tag{12}$$

The formal link between the CPA approximation and the addition of free random variables has been noticed using different methods in [11]. However the physical systems considered there were of a different variety. In the remaining part of this paper we will apply this method to the analysis of the spectrum of the model. In fact it is quite intriguing why the addition method developed for random matrices *i.e.* for 0-dimensional systems, works so well also for the 1-dimensional Hatano–Nelson model. In the appendix we will reconsider the justification of our method using diagrammatic arguments. In particular we will show how one can map a large class of Feynman graphs for the Hatano–Nelson Green's function into an effective random matrix model.

4.

In this section we would like to find the edge of the spectrum on the real axis and the density of localized eigenvalues within our approximation

scheme. To this end we will first derive a formula for the resolvent $G(z)$ along the real axis using the addition law for 'hermitean' matrices. Then the density of localized eigenvalues would follow from the imaginary part of the Green's function $\pi\nu(\lambda) = \mathrm{Im}G(\lambda + i\varepsilon)$, and the edge of the spectrum would correspond to a branch point of $G(z)$ or, equivalently, to $G'(z_e) = \infty$.

With this in mind, the resolvent for the random part is simply given by

$$G_{\mathcal{V}}(z) = \frac{1}{N}\frac{1}{2\Delta}\int_{-\Delta}^{\Delta}\frac{N\,dV}{z-V} = \frac{1}{2\Delta}\ln\,\frac{z+\Delta}{z-\Delta} \tag{13}$$

with an inverse (Blue's function) given by $B_{\mathcal{V}} = \Delta\coth\Delta z$, in agreement with [16]. The resolvent for the deterministic part is

$$G_{H_0}(z) = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{z - t\cos\left(2\pi n/N + ih\right)}\,. \tag{14}$$

In the large N limit $G_{H_0}(z) = 1/\sqrt{z^2 - t^2}$ for z outside the ellipse defined by

$$\left(\frac{x}{\cosh h}\right)^2 + \left(\frac{y}{\sinh h}\right)^2 = t^2 \tag{15}$$

and zero inside. The inverse of the resolvent (Blue's function) is just $B_{H_0} = \sqrt{1/z^2 + t^2}$.

For large z, the resolvent $G(z)$ for (1) along the real axis follows from its functional inverse $B[G(z)] = z$ through the addition law [13], $B = B_{H_0} + B_{\mathcal{V}} - 1/z$. Specifically

$$z = \sqrt{\frac{1}{G^2} + t^2} + \Delta\coth\Delta G - \frac{1}{G} \tag{16}$$

and is h-independent. Along the real axis, the end-points z_e of the spectrum satisfy $z_e \equiv B(z_c)$, with $dB(z_c)/dz = 0$, that is

$$-\frac{1}{z^3}\frac{1}{\sqrt{\frac{1}{z^2} + t^2}} - \frac{\Delta^2}{\sinh^2\Delta z} + \frac{1}{z^2} = 0\,. \tag{17}$$

For $\Delta = t = 1$ this yields $z_c = 1.5752$ or $z_e = 1.63915$.

5.

In this section, we derive analytical conditions for the migration of eigenvalues into the complex plane (see Fig. 1). For the model considered here, this corresponds to the celebrated nonhermitean delocalization [1].

The locus of the eigenvalue distribution appears through the discontinuity of the Green's function. For one dimensional curves in the complex plane the density of delocalized states can be reconstructed from the two-dimensional Gauss law in a standard way. Here we will just restrict ourselves to determining some global characteristics of the spectrum. Recall from Section 3, that the condition for the discontinuity of the Green's function reads

$$F_{in} \stackrel{\text{def}}{\equiv} h + \log(t/2) = F_{\text{out}}(z)\,, \tag{18}$$

where F_{out} is determined by the condition $\partial_z F_{\text{out}}(z) = G(z)$. For that, we analytically continue $G(z)$ to the z-plane minus the ellipse (15), and use it to construct the potential $F_{\text{out}}(z)$ through $F_{\text{out}} = \int dz\; G$, with the integration constant fixed by the appropriate behavior at infinity [12]. In fact this integral can be performed even when the explicit form of $G(z)$ is not known but only using the expression for the Blue's function, that is

$$F_{\text{out}}(z) = zG(z) - \int dG B(G)\,. \tag{19}$$

In this way we obtain

$$\begin{aligned} F_{\text{out}}(z) = \quad & +\Delta G \coth \Delta G - 1 \\ & + \log \frac{1+\sqrt{1+G^2}}{2} - \log \frac{\sinh \Delta G}{\Delta}\,. \end{aligned} \tag{20}$$

The locus of the cusp (18) coincides with the position of the *complex* eigenvalue distribution shown in Fig. 1 as $N \to \infty$. The real (localized) eigenvalues are the remnants of the hermitean addition law discussed above, and disappear at some critical value of $h = h_c^{(2)}$.

Fig. 2 (left) shows the behavior of the critical cusp-line along the real axis, where the eigenvalue spectra have a branching point (x_{branch}), versus the strength h for which (18) holds. At $x_{\text{branch}}(h_c^{(2)}) = z_e$ the localized states disappear from the spectrum. The dots are the numerically generated branching points, while the solid line corresponds to using (18)–(20). Fig. 2 (right) shows the same along the imaginary axis for the topmost point of the spectrum $y_{\max}$. For large h, the resolvent drops to zero in the outer region like $1/z$, and using (16) together with (20) we get $y_{\max} \approx \sinh h - \Delta^2/6 \sinh h$.

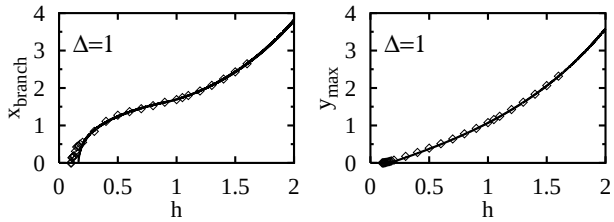


Fig. 2. $x_{\rm branch}$ and $y_{\rm max}$ versus h for $t = \Delta = 1$.

The existence of the solution for given h defines the *critical* value $h_c^{(1)}$ for which the cut starts to develop in the inner ellipse of Fig. 1. The dependence of $h_c^{(1)}$ on Δ is shown in Fig. 3 (solid line). The dotted lines are generated analytically by working out the leading contributions to F in (20), for small values of Δ (dotted line), and large values of Δ (long dashed line).

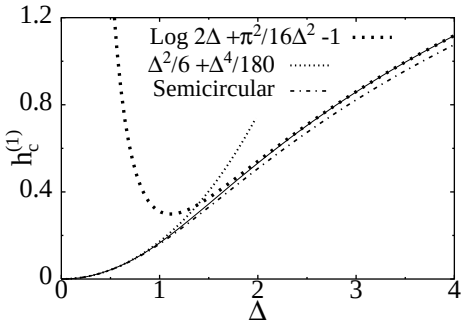


Fig. 3. $h_c^{(1)}$ versus Δ. See text.

These results are also reproduced analytically using a semi-circular distribution for the random part, that is $\pi P(V_i) = \sqrt{4\tau - V_i^2}/2\tau$, when $\tau = \Delta^2/3 \to \infty$, as indicated by the thick dashed line in Fig. 3. Then the equation for the Green's function in the outer region follows from the semicircular Blue's function $B_{\mathcal{V}}^{\rm semicircular} = \tau G + 1/G$:

$$(z - \tau G)^2 G^2 = 1 + t^2 G^2 \,. \tag{21}$$

Similarly, we consider the Cauchy randomness, that is $\pi P(V_i) = \gamma/(\gamma^2 + V_i^2)$. In this case, the results are exact, as originally suggested in [11] for the hermitean case. In particular, (20) is now

$$F(z) = \log\left(i\gamma + z + \sqrt{(i\gamma + z)^2 - t^2}\right) - \log 2 \,. \tag{22}$$

Using (18) with (22) in the outside leads the position of the 'cusp' for Cauchy randomness. In this case, the critical value $h_c^{(1)}$ for the onset of delocalization

is given in closed form through $\sinh h_c^{(1)} = \gamma/t$, as noticed first by Brezin and Zee [18] using different arguments.

Finally, the dependence of the eigenvalue distribution shown in Fig. 1 on $1/N$, may be qualitatively understood by using the circular version of (1). This amounts to trading $t/2e^h \to r$ and $t/2e^{-h} \to 0$. For a uniform distribution of eigenvalues, the secular equation is

$$\prod_i (\lambda - V_i) = -(-r)^N \,. \tag{23}$$

For small values of Δ, hence V_i, (23) can be solved perturbatively, with the ansatz $\lambda_j = re^{i2\pi j/N} + \varepsilon_j$, giving $\varepsilon_j = -\sum_i V_i/N$. Typically, $\langle N^2 \varepsilon_j^2 \rangle = N\langle V_i^2 \rangle = N\Delta^2/3$, showing the corrections to be $\varepsilon \sim 1/\sqrt{N}$ in large N. This result compares well with the numerical estimate using $\delta\lambda = \max(|\lambda_i| - |\lambda_j|)$ between the outer and inner eigenvalues as shown in Fig. 4.

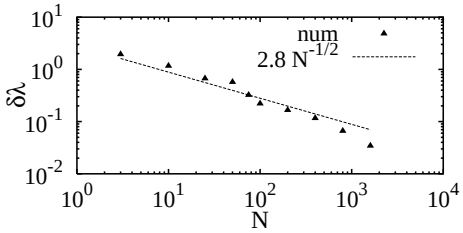


Fig. 4. $\delta\lambda$ versus N (matrix size) for $r = \Delta = 1$.

One should note the intriguing possibility of observing a universal behavior which is exactly characterized by the scale $1/\sqrt{N}$ in the imaginary direction. This regime christened 'weak nonhermiticity' has received much attention lately [2,3], and it would be interesting to extend it to the present context.

6.

The previous analysis borrows on some of the methods discussed in [12] for random matrix models. However, the Hatano–Nelson model differs in an important way from matrix models: it knows about the dimensionality of space. Recently Efetov [3] has used supersymmetric methods to argue that in 0-dimension the model reduces to a nonhermitean random matrix model with weak-nonhermiticity [2], and suggested that the reduction may yield to new developments in the context of oriented quantum chaos.

The reduction is actually understandable from the point of view of continuum quantum mechanics of constant modes. The continuum version of (1) is $H = (p + ih)^2 + \mathcal{V}$, where p is the D-dimensional momentum and $\mathcal{V}$ the

random site potential. The reduction to 0-dimension means that $\mathcal{V}$ is x-independent. Classically this would imply that $p = 0$. Quantum mechanically, however, p fluctuates. In a box of size N, $p \sim \mathcal{W}/\sqrt[D]{N}$ by the uncertainty principle. Hence, the reduction of H to H_* in 0-dimension amounts to

$$H_* = \mathcal{V} + \frac{i2h}{\sqrt[D]{N}}\,\mathcal{W}\,. \tag{24}$$

For sufficiently random hopping, the result is a random matrix model with small nonhermiticity in general. For D=2 this is just the case of *weak* nonhermiticity discussed by Fyodorov *et al.* [2]. For $\mathcal{V}, \mathcal{W}$ chosen in the GOE ensemble as motivated by $\mathcal{V}$ real in the D-dimensional version of (1), our arguments suggest localization for D=2 in (24) as also noted in [3].

The present arguments may also extend to other models. For example, the (massless) QCD Dirac operator in a D-dimensional Euclidean box of volume N at finite chemical potential μ is $H = \gamma^{D+1}(i\gamma^a\nabla^a + i\mu\gamma^D)$, where γ's are Dirac matrices with $a = 1, ..., D$, and ∇ the covariant derivatives with external gauge fields. The squared operator,

$$H_* = (i\nabla^a)(i\nabla^a) + \frac{i}{2}\sigma^{ab}[i\nabla^a, i\nabla^b] + 2i\mu(i\nabla^D) \tag{25}$$

with $\sigma^{ab} = i[\gamma^a, \gamma^b]/2$, is analogous to (24). For sufficiently random hopping and $i\nabla^D \sim 1/\sqrt[D]{N}$, localization may take place for the GOE ensemble in D=2 dimensions at finite μ. Since two-dimensional QCD, in the limit of a large number of colors, exhibits quasi long-range order, this issue is worth investigating.

7.

We have shown that the Hatano–Nelson model (1), may be understood in terms of the addition law for free random variables, and obtained analytical conditions for the end-points of the complex eigenvalue spectra that compare well with numerics. Our diagrammatic approximation satisfies the nonhermitean analogue of the CPA equation, and is exact for Cauchy randomness [18]. Corrections for generalized randomness can be sought in the remaining non-planar graphs in the form of two-site, three-site, etc. rescatterings. In the light of recent discussion [19], it would be interesting to consider the extension of our method to higher dimensions. For two-dimensions this looks possible through (12), by modifying the Blue's function for the deterministic matrix H_0 to incorporate the structure of the two dimensional lattice. The result involves elliptic functions.

Finally, we have presented generic arguments for how the constant modes of the model in D-dimensions and for sufficiently random hopping, relate to random matrix models with weak nonhermiticity. For D=2 our arguments confirm a recent observation by Efetov [3].

We would like to thank Prof. D. Nelson for suggesting that we look at this problem, E. Gudowska-Nowak for discussions. IZ thanks Y. Fyodorov and A. Zee for discussions. RJ thanks the Nuclear Theory Group at Stony Brook, where part of this work was done. This work was supported in part by the US DOE grant DE-FG-88ER40388, by the Polish Government Project (KBN) grants 2P03B04412 and 2P03B00814 and by the Hungarian grants OTKA T022931 and F019689.

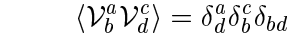
Appendix

In this section, we proceed to motivate the use of free random variables for the Hamiltonian (1) from a diagrammatic standpoint. Note that the physical meaning of our approximation scheme is most clearly spelled out using the equivalence with the Coherent Potential Approximation (CPA. The discussion below is not needed to apply our method to specific cases but may perhaps help in emphasizing the features of the model which are essential for the method to have a chance of working (*e.g.* translational invariance).

As we saw all the information about the behavior of the model (1) is encoded in the Green's function (2). We will now analyze the diagrammatic expansion of (2) and try to reinterpret a large class of graphs as coming from a certain random matrix model.

For that, we note that the diagrammatic expansion for (1) follows that of random matrix models [12] to the exception that the $\mathcal{V}$-propagators are changed to

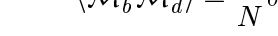
$$\langle \mathcal{V}_b^a \mathcal{V}_d^c \rangle = \delta_d^a \delta_b^c \delta_{bd} = \tag{26}$$



Feynman graph:ˊ

This is, however, markedly different from the random matrix propagator

$$\langle \mathcal{M}_b^a \mathcal{M}_d^c \rangle = \frac{1}{N} \delta_d^a \delta_b^c = \tag{27}$$



Feynman graph:ˆ

In particular there is no 'double line' structure in (26), as visualized by our notation (all the lines are pinched to a point). Also there is no accompanying

factor of $1/N$. This causes the fact that for the Hatano–Nelson Green's function there is no natural separation of graphs into planar and subleading non-planar ones. We will now show that nevertheless a class of graphs for $G(z)$ can be rewritten in a random matrix like way.

Consider an expectation value $\langle \mathcal{V} \cdot \mathcal{C} \cdot \mathcal{V} \rangle$ with $\mathcal{C}$ some translationally invariant matrix. Then we have

$$\left\langle \mathcal{V}_b^a \cdot \mathcal{C}_c^b \cdot \mathcal{V}_d^c \right\rangle = \underset{\substack{\text{Feynman graph:}\\ \text{a b}\quad \text{c d}}}{C} = \underbrace{\delta_d^a \mathcal{C}_a^a}_{\text{no summation!}}$$

$$= \underset{\substack{\text{Feynman graph:}\\ \text{a b}\quad \text{c d}}}{C} = \delta_d^a \cdot \frac{1}{N} \operatorname{tr} \mathcal{C} \tag{28}$$

Note that the result is the same as if it were calculated for a random matrix model with the propagator (27). This enables us to use the double line propagators in the Hatano–Nelson model.

We will now proceed to consider the diagrammatic expansion of the resolvent $G(z) = \operatorname{tr} \mathcal{G}(z)$ for this model

$$\mathcal{G}(z) = \left\langle \frac{1}{z - H_0 - \mathcal{V}} \right\rangle = \frac{1}{z - H_0 - \Sigma(z)}, \tag{29}$$

where $\Sigma(z)$ is the 1-particle irreducible (1PI) self-energy. Typical contributions to $\Sigma(z)$ are shown in Fig. 5. They may be decomposed into planar (gluon-)connected (*pl*, *c*) (see [13] for the precise definition) and non-planar 1PI ones (*np*). Generically,

$$\Sigma(z) = \sum_n (\langle \operatorname{tr} \mathcal{V}\mathcal{G}(z)\mathcal{V} \ldots \mathcal{G}(z)\mathcal{V} \rangle_{\text{pl, c.}} + \langle \operatorname{tr} \mathcal{V}\mathcal{G}(z)\mathcal{V} \ldots \mathcal{G}(z)\mathcal{V} \rangle_{\text{np}}). \tag{30}$$

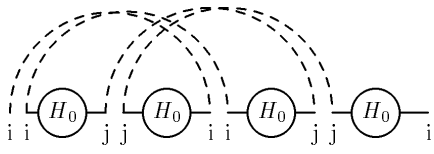
The planar piece is equal to

$$\sum_{i=1}^{N} \langle V_i^n \rangle_{\text{pl}, c} \cdot (\mathcal{G}(z)_{ii})^{n-1} = \langle \operatorname{tr} \mathcal{V}^n \rangle_{\text{pl}, c} \, G(z)^{n-1}. \tag{31}$$

In the last equality we made use of the invariance of H_0 under lattice translations. The non-planar piece is, as it stands, much more complicated. Here, we will *approximate* $\mathcal{G}(z)$ by its *diagonal* part $G(z) \cdot \mathbf{1}$, *i.e.* retaining only the single-site rescattering. Hence

$$\Sigma(z) = \sum_n \left(\langle \operatorname{tr} \mathcal{V}^n \rangle_{\text{pl}, c} + \langle \operatorname{tr} \mathcal{V}^n \rangle_{\text{np}} \right) \cdot G(z)^{n-1}. \tag{32}$$

This is our main approximation. To convince oneself that this is really an approximation one can examine the first graph (lowest order in $1/z$ expansion of the Green's function) that is not included in the 'random-matrix like' diagrammatic expansion of (32). It gives a contribution to the coefficient of the $1/z^9$ term in the Green's function. Indeed the following graph contributing to the expectation value $\langle \operatorname{tr} \mathcal{V} H_0 \mathcal{V} H_0 \mathcal{V} H_0 \mathcal{V} H_0 \rangle$ does not appear in our approximation scheme:



$$= \sum_{i,j} \left({H_0}^{ij} {H_0}^{ji} \right)^2 \neq 0 \,. \tag{33}$$

Fig. 5. Planar and non-planar contributions to $\Sigma(z)$. The lines with a dot represent $\mathcal{G}(z)$.

Having said this, we note that the calculation of the resolvent would be simplified considerably if we could show that there *exists* a random matrix model $\mathcal{M}$ with a measure $P(\mathcal{M})$ such that

$$\langle \operatorname{tr} \mathcal{M}^n \rangle_{\mathrm{pl},\, c} = \left(\langle \operatorname{tr} \mathcal{V}^n \rangle_{\mathrm{pl},\, c} + \langle \operatorname{tr} \mathcal{V}^n \rangle_{\mathrm{np}} \right) . \tag{34}$$

Indeed, then $\Sigma(z)$ and hence $G(z)$ for our problem would simply follow from the *analogue* random matrix model $H_0 + \mathcal{M}$. The main advantage is that now only *planar* graphs in the random matrix model would contribute, for which the whole machinery of free random variables applies [10], in particular the addition law [10,13].

To show the existence of such random matrix model, we first note that all the non-planar parts $\langle \operatorname{tr} \mathcal{V}^i \rangle_{\mathrm{np}}$ can be absorbed into effective connected i-th moments. This amounts to modifying the probability distribution $P(\mathcal{M})$ to be different from the probability distribution of the $\mathcal{V}_A$'s.

We may now use the combinatorial relations discussed in [14], between connected and ordinary moments for *planar* diagrams to reduce the equality (34) just to the equality of moments (dropping the requirement of gluon-connectedness):

$$\langle \operatorname{tr} \mathcal{M}^n \rangle_{\mathrm{pl}} = \langle \operatorname{tr} \mathcal{V}^n \rangle \tag{35}$$

with the pertinent measures for each averaging. This is equivalent to finding a probability distribution $P(\mathcal{M}) = \exp(-N \operatorname{tr} \boldsymbol{V}(\mathcal{M}))$ such that the resolvents and consequently the eigenvalue distributions of $\mathcal{M}$ and $\mathcal{V}$ coincide.

To this end suppose that the eigenvalue distribution of $\mathcal{V}$ has support on the interval $[-1, 1]$. Consider $\rho_{\mathcal{V}}(\lambda)/\sqrt{1-\lambda^2}$. If this function is a polynomial we are done — we may read off the coefficients of the random matrix potential from the formulas of [14]. In the other case we may approximate it by polynomials to an arbitrary degree of accuracy and take the limit of the corresponding probability distributions. In any case an explicit formula for the probability distribution can be derived (see *e.g.* [15]):

$$\frac{d\boldsymbol{V}(z)}{dz} = G_{\mathcal{V}}(z + i\varepsilon) + G_{\mathcal{V}}(z - i\varepsilon)\,. \tag{36}$$

A matrix model with such a potential will satisfy all our requirements. However to apply the methods of free random variables we need only to know that such a model exists, the explicit form of the random matrix potential $\boldsymbol{V}$ through (36) is unnecessary. This concludes our proof of existence.

To summarize, our approximation scheme for (1) in the form of (32) includes all the planar graphs and resums the single-site rescatterings from the non-planar contributions to the resolvent. It is easily implemented in the matrix analogue using the addition law for $H_0 + \mathcal{M}$, as we have now shown.

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