ROLE OF SHORT RANGE POTENTIALS IN SOLVING THE EIGENVALUE PROBLEM FOR THE THREE BODY DIRAC EQUATION

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The eigenvalue problem for a bound state solution of three quarks requires deep analysis to even start a numerical attempt. A power series solution to the three body Dirac equation solved in hypercentral approximation is sought. A scalar linear flux tube three body string potential is used to confine the quarks. In addition one gluon exchange potentials (OGEP) between quark pairs are considered to model the short range interactions. The angular momentum barrier is found to dominate the wave function behavior at the origin when including only the magnetic part of the OGEP. This occurs when the Coulomb part of the OGEP is neglected, or canceled by terms of opposite sign from the scalar potential. Recurrence relations for the power series coefficients are determined. When the Coulomb part of the OGEP is included, the initial ratios of the composite three quark wave function components are also determined. In this case, the Coulomb strength of the OGEP combines with the angular momenta to determine the wave function behavior near the origin.

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1. Introduction

A confined three quark model, with small current quark masses [1], is used to describe the proton. The necessarily relativistic quark dynamics are described by the three body Dirac equation. The masses of the up and down quarks are taken as the same. Hyperspherical coordinates are used which properly handles the center of mass problem in the system rest

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frame [2,3]. The hyperradius squared is defined from the sum of the squared quark pairwise separations as:

$$r^2 = \frac{1}{2}\left(r_{12}^2 + r_{13}^2 + r_{23}^2\right) . \tag{1}$$

A possible set of the five hyperangles are as follows. The location of the three quarks defines a triangle. Two of the interior angles of this triangle are the first two hyperangles. The triangle has a normal. The direction of this normal defines two more hyperangles. The azimuthal orientation of the triangle about the normal is the fifth hyperangle. The hyperspherical expansion utilizes a sum over configurations in describing the three quark composite wave function. This expansion is truncated, in the hypercentral approximation, to a single configuration [4]. The quarks are each assumed to be in the $(1/2^+)^3$ configuration, coupled to a total spin of $1/2$, for the proton ground state.

A set of eight coupled first order radial differential equations in the hyperradius are obtained, after integration over the hyperangles, for the eight components of the composite three quark wave function. With equal masses, equivalent sets of quantum numbers for each quark, and flavor independent forces, symmetry conditions reduces this to a set of four coupled first order differential equations involving four unknown components. The differential variable is the hyperradius. A scalar linear confining three body potential, V, plus one gluon exchange potentials (OGEP) V_{12}, V_{13}, V_{23} between quark pairs is considered. The scalar potential is parameterized as $V = bS$, where S is the minimum flux tube length between the three quarks.

Such a potential is common [5–7] but the OGEP contribution is often included as a perturbation [8,9]. Here the OGEP contributions are sought to be kept completely in all stages of the search for a solution. The string constant b, is $0.18\mathrm{GeV}^2$ [10] for quarks.

The one gluon exchange potential [10] between quark pairs is:

$$V_{12} = -(2\alpha_s/3r_{12})(1 + \vec{\alpha_1}\cdot\vec{\alpha_2}) . \tag{2}$$

The subscripts are particle labels, and the $\vec{\alpha}$ and β are Dirac matrices associated with a given particle. α_s is the strong coupling constant, viewed here as a potential parameter. The $-(2\alpha_s/3r_{12})(1)$ term in the OGEP above is called the Coulomb term, the rest of the OGEP is called the magnetic term. In the representation used, the Dirac $\vec{\alpha}$ matrix is just the Pauli $\vec{\sigma}$ matrices along the skew diagonal.

A numerical solution of these equations is an eigenvalue problem where given one of either the potential or the system rest energy, the other is guessed at until a solution is found that satisfies appropiate boundary conditions at the origin and at infinity.

The difficulty is that this is a problem in a multi-parameter space. The overall wave function normalization can set the scale of one of the wave function components. But one must still guess the ratio of three of the other components to the value of the first component at, or near, the origin to even start off an attempted numerical solution. If such an attempt fails to satisfy the boundary conditions, the failure can be due to the ratios assumed, or to the rest energy guess utilized by the numerical solution.

If the wave function components are all analytic, and therefore expandable in a power series of say, ascending powers of the hyperradius, then the multiparameter aspect of this eigenvalue problem can be avoided. If the coefficients of the power series expansions can all be determined recursively in terms of the rest energy of the system and the potential parameters, then the eigenvalue problem reduces to just one parameter, the value of the rest energy utilized. The power series solution can then be used for any hyperradial range if enough terms in the series are included. Including only a small finite number of terms of the series will still provide wave function components of sufficient accuracy for small hyperradial values. These components can then be used to determine the starting values of the component ratios for the numerical solution of the coupled differential equations, thereby avoiding the multiparameter aspect of the eigenvalue problem.

The main point of this paper is to show that this can be done for the three body Dirac equation solved in hypercentral approximation. A scalar linear flux tube three body string potential is used to confine the quarks. In addition, one gluon exchange potentials (OGEP) between quark pairs are considered to model the short range interactions. The Coulomb contribution of the OGEP affects the wave function behavior at the origin, but not so for the magnetic contribution. The Coulomb contribution dominates over the angular momentum barrier near the origin if the one gluon exchange potential is completely kept. If just the magnetic part of the OGEP is included, then the angular momentum barier dominates the wave function behavior near the origin.

2. Analysis of radial Dirac equation for 3 quarks

The three body Dirac equation is:

$$\left\{ \begin{array}{c} (\vec{\alpha_1}\cdot\vec{p_1}+m_1\beta_1)+V_{12} \\ +(\vec{\alpha_2}\cdot\vec{p_2}+m_2\beta_2)+V_{13}+V_{23} \\ +(\vec{\alpha_3}\cdot\vec{p_3}+m_3\beta_3)+\beta_1\beta_2\beta_3 V \end{array} \right\} \Psi = E\Psi\,. \tag{3}$$

After the hyperangular integration [2], a set of eight coupled linear differential equations are obtained [11,12] in the hypercentral approximation. With f,g denoting the upper and lower component in the two component

Dirac notation for a single quark wave function, the eight components, labeled R_1 through R_8 of the composite three quark wave function, denote the composite combinations $fff, gff, fgf, ggf, ffg, gfg, fgg$, and ggg. The normalization of these components is the same as in [4,13]. The first component is the one that survives in the non-relativistic limit. For equal mass quarks, and flavor independent forces, there is a symmetry such that we have the component relations [12], $R_2 = R_3 = R_5$, and that $R_4 = R_6 = R_7$. The coupled differential equations for the four unknown components can be represented as the matrix equation, $M\Psi = 0$, where Ψ is the column vector, R_1, R_2, R_4, and R_8. The matrix equation is:

$$\begin{bmatrix} M_1 & \text{-D(5)} & 0 & 0 \\ \text{D(0)} & M_2 & \text{-D(6)/2} & 0 \\ 0 & \text{2D(-1)} & M_4 & \text{-D(7)/5} \\ 0 & 0 & \text{3D(-2)} & M_8 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_4 \\ R_8 \end{bmatrix} = 0\,. \tag{4}$$

The relativistic kinetic energy appears in the operator

$$D(n) = \frac{d}{dr} + \frac{n}{r}\,. \tag{5}$$

The diagonal matrix elements are combinations of the mass terms, the energy, the scalar potential, the Coulomb part and the magnetic part of the OGEP terms. The magnetic part has off diagonal matrix elements in the 8 by 8 representation, but due to the symmetry relations of the components, it appears along the diagonal of the 4 by 4 system of equations. It is convenient to have a dimensionless variable, so we define $y = Lr$, where L is an arbitrary non zero energy (or inverse length) unit and y is dimensionless. We divide the equations above by L, so that their forms remains the same if we now take the relativistic kinetic energy operator as:

$$D(n) = \frac{d}{dy} + \frac{n}{y}\,. \tag{6}$$

The diagonal matrix elements are $M_1 = -e_1 + K_1 y - c_1/y$, where the scalar potential contribution is in $K_1 = \text{b } 1.59\ \sqrt{(2/3)}/L^2$. b is the string constant and the coefficients appearing in K_1 come from the hyperangular integration over the string length, S, and from the change of variables from r to y. The energy and mass terms are in $e_1 = (E - 3M)/L$, where E is the system rest energy, and M is the quark rest mass energy. The Coulomb part of the OGEP is reflected in $c_1 = 48a_0$, where

$$a_0 = \alpha_s \frac{\sqrt{3/2}}{\pi}\,. \tag{7}$$

The second diagonal matrix element is $M_2 = -e_2 + K_2 y - c_2/y$. Here K_2 is minus K_1, as the scalar contribution has flipped sign in the Hamiltonian representation. The energy and mass term is $e_2 = (E - M)/L$. c_2 has a contribution of $264a_0/9$ from the Coulomb, and an additional $(32/9)a_0$ from the magnetic part of the OGEP.

The third diagonal matrix element is $M_4 = -e_4 + K_4 y - c_4/y$. Here K_4 is equal to K_1, and the energy term is $e_4 = (E + M)/L$. c_4 has a contribution of $(208/27)a_0$ and also $(32/27)a_0$ from the Coulomb and magnetic parts of the OGEP, respectively. The last diagonal matrix element is $M_8 = -e_8 + K_8 y - c_8/y$. Here the scalar contribution is again the negative, K_8= -K_1. The energy and mass term is $e_8 = (E + 3M)/L$. $c_8 = (8/9)a_0$, all from the Coulomb part of the OGEP, none from the magnetic part.

The matrix equation of coupled differential relations can be solved for the derivative terms resulting in the matrix equation:

$$d\Psi/dy = B\Psi \,, \tag{8}$$

where B is the matrix:

$$\begin{bmatrix} 0 & -M_2 & 4/\mathrm{y} & -M_8/6 \\ M_1 & \text{-}5/\mathrm{y} & 0 & 0 \\ 0 & 0 & 2/\mathrm{y} & -M_8/3 \\ 10M_1 & \text{-}60/\mathrm{y} & 5M_4 & \text{-}7/\mathrm{y} \end{bmatrix} . \tag{9}$$

3. The power series solution

We will now seek and find a power series solution to these differential equations. We set $R_1 = \Sigma A_n y^n$, where the sum is over n going from zero through positive integers. Also we expand $R_2 = \Sigma B_n y^n$, $R_4 = \Sigma C_n y^n$, and $R_8 = \Sigma D_n y^n$. The derivative with respect to y of R_1 is then just $\Sigma n A_n y^{n-1}$, with similar expressions for the derivatives of the other components. The coefficients of these power series expansions are unknown, and the goal is to determine all of the coefficients recursively. These power series expansions are substituted into the above matrix equation solved for the derivative terms. The coefficients of like powers of y are then equated in the matrix relations. One then obtains the relations:

$$\text{I} \quad (n+1)A_{n+1} = e_2 B_n - K_2 B_{n-1} + c_2 B_{n+1} + 4C_{n+1} + \frac{e_8 D_n - K_8 D_{n-1} + c_8 D_{n+1}}{6}$$

$$\text{II} \quad (n+6)B_{n+1} = -e_1 A_n + K_1 A_{n-1} - c_1 A_{n+1}$$

$$\text{III} \quad (n-1)C_{n+1} = \frac{e_8 D_n - K_8 D_{n-1} + c_8 D_{n+1}}{3}$$

IV $(n+8)D_{n+1} = -10e_1 A_n + 10K_1 A_{n-1} - 10c_1 A_{n+1} - 60B_{n+1} - 5e_4 C_n + 5K_4 C_{n-1} - 5c_4 C_{n+1}$.

Now use II and III to replace the B_{n+1} and C_{n+1} in the right hand side of IV. Then solve the resulting equation for D_{n+1}. These are the recurrence relations for the unknown coefficients of the series expansions of the wave function components.

Set all expansion coefficients with negative subscripts to zero. Set the coefficient, A_0 to unity. This is equivalent to the normalization condition. After a solution has been found, the components can all be rescaled by a common factor to satisfy a desired normalization. A solution where the angular momentum dominates the wave function behavior at the origin is found only if c_1 is zero. From II, with n set to -1, we get $B_0 = -c_1/5$. From I, with $n = -1$, we get $C_0 = -c_2 B_0/4 = c_1 c_2/20$. From III, with $n = -1$, we get $D_0 = -6c_2 c_1/20c_8$. From IV, with $n = -1$, we get $7D_0 = -10c_1 A_0 - 60B_0 - 5c_4 C_0$.

These must be simultaneously true, so we obtain from the last relation,

$$c_1 \left(2 + c_2 \frac{2.1}{c_8} - \frac{c_4}{4} \right) = 0\,. \tag{10}$$

For the angular momentum barrier to dominate at small y, we expect R_2 to be proportional to y there, and R_4 and R_8 proportional to y^2 and y^3 respectively. This happens only if c_1 is zero. If c_1 is not zero, then the index power, g, of the power series must instead be first determined. See below for that case. With c_1 set to zero, then the coefficients B_0, C_0, and D_0 are all zero. With this restriction, then II becomes:

IIb $(n+6)B_{n+1} = -e_1 A_n + K_1 A_{n-1}$.

And for n set to 0, we have,

$$B_1 = \frac{-e_1 A_0}{6}. \tag{11}$$

With $n = 0$, III yields,

$$C_1 = \frac{-c_8 D_1}{3} \tag{12}$$

and IV, with $n = 0$ yields:

$$8D_1 = -5c_4 C_1\,. \tag{13}$$

These are simultaneously satisfied only if both C_1 and D_1 are zero. For $n = 1$, III demands that D_2 =0. I with $n = 0$ yields that $A_1 = c_2 B_1$, and thus the equations IIb, IV, III, and I can be used recursively for successive values of n to determine the coefficients of the power series solutions. This solution also satisfies the expectation, from the angular momentum dominance at the origin, that the components R_2, R_4, and R_8 vanish near the origin, with the first, second and third power of the radius respectively.

4. Discussion of the Coulomb term c_1 being zero

There are two ways for c_1 to be zero. One can neglect the Coulomb part of the OGEP, arguing that the magnetic part of this potential is all that needs to be included in attempts to explain the Delta-proton mass difference. For that mass difference, the Coulomb part of the OGEP will contribute zero anyway. The other way is to add to the scalar potential a term that cancels the Coulomb OGEP contribution to c_1. This can be easily done, but involves changing simultaneously the other values, c_2, c_4, and c_8 of the diagonal matrix elements. This also involves using a potential not inspired from QCD considerations. Certain combinations of a scalar and vector potential that has both linear confining terms and magnetic OGEP type terms [12,13] yield analytic wave function components. The power series solutions found here by including the first 40 terms, not shown, well reproduce these analytic solutions for values of y less than 10.

The power series solution, dominated at the origin by the Coulomb term when c_1 is not zero, is now discussed. This case of including the Coulomb term of the OGEP seems reminiscent of the slight divergence at the origin of the Dirac equation solution to the Hydrogen Coulomb problem [14]. If one includes only the $1/y$ terms in the matrix B above, near the origin, one can ask that each component is proportional to y^g, where g is the index power to be determined. Substituting this anzatz into the equation, $d\Psi/dy = B\Psi$, one finds that the wave function components constants of proportionality are all zero unless the determinant of the coefficients is zero. Therefore, one has det $G = 0$, where G is the matrix:

$$G = \begin{bmatrix} -g & c_2 & 4 & c_8/6 \\ -c_1 & (-g\text{-}5) & 0 & 0 \\ 0 & 0 & (-g+2) & c_8/3 \\ -10c_1 & -60 & -5c_4 & (-g\text{-}7) \end{bmatrix}. \tag{14}$$

The requirement that det $G = 0$ results in a quartic equation for g. Physical reasoning must be applied to determine which of the roots of the quartic is allowed. For instance, we expect g to be real. As the hyperspherical volume element contains $r^5 dr$, g cannot be more negative than -5/2, from normalization considerations. If c_8 is small, because the strong coupling constant is small, or if c_8 is viewed as small compared to c_1, (c_8/c_1 =1/54) then the quartic equation for g simplifies to a quadratic, resulting in

$$g = \frac{-5 + \sqrt{25 - 4c_1 c_2}}{2}. \tag{15}$$

In this limit, g is small and negative and satisfies the normalization constraint. If c_1 is zero, of course we see that the index power is zero, as in the power series solution above. If c_1 is not small, then the smallest real root of the quartic larger than -5/2 is the desired root. If c_1 is too large only complex roots for g are found. This corresponds to the strong coupling constant, α_s exceeding about 0.18 from the quadratic approximation for g in the above equation. Then the method fails, as does the Hydrogen Coulomb problem with large Z [14]. Once g is determined, then the constants of proportionality can be determined in terms of the proportionality constant chosen for the first wave function component. Knowing these ratios and g, one is prepared to numerically solve the eigenvalue problem for the coupled radial differential equations in the Coulomb case.

5. Summary

A power series solution for the three body Dirac equation solved in hypercentral approximation has been found. Such a solution is necessary to begin numerical studies of three quark models of the baryons. One has an eigenvalue problem to solve where the boundary conditions at the origin and at infinity must be met. The numerical approach to a solution requires the component ratios near the origin where the boundary conditions are applied to start the numerical solution. The power series solution allows the ratios of the various components of the composite three quark wave function to be determined near the origin. These ratios depend only on the energy and the potential parameters, and the quark masses. Determining these ratios by a power series allows the numerical eigenvalue problem to be reduced to one variable, the system rest energy. Without the power series solutions, the numerical solution of the coupled differential equations requires one to guess the ratios of the components to the composite wave function, while trying to guess the system rest energy.

When the angular momentum dominates the wave function behavior near the origin, recursion relations for obtaining such a power series solution can be found. This happens when the potential does not diverge as $1/r$ at the origin for the large component of the composite wave function. This is the component that survives in the non-relativistic limit. To bring this about, the Coulomb part of the OGEP must be neglected or partially cancelled out by a scalar term introduced for that purpose. If the Coulomb term is included, then it dominates the wave function behavior at the origin jointly with the angular momentum. In this case, the initial hyperradial behavior of the wave function components as well as their ratios depend on the strong coupling constant. This analysis determines the composite wave function component ratios needed to obtain the starting values for a numerical solution to the eigenvalue problem.

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