ON POLYNOMIAL APPROXIMATION OF THE STATIC VORTEX IN ABELIAN HIGGS MODEL*

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The static vortex solution in Abelian Higgs model with small ratio of vector and Higgs particle masses is considered. Several formulae approximating this solution are discussed. The accuracy of these approximations is tested by numerical computations.

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1. Introduction

Nowadays vortex solutions are found to be interesting in many areas of physics. Their investigations can help in better understanding of some phenomena in field theory, cosmology and condensed matter physics [1]. It is rather difficult to obtain vortex solutions since one has usually to solve highly non-linear, very complicated differential equations. Therefore appropriate analytical and numerical methods must be worked out and applied to make some progress in this area.

One of the simplest systems possessing vortex solution is the Abelian Higgs model. An exact vortex solution, in the form of infinite convergent series, was found for this model in the so-called Bogomolny limit (the masses of the scalar and vector bosons are the same) [2]. However in this limit the equations of motion reduce to the first order differential equations and the underlying methods can not be simply applied in general case.

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Recently in the Abelian Higgs model a polynomial approximation was extensively used to investigate excited vortex [3,4]. This method gives analytical formulae simple enough to be applied in further computations. However, it is necessary to estimate the error of this approximation and this can be done by a comparison with the numerical solution.

The goal of the present paper is twofold. First we would like to test some approximating formulae for the static vortex solution proposed in paper [3]. These analytic formulae were obtained in the limit $\kappa \to 0$ where κ is the ratio of vector and Higgs particle masses. Our investigations are also limited to small values of κ. Second we would like to present more precise analytical approximations. The accurate analytical and numerical results for the static vortex solution can be useful in many problems apart from the excited vortex. For example, equations governing evolution of a curved vortex [5] involve constants which are determined by the static vortex solution.

Our paper is organized as follows. In Sec. 2 we shortly review the simple analytic formulae approximating the static vortex solution. As mentioned above these formulae were introduced and discussed in more detail in [3]. We also compare the results obtained this way with numerical approximations to test the accuracy of the algorithm. In Sec. 3 we propose several improvements of this method to get more precise analytical approximations and numerical results valid in wider region than the previous ones. Finally in Sec. 4 we present some general remarks and conclusions summarizing our paper.

2. Simple approximations

The Abelian Higgs model in $3+1$-dimensional Minkowski space-time is described by the following Euler–Lagrange equations

$$(\partial_\mu + iqA_\mu)(\partial^\mu + iqA^\mu)\Phi + \frac{\lambda}{2}\Phi\left(|\Phi|^2 - \frac{2m^2}{\lambda}\right) = 0, \tag{1}$$

$$\partial_\mu F^{\mu\nu} = iq(\Phi^*\partial^\nu\Phi - \Phi\partial^\nu\Phi^*) - 2q^2A^\nu|\Phi|^2. \tag{2}$$

The Higgs field $\Phi(x)$ is complex valued, the star denotes the complex conjugation. The signature of metric of the space-time is $(-+++)$. The static Abrikosov–Nielsen–Olesen solution [6] represents a straight linear, infinite vortex. The unit topological charge vortex lying along the z-axis can be obtained from the equations (1), (2) by imposing on them the axially symmetric Ansatz

$$\begin{aligned}
\Phi(\rho,\phi) &= \sqrt{\frac{2m^2}{\lambda}}\mathrm{e}^{i\theta}F(\rho)\,,\\
A_0 &= 0\,, \quad A_3 = 0\,,\\
A_1 &= \sin(\theta)\frac{\chi(\rho)-1}{q\rho}\,,\\
A_2 &= -\cos(\theta)\frac{\chi(\rho)-1}{q\rho}\,.
\end{aligned} \tag{3}$$

Here $\rho = \sqrt{(x^1)^2+(x^2)^2}$ and $\theta = \arctan(x^2/x^1)$ are the standard polar coordinates in the (x^1,x^2) plane.

The Ansatz (3) together with the rescaling of the ρ variable

$$r = 2m^2\rho\,, \tag{4}$$

leads to the second order differential equations for the fields $F(r)$ and $\chi(r)$

$$F'' + \frac{F'}{r} - \frac{\chi^2}{r^2}F + \frac{1}{2}(F - F^3) = 0\,, \tag{5}$$

$$\chi'' - \frac{\chi'}{r} - \kappa^2F^2\chi = 0\,, \tag{6}$$

where prime denotes differentiation with respect to r and $\kappa = \sqrt{2q^2/\lambda}$ is the only remaining free parameter.

To guarantee the vortex solution to be regular on x^3-axis and to have finite energy per unit of length in x^3 direction the above equations must be supplemented by the boundary conditions

$$F(0) = 0\,, \quad F(\infty) = 1\,, \tag{7}$$

$$\chi(0) = 1\,, \quad \chi(\infty) = 0\,. \tag{8}$$

The asymptotic form of χ can be easily obtained from Eqs (6), (8) simplified by putting $F = 1$. Thus we get

$$\chi_{\mathrm{asym}}(r) = c_0 r K_1(\kappa r) \approx c_0\sqrt{\frac{\pi r}{2}}\exp(-\kappa r)\,, \tag{9}$$

where K_1 is the modified Bessel function [7] and c_0 is a constant. The asymptotic behaviour of $F(r)$ strongly depends on κ [8]. For $\kappa < \frac{1}{2}$ it is determined mainly by the term χ^2F/r^2 and can be obtained from Eqs (5), (7)

$$F_{\mathrm{asym1}} = 1 - \frac{c_0^2\pi}{\kappa(1-4\kappa^2)r}\exp(-2\kappa r)\,, \tag{10}$$

while for $\kappa > \frac{1}{2}$ the Higgs term $(F - F^3)/2$ is more important. In this case Eq. (5) linearized in $(F-1)$ gives the following result

$$F_{\text{asym2}} = 1 + c_1 K_0(r) \approx 1 + c_1 \sqrt{\frac{\pi}{2r}} \exp(-r)\,. \tag{11}$$

Here K_0 denotes the zero-order modified Bessel function [7] and c_1 is a constant.

The asymptotic forms of F and χ were used in [3] to get the approximate static vortex solution. In the neighbourhood of $r = 0$ the fields F and χ were approximated by low order polynomials obtained as power series solutions of Eqs (5), (6). These polynomials were smoothly matched with the appropriate asymptotics at some point $r = r_0$ *i.e.* the functions $F(r)$ and $\chi(r)$ were required to be continuous at $r = r_0$ together with their first derivatives

$$F_{\text{poly}}(r_0) = F_{\text{asym}}(r_0)\,, \quad F'_{\text{poly}}(r_0) = F'_{\text{asym}}(r_0)\,, \tag{12}$$

$$\chi_{\text{poly}}(r_0) = \chi_{\text{asym}}(r_0)\,, \quad \chi'_{\text{poly}}(r_0) = \chi'_{\text{asym}}(r_0). \tag{13}$$

In the simplest version proposed in [3] the static vortex solution was approximated by the following formulae

$$F(r) = \begin{cases} f_1 r - \frac{1}{3!} f_3 r^3 & \text{if } r < r_0, \\ 1 & \text{if } r > r_0\,, \end{cases} \tag{14}$$

$$\chi(r) = \begin{cases} 1 - \frac{1}{2!}\chi_2 r^2 + \frac{1}{4!}\chi_4 r^4 - \frac{1}{6!}\chi_6 r^6 & \text{if } r < r_0\,, \\ c_0 r K_1(r) & \text{if } r > r_0\,, \end{cases} \tag{15}$$

involving four constants r_0, f_1, χ_2, c_0. These constants were fixed by applying the matching conditions described above. The other constants are given by recurrence relations (18) below.

In order to get more accurate solutions the formula for F in the region $r > r_0$ was replaced with a more subtle one

$$F = \sqrt{1 - 2\left(\frac{\chi}{r}\right)^2}\,, \tag{16}$$

obtained from Eq. (5) simplified by neglecting the terms with the derivatives of F. In this case the polynomial approximation of F must be completed with the term proportional to r^5

$$F = f_1 r - \frac{1}{3!} f_3 r^3 + \frac{1}{5!} f_5 r^5\,, \tag{17}$$

while the formulae for χ remain unchanged although the values of the particular parameters are different. The equations (5), (6) lead to the following recurrence relations for coefficients of the polynomials

$$\begin{aligned}
f_3 &= \frac{3}{4}\left(\frac{1}{2}+\chi_2\right) f_1, \\
\chi_4 &= 3\kappa^2 f_1^2, \\
f_5 &= \frac{5}{6}\left(\frac{1}{2}+\chi_2\right) f_3 + \frac{5}{2}\left(\frac{1}{6}\chi_4 + \frac{1}{2}\chi_2^2 + f_1^2\right) f_1\,, \\
\chi_6 &= 5\kappa^2 f_1\left(2f_3 + 3\chi_2 f_1\right)\,. \qquad (18)
\end{aligned}$$

In Fig. 1 we have compared the described above approximation of the Higgs field F with its numerical values obtained by applying standard algorithms for stiff differential equations [9]. As we are interested mainly in small values of κ we have limited ourselves to $\kappa = 0.05$, $\kappa = 0.1$ and $\kappa = 0.2$. Since the values of the field χ obtained from the numerical computations and approximate formula differ very slightly we have plotted their differences in Fig. 2 and the numerical values themselves in Fig. 3. The numerical values of the free parameters f_1, χ_2, r_0, c_0 are given in Table I and Table II.

TABLE I

$F_{\rm asym} = 1$

κ	f_1	h_2	r_0	c_0
0.02	0.6505427	0.00157722	2.305767	0.020004
0.1	0.6646855	0.02362296	2.256706	0.1005027
0.2	0.6929167	0.06904716	2.164762	0.2040623

TABLE II

$F_{\rm asym} = \sqrt{1-2(\chi/r)^2}$

κ	f_1	h_2	r_0	c_0
0.02	0.4285536	0.001431718	2.106883	0.0200066
0.1	0.443162	0.02028046	2.068759	0.1007802
0.2	0.469365	0.05754442	2.000099	0.2056864

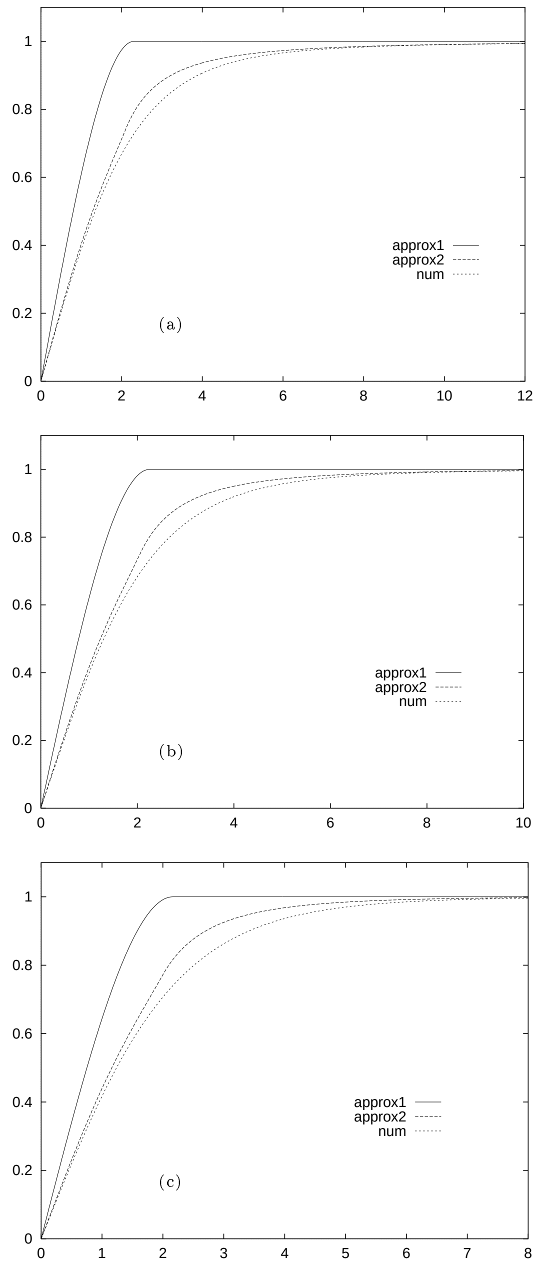

Fig. 1. The approximate and numerical values of the field F (a) — $\kappa = 0.05$, (b) — $\kappa = 0.1$, (c) — $\kappa = 0.2$.

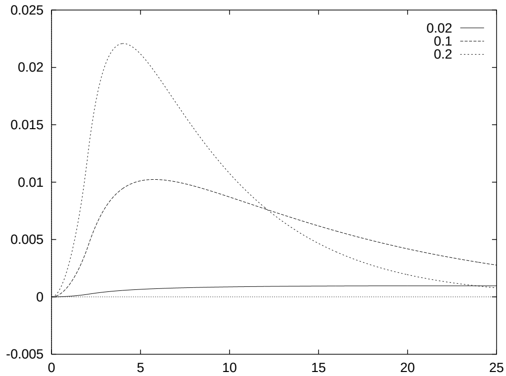

Fig. 2. The differences between approximate and numerical values of the field χ for $\kappa = 0.05, 0.1, 0.2$.

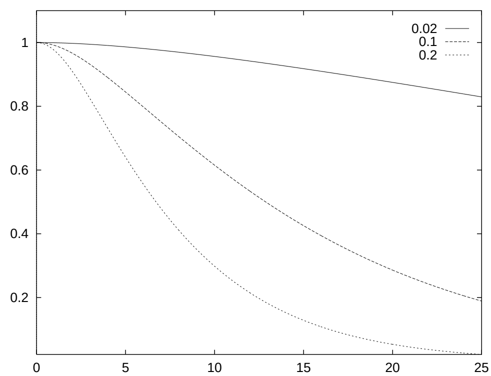

Fig. 3. The numerical values of the field χ for $\kappa = 0.05, 0.1, 0.2$.

Let us note that the approximate formula for the function χ is quite good while the approximation of F is much worse. This is the price for simplicity of the analytical expressions. The field F tends very quickly to its asymptotic form and such behaviour can be hardly described by simple analytical formula.

3. Improved approximate solutions

The main defect of the approximate formulae considered in the previous section is the behaviour of the field F in the region of intermediate values of r particularly in the neighbourhood of the matching point r_0. One can try to improve that approximation by using the higher order polynomial solution for the functions F and χ in the interval $(0, r_0)$. However the practical effect of such improvement seems to be rather small. A better accuracy can be reached by changing the approximation for the function F in the region (r_0, ∞). The more accurate asymptotics [3] is given by

$$F_{\text{asym}} = \sqrt{1 - 2\left(\frac{\chi_{\text{asym}}}{r}\right)^2} + c_1 K_0(r). \tag{19}$$

This formula involves a new parameter c_1 and an extra condition is necessary to determine it. Therefore we have used an additional matching condition ensuring the continuity of the second order derivative of F at $r = r_0$

$$F''_{\text{poly}}(r_0) = F''_{\text{asym}}(r_0). \tag{20}$$

We were able to satisfy the matching conditions if the polynomials approximating the functions F and χ were of order fifteen and fourteen or nineteen and eighteen, respectively. We do not present them as their forms are very complicated and the numerical results not excellent as is shown in Figs 4 and 5.

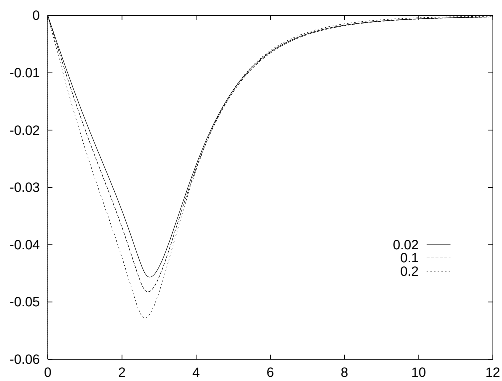

Fig. 4. The differences between approximate and numerical values of the field F for $\kappa = 0.05, 0.1, 0.2$ (second order derivative method).

There is also another simple possibility to determine the values of five parameters c_0, c_1, f_1, h_2, r_0. One can solve the four matching conditions (12), (13) for the fixed value of the radius r_0 and repeat this procedure for several values of r_0 in some interval. Thus we get the four parameters c_0, c_1, f_1, h_2 as the numerical functions of r_0 . The last step was to compare the approximations for F an χ obtained this way with numerical calculations and fix the value of r_0 which gives the best fitting. It turned out that in this case it was enough to approximate F and χ by polynomials of order five and four respectively, see Eqs (15), (17), (18) with neglected χ_6-terms.

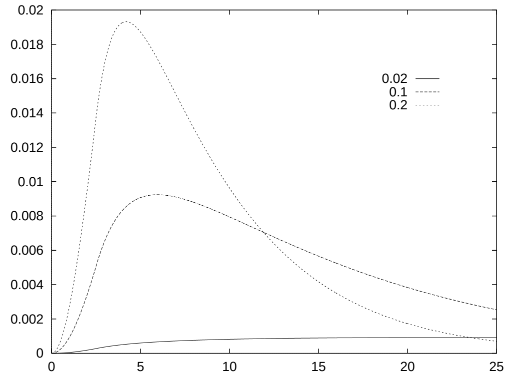

Fig. 5. The differences between approximate and numerical values of the field χ for $\kappa = 0.05, 0.1, 0.2$ (second order derivative method).

The differences between approximated and numerical values of F and χ are presented in Figs 6 and 7. The numerical values of the parameters f_1, χ_2, c_0, c_1 are given in Table III while $r_0 = 2.5$.

TABLE III

κ	f_1	h_2	c_0	c_1
0.02	0.3811035	0.001427667	0.0200068	1.368773
0.1	0.3885089	0.02025245	0.1007819	1.530144
0.2	0.3998954	0.05794554	0.2052965	1.851151

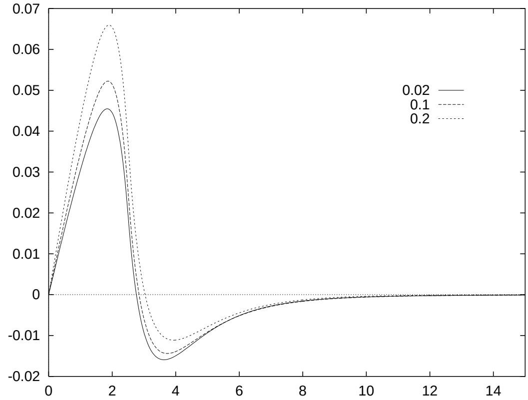

Fig. 6. The differences between approximate and numerical values of the field F for $\kappa = 0.05, 0.1, 0.2$ (fixed matching point algorithm).

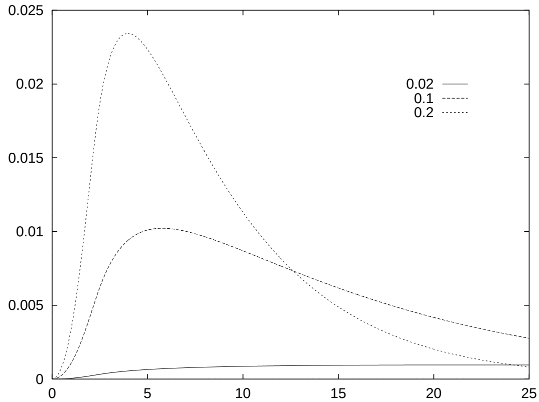

Fig. 7. The differences between approximate and numerical values of the field χ for $\kappa = 0.05, 0.1, 0.2$ (fixed matching point algorithm).

The vortex solutions considered so far were obtained by smooth matching of some polynomials approximating the vortex core with asymptotic formulae valid in the outer region. More accurate approximations can be obtained by dividing the whole area into more pieces and approximating the solution in each sector separately. We have chosen $k+1$ points $0 < r_0 < r_1 < \ldots < r_k$. The central part of the vortex in the interval $(0, r_0)$ was approximated by

$$F = f_1 r - \frac{1}{3!} f_3 r^3 + \frac{1}{5!} f_5 r^5 + \ldots + \frac{(-1)^n}{(2n+1)!} f_{2n_0+1} r^{2n_0+1} , \qquad (21)$$

$$\chi = 1 - \frac{1}{2!}\chi_2 r^2 + \frac{1}{4!}\chi_4 r^4 + \ldots + \frac{(-1)^n}{(2n)!}\chi_{2n_0} r^{2n_0} , \qquad (22)$$

while in (r_j, r_{j+1}) for $j = 0, 1, 2, \ldots, k-1$ we have used the truncated Taylor series expansions

$$F = \widetilde{f}_{0j} + \widetilde{f}_{1j}(r - r_j) + \frac{1}{2!}\widetilde{f}_{2j}(r - r_j)^2 + \ldots + \frac{1}{n!}\widetilde{f}_{nj}(r - r_j)^{n_j} , \quad (23)$$

$$\chi = \widetilde{\chi}_{0j} + \widetilde{\chi}_{1j}(r - r_j) + \frac{1}{2!}\widetilde{\chi}_{2j}(r - r_j)^2 + \ldots + \frac{1}{n!}\widetilde{\chi}_{nj}(r - r_j)^{n_j} . \quad (24)$$

In the region (r_k, ∞) the previous asymptotic formulae (9), (19) were applied. We have required the functions F and χ together with their first derivatives to be continuous in the matching points $r_0, r_1, \ldots, r_k$. These conditions and Eqs (5), (6) are enough to determine all the coefficients $f_j, \chi_j, f_{ij}, \chi_{ij}, c_0, c_1$. Let us note that passing from $r = 0$ to $r = r_k$ resembles the process of analytic continuation and is rather easy to perform. The main difficulty is to bind these solutions with their appropriate asymptotics.

The above formulae were applied in two ways. In equations (23), (24) we have firstly put the expansion order to 4 ($n_0 = 2, n_1 = 4$) and have divided the whole region of r into three pieces ($r_0 = 2, r_1 = 3$). In the region $(0, r_0)$ the formulae (21), (22) for $n_0 = 2$ reduce to (15), (17), (18) with neglected χ_6-terms while in the interval (r_0, r_1) the following recurrence relations are valid in Eqs. (23), (24)

$$
\begin{aligned}
\widetilde{f}_2 &= \frac{-\widetilde{f}_0}{2} + \frac{\widetilde{f}_0^3}{2} - \frac{\widetilde{f}_1}{r_0} + \frac{\widetilde{f}_0\widetilde{\chi}_0^2}{r_0^2}, \\
\widetilde{\chi}_2 &= \kappa^2 \widetilde{f}_0^2 \widetilde{\chi}_0 + \frac{\widetilde{\chi}_1}{r_0}, \\
\widetilde{f}_3 &= \frac{\widetilde{f}_1}{r_0^2} + \frac{2\widetilde{f}_0\widetilde{\chi}_0\widetilde{\chi}_1}{r_0^2} + \frac{\widetilde{\chi}_0^2\widetilde{f}_1}{r_0^2} - \frac{\widetilde{f}_2}{r_0} - \frac{\widetilde{f}_1}{2} + \frac{3\widetilde{f}_0^2\widetilde{f}_1}{2} - \frac{2\widetilde{f}_0\widetilde{\chi}_0^2}{r_0^3}, \\
\widetilde{\chi}_3 &= -\frac{\widetilde{\chi}_1}{r_0^2} + \kappa^2 \widetilde{f}_0^2 \widetilde{\chi}_1 + 2\kappa^2 f_0 \widetilde{f}_1 \widetilde{\chi}_0 + \frac{\widetilde{\chi}_2}{r_0}, \\
\widetilde{f}_4 &= -\frac{2\widetilde{f}_1}{r_0^3} + \frac{6\widetilde{f}_0\widetilde{\chi}_0^2}{r_0^4} - \frac{\widetilde{f}_3}{r_0} - \frac{f_2}{2} + \frac{3\widetilde{f}_2\widetilde{f}_0^2}{2} + 3\widetilde{f}_1^2\widetilde{f}_0 \\
&\quad + \frac{2\widetilde{f}_0\widetilde{\chi}_0\widetilde{\chi}_2}{r_0^2} + \frac{2\widetilde{f}_0\widetilde{\chi}_1^2}{r_0^2} + \frac{\widetilde{f}_2\widetilde{\chi}_0^2}{r_0^2} + \frac{4\widetilde{f}_1\widetilde{\chi}_0\widetilde{\chi}_1}{r_0^2} + \frac{2\widetilde{f}_2}{r_0^2} \\
&\quad - \frac{8\widetilde{f}_0\widetilde{\chi}_0\widetilde{\chi}_1}{r_0^3} - \frac{4\widetilde{f}_1\widetilde{\chi}_0^2}{r_0^3}, \\
\widetilde{\chi}_4 &= -\frac{2\widetilde{\chi}_2}{r_0^2} + \frac{2\widetilde{\chi}_1}{r_0^3} + \kappa^2 \widetilde{f}_0^2 \widetilde{\chi}_2 + 4\kappa^2 \widetilde{f}_0 \widetilde{f}_1 \widetilde{\chi}_1 \\
&\quad + 2\kappa^2 \widetilde{f}_0 \widetilde{\chi}_0 \widetilde{f}_2 + 2\kappa^2 \widetilde{\chi}_0 \widetilde{f}_1^2 + \frac{\widetilde{\chi}_3}{r_0},
\end{aligned}
$$

where the second index ($j = 0$) is omitted for simplicity. The numerical values of the remaining parameters are found from the matching conditions and they are given in Table IV and Table V. The rather simple but quite accurate analytical approximation of the static vortex solution was obtained this way, as is presented in Figs 8 and 9.

TABLE IV

κ	f_1	h_2	c_0	c_1
0.02	0.3980053	0.1392637	0.02000953	1.273493
0.1	0.4078238	0.01947355	0.1011135	1.34181
0.2	0.4238514	0.05537332	0.2079306	1.479463

TABLE V

κ	$\widetilde{f}_0$	$\widetilde{f}_1$	$\widetilde{\chi}_0$	$\widetilde{\chi}_1$
0.02	0.6551700	0.2454606	0.9973415	-0.002531821
0.1	0.6676333	0.2496415	0.9643793	-0.03229428
0.2	0.6863076	0.2557591	0.9036254	-0.08200263

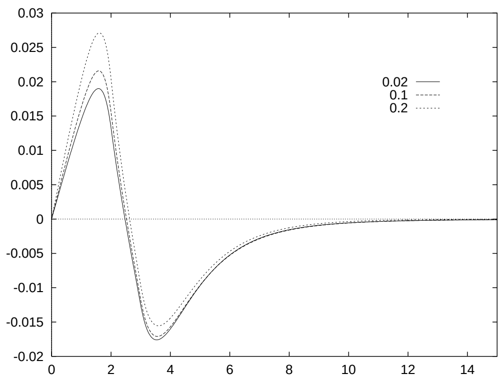

Fig. 8. The differences between approximate and numerical values of the field F for $\kappa = 0.05, 0.1, 0.2$ (two matching points method).

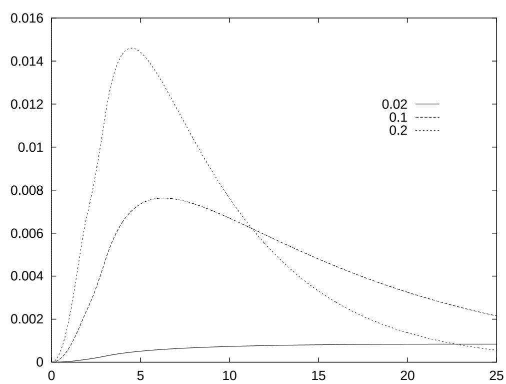

Fig. 9. The differences between approximate and numerical values of the field χ for $\kappa = 0.05, 0.1, 0.2$ (two matching points method).

In the second case we have tried to get the possibly most accurate numerical results. Therefore we have put large $n_0 = 9$, $n_j = 10$ and very small value of $r_{j+1} - r_j = 0.01$ for $r_0 = 1$, $r_k = 20$. The corresponding results are shown in Figs 10 and 11.

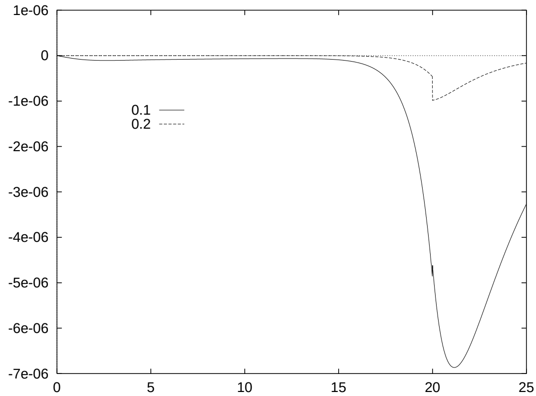

Fig. 10. The differences between approximate and numerical values of the field F for $\kappa = 0.1, 0.2$ (analytic continuation algorithm).

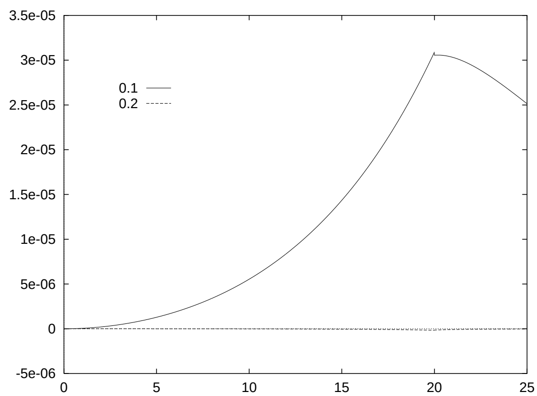

Fig. 11. The differences between approximate and numerical values of the field χ for $\kappa = 0.1, 0.2$ (analytic continuation algorithm).

4. Remarks

In the present paper we have considered several formulae approximating the vortex solution in the Abelian Higgs model. We started with simple analytical formulae presented in [3] and compared them with numerical computations. It turned out that the approximation of the Higgs field in the neighbourhood of the vortex core is rather rough and should be improved to get more accurate results. We have tried several methods to reach this goal.

First of all we have changed the formula describing the Higgs field in the outer region. This formula should not be interpreted as the better asymptotics only. Perhaps more important is the fact that this expression involves a new free parameter which lets us improve the whole algorithm. We have used this possibility in several ways. At this point it is worth noting that the relative error of the function $F-1$ with F given by the approximate formula (16) does not tend to zero for large r. This can be easily seen by comparing Eqs (9), (10) and (16).

Our first trial to improve the accuracy of the approximation was the algorithm with an additional matching condition ensuring the continuity of the second derivative of the Higgs field. The next possibility we have tried was to solve the matching conditions in some fixed point and repeat this step several times in different points to choose finally the best matching point on the basis of numerical results.

At last we have modified the algorithm by solving the equations of motion approximately as the truncations of the Taylor series expansions around an arbitrary point. We have used these solutions in the manner resembling the process of analytical continuation. This way we have obtained both: our best numerical approximations of the static vortex solutions and quite simple but accurate analytical formulae generalizing those from [3]. It was possible because the final version of the algorithm turned out to be very flexible and could be applied to reach apparently different purposes: analytical simplicity of expressions and numerical accuracy of computer calculations.

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