A SLOW AND CAREFUL LEGENDRE TRANSFORMATION FOR SINGULAR LAGRANGIANS* **

Włodzimierz M. Tulczyjew

Istituto Nazionale di Fisica Nucleare
Sezione di Napoli, Italy
e-mail: tulczy@camserv.unicam.it

and Paweł Urbański

Division of Mathematical Methods in Physics
University of Warsaw
Hoża 74, 00-682 Warszawa, Poland
e-mail: urbanski@fuw.edu.pl

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Dedicated to the memory of Leopold Infeld, our teacher

A comparative analysis of two different versions of the Legendre transformation is presented. A complete geometric background for this analysis is provided. Examples of applications to a number of physical systems are given.

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1. Introduction

In the present note we address the issue of singular Lagrangians in analytical mechanics. Deriving Hamiltonian formulations of physical systems with singular Lagrangians was attempted by Dirac and Bergmann [1]. The aim was to obtain Hamiltonian formulations of relativistic field theories although Dirac formulated his theory in terms of finite dimensional geometry. Applying Dirac procedures to relativistic mechanical systems we find that in most cases the resulting Hamiltonian description contains less information

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than was available in the Lagrangian formulation. We propose a version of the Legendre transformation without this defect.

In a recent paper Cendra, Holm, Hoyle, and Marsden [2] express the opinion that Lagrangian systems and Hamiltonian systems offer different representations of the same object. The Legendre transformation is the passage from one of these representations to the other. We agree with these concepts. We also agree with the statement that "one should *do the Legendre transformation slowly and carefully* when there are degeneracies". We think that our Legendre transformation is slow and careful enough to provide the correct Hamiltonian representation of relativistic mechanical systems.

We provide an almost complete although somewhat superficial review of the geometric background for analytical mechanics. Complete coordinate characterizations of all structures are provided. Intrinsic constructions of most of the objects are given. A more rigorous version of this material is in preparation. Related material can be found in [10,12,13].

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2. Geometry of tangent and cotangent bundles

Let Q be a differential manifold of dimension m. We use a coordinate system or a chart

$$\begin{aligned}(q^\kappa) \ &: \ Q \to \mathbb{R}^m \\ &: \ x \mapsto (q^\kappa)(x) = (q^1(x), \dots, q^m(x)).\end{aligned} \tag{1}$$

Each individual coordinate is a function

$$q^\kappa \colon Q \to \mathbb{R}. \tag{2}$$

We ignore the fact that the domain of a chart could be an open submanifold of Q and not all of Q.

Let F be a differentiable function on Q. The function

$$F \circ (q^\kappa)^{-1} \colon \mathbb{R}^m \to \mathbb{R} \tag{3}$$

is the coordinate expression of the function F. It is a function of the coordinates $(q^\kappa(q)) \in \mathbb{R}^m$ of a point $q \in Q$. We define partial derivatives

$$\partial_\kappa F = \frac{\partial (F \circ (q^\mu)^{-1})}{\partial q^\kappa(v)} \circ (q^\mu). \tag{4}$$

These partial derivatives are functions on Q.

The *tangent bundle* of a manifold Q is a manifold $\mathsf{T}Q$. There is a mapping

$$\tau_Q\colon \mathsf{T}Q \to Q \tag{5}$$

called the *tangent fibration*. Tangent vectors (elements of $\mathsf{T}Q$) are equivalence classes of curves in Q. Two curves $\gamma\colon \mathbb{R} \to Q$ and $\gamma'\colon \mathbb{R} \to Q$ are equivalent if $\gamma'(0) = \gamma(0)$ and $\mathrm{D}(f \circ \gamma')(0) = \mathrm{D}(f \circ \gamma)(0)$ for each function $f\colon Q \to \mathbb{R}$. The equivalence class of a curve $\gamma\colon \mathbb{R} \to Q$ will be denoted by $\mathsf{t}\gamma(0)$. Coordinates

$$\begin{aligned}(q^\kappa, \delta q^\lambda) \;&:\; \mathsf{T}Q \to \mathbb{R}^{2m} \\ &:\; v \mapsto (q^1(v), \ldots, q^m(v), \delta q^1(v), \ldots, \delta q^m(v))\end{aligned} \tag{6}$$

are induced by coordinates (q^κ) in Q. If γ is a representative of a vector v, then $q^\kappa(v) = q^\kappa(\gamma(0))$ and $\delta q^\lambda(v) = \mathrm{D}(q^\lambda \circ \gamma)(0)$. The tangent fibration is defined by

$$\tau_Q(\mathsf{t}\gamma(0)) = \gamma(0). \tag{7}$$

Fibres of this fibration are vector spaces. We have operations

$$+\colon \mathsf{T}Q \underset{(\tau_Q, \tau_Q)}{\times} \mathsf{T}Q \to \mathsf{T}Q \tag{8}$$

and

$$\cdot\colon \mathbb{R} \times \mathsf{T}Q \to \mathsf{T}Q \tag{9}$$

with coordinate representations

$$(q^\kappa, \delta q^\lambda)(v_1 + v_2) = (q^\kappa(v_1), \delta q^\lambda(v_1) + \delta q^\lambda(v_2)) \tag{10}$$

and

$$(q^\kappa, \delta q^\lambda)(k \cdot v) = (q^\kappa(v), k\delta q^\lambda(v)). \tag{11}$$

We denote by $\mathsf{T}Q \times (\tau_Q, \tau_Q)\mathsf{T}Q$ the set

$$\{(v_1, v_2) \in \mathsf{T}Q \times \mathsf{T}Q;\ \tau_Q(v_1) = \tau_Q(v_2)\}. \tag{12}$$

Since representatives of vectors (curves in Q) can not be added the construction of linear operations in fibres of τ_Q is somewhat indirect. Let $v = \mathsf{t}\gamma(0)$, $v_1 = \mathsf{t}\gamma_1(0)$, and $v_2 = \mathsf{t}\gamma_2(0)$ be elements of the same fibre $\mathsf{T}_qQ = \tau_Q^{-1}(q)$. We write

$$v = v_1 + v_2 \tag{13}$$

if

$$\mathrm{D}(f \circ \gamma)(0) = \mathrm{D}(f \circ \gamma_1)(0) + \mathrm{D}(f \circ \gamma_2)(0) \tag{14}$$

for each function f on Q. We have defined a relation between three elements of a fibre T_qQ. This relation will turn into a binary operation if we show that for each pair $(v_1, v_2) \in \mathsf{T}_qQ \times \mathsf{T}_qQ$ there is an unique vector $v \in \mathsf{T}_qQ$ such that $v = v_1 + v_2$. The coordinate construction

$$(q^\kappa \circ \gamma)(s) = (q^\kappa(v_1) + (\delta q^\kappa(v_1) + \delta q^\kappa(v_2))s) \tag{15}$$

of a representative γ of v proves existence. Let $v = \mathsf{t}\gamma(0)$ and $v' = \mathsf{t}\gamma'(0)$ be in relations $v = v_1 + v_2$ and $v' = v_1 + v_2$ with $v_1 = \mathsf{t}\gamma_1(0)$ and $v_2 = \mathsf{t}\gamma_2(0)$. Then

$$\mathrm{D}(f \circ \gamma')(0) = \mathrm{D}(f \circ \gamma)(0) = \mathrm{D}(f \circ \gamma_1)(0) + \mathrm{D}(f \circ \gamma_2)(0) \tag{16}$$

for each function f on Q. It follows that γ' and γ represent the same vector $v' = v$. This proves uniqueness. Let $v = \mathsf{t}\gamma(0)$ and $u = \mathsf{t}\lambda(0)$ be elements of T_qQ and let k be a number. We write

$$v = ku \tag{17}$$

if

$$\mathrm{D}(f \circ \gamma)(0) = k\mathrm{D}(f \circ \lambda)(0) \tag{18}$$

for each function f on Q. The coordinate construction

$$(q^\kappa \circ \gamma)(s) = (q^\kappa(u) + k\delta q^\kappa(u)s) \tag{19}$$

shows that for each $k \in \mathbb{R}$ and $u \in \mathsf{T}_qQ$ there is a vector $v \in \mathsf{T}_qQ$ such that $v = ku$. If $v = \mathsf{t}\gamma(0)$ and $v' = \mathsf{t}\gamma'(0)$ are two such vectors, then

$$\mathrm{D}(f \circ \gamma')(0) = \mathrm{D}(f \circ \gamma)(0) = k\mathrm{D}(f \circ \lambda)(0). \tag{20}$$

It follows that the vector v is unique.

Each curve $\gamma\colon \mathbb{R} \to Q$ has a *tangent prolongation*

$$\begin{aligned} \mathsf{t}\gamma \ &: \ \mathbb{R} \to \mathsf{T}Q \\ &: \ s \mapsto \mathsf{t}\gamma(\cdot + s)(0). \end{aligned} \tag{21}$$

The curve $\gamma(\cdot + s)$ is the mapping

$$\begin{aligned} \gamma(\cdot + s) \ &: \ \mathbb{R} \to Q \\ &: \ s' \mapsto \gamma(s' + s). \end{aligned} \tag{22}$$

The vector $\mathsf{t}\gamma(s)$ is the vector tangent to γ at $\gamma(s)$. The coordinate description of the prolongation is given by

$$(q^\kappa, \delta q^\lambda) \circ \mathsf{t}\gamma = (q^\kappa \circ \gamma, \mathrm{D}(q^\lambda \circ \gamma)). \tag{23}$$

A mapping $X\colon Q \to \mathsf{T}Q$ such that $\tau_Q \circ X\colon Q \to Q$ is the identity mapping is called a *section* of the fibration τ_Q. A section of the tangent fibration is called a *vector field*.

Let P be a differential manifold with coordinates

$$(p^i)\colon P \to \mathbb{R}^n. \tag{24}$$

For each differentiable mapping

$$\alpha\colon Q \to P \tag{25}$$

we have the *tangent mapping*

$$\mathsf{T}\alpha\colon \mathsf{T}Q \to \mathsf{T}P. \tag{26}$$

If $\gamma\colon \mathbb{R} \to Q$ is a representative of a vector $v \in \mathsf{T}Q$, then $\alpha \circ \gamma\colon \mathbb{R} \to P$ is a representative of the vector $\mathsf{T}\alpha(v) \in \mathsf{T}P$:

$$\mathsf{T}\alpha(\mathsf{t}\gamma(0)) = \mathsf{t}(\alpha \circ \gamma)(0). \tag{27}$$

The coordinate definition of the tangent mapping is given by

$$(p^i, \delta p^j) \circ \mathsf{T}\alpha = (\alpha^i \circ \tau_Q, (\partial_\kappa \alpha^j \circ \tau_Q)\delta q^\kappa) \tag{28}$$

with $\alpha^i = p^i \circ \alpha$ or by a simplified formula

$$(p^i, \delta p^j) \circ \mathsf{T}\alpha = (\alpha^i, \partial_\kappa \alpha^j \delta q^\kappa). \tag{29}$$

Einstein's summation convention is used. The commutative diagram

$$\begin{array}{ccc} \mathsf{T}Q & \xrightarrow{\quad\mathsf{T}\alpha\quad} & \mathsf{T}P \\ {\scriptstyle\tau_Q}\big\downarrow & & \big\downarrow{\scriptstyle\tau_P} \\ Q & \xrightarrow{\quad\alpha\quad} & P \end{array} \tag{30}$$

is a vector fibration morphism.

A differentiable mapping $\sigma\colon T \to Q$ is called an immersion if at each point $t \in T$ the linear mapping $\mathsf{T}_t\sigma\colon \mathsf{T}_tT \to \mathsf{T}_{\sigma(t)}Q$ obtained by restricting the mapping $\mathsf{T}\sigma$ to the fibre $\mathsf{T}_tT = \tau_T^{-1}(t)$ is injective. If

$$(t^i)\colon T \to \mathbb{R}^k \tag{31}$$

are coordinates in T and $\sigma^\kappa = q^\kappa \circ \sigma$, then σ is an immersion if the matrix $(\partial_i \sigma^\kappa)$ is of maximal rank k. The image $S = \mathrm{im}(\sigma) \subset Q$ is called an

(immersed) *submanifold* of Q of dimension k. A submanifold $S \subset Q$ is frequently given as a set

$$S = \{q \in Q;\ \forall_A F_A(q) = 0\}\,, \tag{32}$$

where F_A are $m-k$ functions on Q such that the matrix $(\partial_\kappa F_A)$ is of maximal rank $m-k$ at points of S. A set S specified in this way is called an *embedded submanifold*. Submanifolds are usually assumed to be embedded. We will adopt the standard practice of not distinguishing elements of geometric spaces from their coordinates. Functions defined on these geometric spaces will be considered functions of coordinates. Instead of writing a formula (32) we will say that S satisfies equations $F_A(q^\kappa) = 0$. The *tangent set* of a subset $S \subset Q$ (not necessarily a submanifold) is a subset of $\mathsf{T}Q$. A vector v is in $\mathsf{T}S$ if there is a curve $\gamma\colon \mathbb{R} \to Q$ such that $v = \mathrm{t}\gamma(0)$ and $\gamma(s) \in S$ for each s in a neighbourhood of $0 \in \mathbb{R}$. We have $\tau_Q(\mathsf{T}S) = S$. If S is the image of an immersion $\sigma\colon T \to Q$, then $\mathsf{T}S$ is the image of $\mathsf{T}\sigma\colon \mathsf{T}T \to \mathsf{T}Q$. The coordinates $(q^\kappa, \delta q^\lambda)$ of elements of $\mathsf{T}S$ are related to coordinates $(t^i, \delta t^j)$ by

$$q^\kappa = \sigma^\kappa(t^i),\ \delta q^\lambda = \partial_j \sigma^\lambda(t^i)\delta t^j\,. \tag{33}$$

If S satisfies equations $F_A(q^\kappa) = 0$, then $\mathsf{T}S$ satisfies equations $\partial_\kappa F_A \delta q^\kappa = 0$ in addition to $F_A(q^\kappa) = 0$.

A 0-*form* on Q is a function on Q. A 1-*form* on Q is a mapping

$$\begin{aligned} A\ &:\ \mathsf{T}Q \to \mathbb{R} \\ &:\ v \mapsto \langle A, v\rangle \end{aligned} \tag{34}$$

linear on fibres of τ_Q. The product of a 0-form F with a 1-form A is a 1-form FA defined by

$$\langle FA, v\rangle = F(\tau_Q(v))\langle A, v\rangle. \tag{35}$$

The *differential* $\mathrm{d}F$ of a function F on Q is 1-form defined by

$$\langle \mathrm{d}F, \mathrm{t}\gamma(0)\rangle = \mathrm{D}(F \circ \gamma)(0). \tag{36}$$

The differential of the product FG of two functions is the 1-form $F\mathrm{d}G + G\mathrm{d}F$. Coordinates (δq^κ) in $\mathsf{T}Q$ are 1-forms. They are the differentials $(\mathrm{d}q^\kappa)$ of coordinates (q^κ) in Q. Each 1-form A can be expressed as a combination

$$A = A_\kappa \mathrm{d}q^\kappa \tag{37}$$

of these differentials. The coefficients A_κ are 0-forms obtained from

$$\langle A, v\rangle = A_\kappa(v)\delta q^\kappa(v) \tag{38}$$

for each $v \in \mathsf{T}Q$. The differential of a function $F(q^\kappa)$ is the 1-form

$$\mathrm{d}F = \partial_\lambda F(q^\kappa)\mathrm{d}q^\lambda. \tag{39}$$

A 2-*form* on Q is a function

$$\begin{aligned} B \;:\; & \mathsf{T}Q \underset{(\tau_Q,\tau_Q)}{\times} \mathsf{T}Q \to \mathbb{R} \\ :\; & (v_1, v_2) \mapsto \langle B, v_1 \wedge v_2\rangle, \end{aligned} \tag{40}$$

which is antisymmetric:

$$\langle B, v_1 \wedge v_2\rangle + \langle B, v_2 \wedge v_1\rangle = 0 \tag{41}$$

and linear in its first argument:

$$\langle B, (kv_1 + k'v_1') \wedge v_2\rangle = k\langle B, v_1 \wedge v_2\rangle + k'\langle B, v_1' \wedge v_2\rangle. \tag{42}$$

Linearity in the first argument and antisymmetry imply linearity in the second argument. The product of 0-form with a 2-form is a 2-form. The *exterior product* of 1-forms A^1 and A^2 is a 2-form $A^1 \wedge A^2$ defined by

$$\langle A^1 \wedge A^2, v_1 \wedge v_2\rangle = \langle A^1, v_1\rangle\langle A^2, v_2\rangle - \langle A^1, v_2\rangle\langle A^2, v_1\rangle. \tag{43}$$

Each 2-form B is a combination

$$B = \frac{1}{2}B_{\kappa\lambda}\mathrm{d}q^\kappa \wedge \mathrm{d}q^\lambda. \tag{44}$$

The coefficients $B_{\kappa\lambda}$ are 0-forms characterized by

$$\langle B, v_1 \wedge v_2\rangle = \frac{1}{2}B_{\kappa\lambda}(\delta q^\kappa(v_1)\delta q^\lambda(v_2) - \delta q^\kappa(v_2)\delta q^\lambda(v_1)) \tag{45}$$

and

$$B_{\kappa\lambda} + B_{\lambda\kappa} = 0. \tag{46}$$

The *exterior differential* of a 1-form A is a 2-form $\mathrm{d}A$. In order to construct the exterior differential we associate with each pair $(v_1, v_2) \in \mathsf{T}Q \underset{(\tau_Q,\tau_Q)}{\times} \mathsf{T}Q$ a mapping $\chi\colon \mathbb{R}^2 \to Q$ such that $v_1 = \mathsf{t}\chi(\cdot, 0)$ and $v_2 = \mathsf{t}\chi(0, \cdot)$. The coordinate construction

$$\chi^\kappa(s_1, s_2) = q^\kappa(\chi(s_1, s_2)) = q^\kappa(v_1) + \delta q^\kappa(v_1)s_1 + \delta q^\kappa(v_2)s_2 \tag{47}$$

proves the existence of such mappings. We define curves

$$\begin{aligned} \xi_1 \;:\; & \mathbb{R} \to \mathsf{T}Q \\ :\; & s \mapsto \mathsf{t}\chi(\cdot, s)(0) \end{aligned} \tag{48}$$

and

$$\begin{aligned}\xi_2 &: \mathbb{R} \to \mathsf{T}Q \\ &: s \mapsto \mathsf{t}\chi(s,\cdot)(0)\end{aligned} \tag{49}$$

with coordinate representations

$$(\xi_1^\kappa(s_2), \delta\xi_1^\lambda(s_2)) = (q^\kappa(\xi_1(s_2)), \delta q^\lambda(\xi_1(s_2))) = (\chi^\kappa(0,s_2), \partial_{s_1}\chi^\lambda(0,s_2)) \tag{50}$$

and

$$(\xi_2^\kappa(s_1), \delta\xi_2^\lambda(s_1)) = (q^\kappa(\xi_2(s_1)), \delta q^\lambda(\xi_2(s_1))) = (\chi^\kappa(s_1,0), \partial_{s_2}\chi^\lambda(s_1,0)). \tag{51}$$

For the mapping defined in (47) we have

$$(\xi_1^\kappa(s), \delta\xi_1^\lambda(s)) = (q^\kappa(v_1) + \delta q^\kappa(v_2)s, \delta q^\lambda(v_1)) \tag{52}$$

and

$$(\xi_2^\kappa(s), \delta\xi_2^\lambda(s)) = (q^\kappa(v_1) + \delta q^\kappa(v_1)s, \delta q^\lambda(v_2)). \tag{53}$$

The exterior differential is defined by

$$\langle \mathrm{d}A, v_1 \wedge v_2\rangle = \mathrm{D}\langle A, \xi_2\rangle(0) - \mathrm{D}\langle A, \xi_1\rangle(0). \tag{54}$$

Relations

$$A^1 \wedge A^2 + A^2 \wedge A^1 = 0, \tag{55}$$

$$\mathrm{d}(FA) = \mathrm{d}F \wedge A + F\mathrm{d}A, \tag{56}$$

and

$$\mathrm{d}\mathrm{d}F = 0 \tag{57}$$

are easily established for an arbitrary 0-form F and arbitrary 1-forms A, A^1, and A^2. The exterior differential of a 1-form $A = A_\lambda \mathrm{d}q^\lambda$ is the 2-form

$$\mathrm{d}A = \mathrm{d}A_\lambda \wedge \mathrm{d}q^\lambda = \partial_\kappa A_\lambda \mathrm{d}q^\kappa \wedge \mathrm{d}q^\lambda = \frac{1}{2}(\partial_\kappa A_\lambda - \partial_\lambda A_\kappa)\mathrm{d}q^\kappa \wedge \mathrm{d}q^\lambda. \tag{58}$$

A 2-form which is the differential of a 1-form is said to be *exact*.

A 1-form A is said to be *closed* if $\mathrm{d}A = 0$. If A is closed, then there is a neighbourhood V of each point q_0 and a 0-form F on V such that $A|V = \mathrm{d}F$. This is as a consequence of the Poincaré lemma.

Let P be a differential manifold with coordinates (p^i) and let $\alpha\colon Q \to P$ be a differentiable mapping. Let $\alpha^i = p^i \circ \alpha$. The *pull back* of a 0-form F on P is the 0-form $F \circ \alpha$ on Q. The *pull back* of a 1-form A on P is the 1-form $\alpha^* A$ on Q defined by

$$\langle \alpha^* A, v\rangle = \langle A, \mathsf{T}\alpha(v)\rangle. \tag{59}$$

If

$$A = A_i \mathrm{d}p^i, \tag{60}$$

then

$$\alpha^* A = A_i \partial_\kappa \alpha^i \mathrm{d}q^\kappa. \tag{61}$$

The *pull back* of a 2-form B on P is the 2-form $\alpha^* B$ on Q defined by

$$\langle \alpha^* B, v_1 \wedge v_2\rangle = \langle B, \mathsf{T}\alpha(v_1) \wedge \mathsf{T}\alpha(v_1)\rangle. \tag{62}$$

If

$$B = \frac{1}{2} B_{ij} \mathrm{d}p^i \wedge \mathrm{d}p^j, \tag{63}$$

then

$$\alpha^* B = \frac{1}{2} B_{ij} \partial_\kappa \alpha^i \partial_\lambda \alpha^j \mathrm{d}q^\kappa \wedge \mathrm{d}q^\lambda. \tag{64}$$

The relations

$$\mathrm{d}(\alpha^* F) = \alpha^* \mathrm{d}F \tag{65}$$

and

$$\mathrm{d}(\alpha^* A) = \alpha^* \mathrm{d}A \tag{66}$$

hold for a 0-form F and a 1-form A. Let $C \subset Q$ be a submanifold. The mapping

$$\begin{aligned} \iota_C &: C \to Q \\ &: q \mapsto q \end{aligned} \tag{67}$$

is the *canonical injection*. The pull backs $\iota_C^* F$, $\iota_C^* A$, and $\iota_C^* B$ are denoted by $F|C$, $A|C$, and $B|C$, respectively.

The *cotangent bundle* of a manifold Q is a manifold T^*Q. The *cotangent fibration*

$$\pi_Q \colon \mathsf{T}^*Q \to Q \tag{68}$$

is the vector fibration dual to the tangent fibration τ_Q. The *canonical pairing* is a bilinear mapping

$$\begin{aligned} \langle\, , \,\rangle &: \mathsf{T}^*Q \underset{(\pi_Q, \tau_Q)}{\times} \mathsf{T}Q \to \mathbb{R} \\ &: (f, v) \mapsto \langle f, v\rangle \end{aligned} \tag{69}$$

defined on the set

$$\mathsf{T}^*Q \underset{(\pi_Q, \tau_Q)}{\times} \mathsf{T}Q = \{(f, v) \in \mathsf{T}^*Q \times \mathsf{T}Q; \quad \pi_Q(f) = \tau_Q(v)\}. \tag{70}$$

Each *covector* $f \in \mathsf{T}^*_qQ = \pi_Q^{-1}(q)$ is the differential $\mathrm{d}F(q)$ of a function $F\colon Q \to \mathbb{R}$. Differentials $(\mathrm{d}q^\kappa(q))$ form a basis of the vector space T^*_qQ. Let $(e_\kappa(q))$ be the basis of the vector space T_qQ dual to the base $(\mathrm{d}q^\kappa(q))$ in the sense that

$$\langle \mathrm{d}q^\kappa(q), e_\lambda(q)\rangle = \delta^\kappa{}_\lambda. \tag{71}$$

Coordinates

$$(q^\kappa, f_\lambda)\colon \mathsf{T}^*Q \to \mathbb{R}^{2m} \tag{72}$$

are defined by

$$(q^\kappa, f_\lambda)(f) = (q^\kappa(\pi_Q(f)), \langle f, e_\lambda(\pi_Q(f))\rangle). \tag{73}$$

The canonical pairing has the coordinate expression

$$\langle f, v\rangle = f_\lambda(f)\delta q^\lambda(v). \tag{74}$$

For the tangent bundle TT^*Q of the cotangent bundle T^*Q we have the tangent fibration

$$\tau_{\mathsf{T}^*Q}\colon \mathsf{TT}^*Q \to \mathsf{T}^*Q \tag{75}$$

and the tangent mapping

$$\mathsf{T}\pi_Q\colon \mathsf{TT}^*Q \to \mathsf{T}Q \tag{76}$$

of the cotangent fibration $\pi_Q\colon \mathsf{T}^*Q \to Q$. The diagram

$$\begin{array}{ccc} \mathsf{TT}^*Q & \xrightarrow{\mathsf{T}\pi_Q} & \mathsf{T}Q \\ {\scriptstyle \tau_{\mathsf{T}^*Q}}\downarrow & & \downarrow{\scriptstyle \tau_Q} \\ \mathsf{T}^*Q & \xrightarrow{\pi_Q} & Q \end{array} \tag{77}$$

is commutative. Hence, $(\tau_{\mathsf{T}^*Q}(w), \mathsf{T}\pi_Q(w)) \in \mathsf{T}^*Q \underset{(\pi_Q, \tau_Q)}{\times} \mathsf{T}Q$ for each $w \in \mathsf{TT}^*Q$. A canonical 1-form ϑ_Q on T^*Q, called the *Liouville form*, is defined by

$$\langle \vartheta_Q, w\rangle = \langle \tau_{\mathsf{T}^*Q}(w), \mathsf{T}\pi_Q(w)\rangle. \tag{78}$$

In the manifold TT^*Q we have coordinates

$$(q^\kappa, p_\lambda, \delta q^\mu, \delta p_\nu)\colon \mathsf{TT}^*Q \to \mathbb{R}^{4m} \tag{79}$$

related to the coordinates (q^κ, f_λ) as the coordinates $(q^\kappa, \delta q^\lambda)$ in $\mathsf{T}Q$ are related to the coordinates (q^κ) in Q. In terms o these coordinates, coordinates

(q^κ, f_λ) in T^*Q, and coordinates $(q^\kappa, \delta q^\lambda)$ in $\mathsf{T}Q$ we have the coordinate definitions of the fibrations τ_{T^*Q} and $\mathsf{T}\pi_Q$:

$$(q^\kappa, f_\lambda) \circ \tau_{\mathsf{T}^*Q} = (q^\kappa, f_\lambda) \tag{80}$$

and

$$(q^\kappa, \delta q^\lambda) \circ \mathsf{T}\pi_Q = (q^\kappa, \delta q^\lambda). \tag{81}$$

It follows that

$$\langle \vartheta_Q, w \rangle = f_\kappa(w) \delta q^\kappa(w). \tag{82}$$

Hence,

$$\vartheta_Q = f_\kappa \mathrm{d}q^\kappa. \tag{83}$$

A 1-form A on Q is a function on $\mathsf{T}Q$ but it can be interpreted as a section $A\colon Q \to \mathsf{T}^*Q$ of the cotangent fibration. In terms of this dual interpretation we state the following fundamental property of the Liouville form:

$$A^* \vartheta_Q = A. \tag{84}$$

A manifold P and an exact, non degenerate 2-form ω form an (exact) *symplectic manifold* (P, ω). The 2-form ω defines a mapping $\beta_{(P,\omega)}\colon \mathsf{T}P \to \mathsf{T}^*P$ characterized by the equality

$$\langle \beta_{(P,\omega)}(u), v \rangle = \langle \omega, u \wedge v \rangle \tag{85}$$

for vectors $u \in \mathsf{T}P$ and $v \in \mathsf{T}P$ such that $\tau_P(v) = \tau_P(u)$. The 2-form ω is said to be *non degenerate* if the mapping $\beta_{(P,\omega)}$ is invertible. The cotangent bundle T^*Q together with the 2-form

$$\omega_Q = \mathrm{d}\vartheta_Q = \mathrm{d}f_\kappa \wedge \mathrm{d}q^\kappa \tag{86}$$

form a symplectic manifold $(\mathsf{T}^*Q, \omega_Q)$. In the cotangent bundle $\mathsf{T}^*\mathsf{T}^*Q$ we use coordinates

$$(q^\kappa, f_\lambda, a_\mu, b^\nu)\colon \mathsf{T}^*\mathsf{T}^*Q \to \mathbb{R}^{4m} \tag{87}$$

induced by coordinates (q^κ, f_λ) in T^*Q. The coordinate definition of the mapping

$$\beta_{(\mathsf{T}^*Q,\omega_Q)}\colon \mathsf{T}\mathsf{T}^*Q \to \mathsf{T}^*\mathsf{T}^*Q \tag{88}$$

is given by

$$(q^\kappa, f_\lambda, a_\mu, b^\nu) \circ \beta_{(\mathsf{T}^*Q,\omega_Q)} = (q^\kappa, f_\lambda, \delta f_\mu, -\delta q^\nu). \tag{89}$$

This mapping is invertible. Its inverse

$$\beta^{-1}_{(\mathsf{T}^*Q,\omega_Q)}\colon \mathsf{T}^*\mathsf{T}^*Q \to \mathsf{T}\mathsf{T}^*Q \tag{90}$$

is defined by

$$(q^\kappa, f_\lambda, \delta q^\mu, \delta f_\nu) \circ \beta^{-1}_{(\mathsf{T}^*Q,\omega_Q)} = (q^\kappa, f_\lambda, -b^\mu, a_\nu). \tag{91}$$

The *Poisson bracket*

$$\{F, G\}\colon \mathsf{T}^*Q \to \mathbb{R} \tag{92}$$

of two functions F and G on T^*Q is defined by

$$\{F, G\}(f) = \langle \mathrm{d}G(f), \beta^{-1}_{(\mathsf{T}^*Q,\omega_Q)}(\mathrm{d}F(f))\rangle. \tag{93}$$

It follows from the coordinate relation (91) that the Poisson bracket $\{F, G\}$ of two functions $F(q^\kappa, f_\lambda)$ and $G(q^\kappa, f_\lambda)$ is the function

$$\frac{\partial F}{\partial q^\kappa}\frac{\partial G}{\partial f_\kappa} - \frac{\partial G}{\partial q^\kappa}\frac{\partial F}{\partial f_\kappa} \tag{94}$$

or

$$\partial_\kappa F \partial^\kappa G - \partial_\kappa G \partial^\kappa F \tag{95}$$

with the symbol ∂^κ used to denote the partial derivative with respect to f_κ.

3. Lagrangian submanifolds

A *Lagrangian submanifold* of a general symplectic manifold (P, ω) is a submanifold $S \subset P$ of dimension $\dim(S) = \frac{1}{2}\dim(P)$ such that $\omega|S = 0$. This last condition means that the symplectic form ω evaluated on two vectors tangent to S vanishes. If S is the image of an immersion $\sigma\colon T \to P$, then $\omega|S = 0$ is equivalent to $\sigma^*\omega = 0$.

A Lagrangian submanifold of $(\mathsf{T}^*Q, \omega_Q)$ is a submanifold $S \subset \mathsf{T}^*Q$ of dimension m such that $\omega_Q|S = 0$. If S is the image of an immersion $\sigma\colon T \to \mathsf{T}^*Q$ from a manifold T with coordinates (t^α) and

$$(q^\kappa, f_\lambda) \circ \sigma = (\sigma^\kappa, \sigma_\lambda), \tag{96}$$

then

$$\sigma^*\omega_Q = \partial_\alpha\sigma_\kappa\partial_\beta\sigma^\kappa \mathrm{d}t^\alpha \wedge \mathrm{d}t^\beta = \frac{1}{2}\left(\partial_\alpha\sigma_\kappa\partial_\beta\sigma^\kappa - \partial_\beta\sigma_\kappa\partial_\alpha\sigma^\kappa\right)\mathrm{d}t^\alpha \wedge \mathrm{d}t^\beta. \tag{97}$$

If S is a Lagrangian submanifold, then the *Lagrange brackets*

$$\partial_\alpha\sigma_\kappa\partial_\beta\sigma^\kappa - \partial_\beta\sigma_\kappa\partial_\alpha\sigma^\kappa \tag{98}$$

vanish. Let $f \in S$ and let $\mathsf{T}_f S \subset \mathsf{T}_f\mathsf{T}^*Q$ denote the space of vectors tangent to S at f. Let

$$\mathsf{T}^\circ_f S = \left\{a \in \mathsf{T}^*_f\mathsf{T}^*Q;\ \forall_{w\in\mathsf{T}_f S}\langle a, w\rangle = 0\right\} \tag{99}$$

be the *polar* of $\mathsf{T}_f S$. If $u \in \mathsf{T}_f S$, then

$$\langle \beta_{(\mathsf{T}^*Q,\omega_Q)}(u), w\rangle = \langle \omega_Q, u \wedge w\rangle = 0 \tag{100}$$

for each $w \in \mathsf{T}_f S$. Hence, $\beta_{(\mathsf{T}^*Q,\omega_Q)}(\mathsf{T}_f S) \subset \mathsf{T}^\circ_f S$. Since $\dim(\beta_{(\mathsf{T}^*Q,\omega_Q)}(\mathsf{T}_f S)) = \dim(\mathsf{T}_f S) = m$ and $\dim(\mathsf{T}^\circ_f S) = \dim(\mathsf{T}^*_f \mathsf{T}^*Q) - \dim(\mathsf{T}_f S) = m$, the spaces $\mathsf{T}^\circ_f S$ and $\beta_{(\mathsf{T}^*Q,\omega_Q)}(\mathsf{T}_f S)$ are equal. If F and G are functions on T^*Q constant on S, then $\mathrm{d}F(f)$ and $\mathrm{d}G(f)$ are in $\mathsf{T}^\circ_f S$ for each $f \in S$. It follows that

$$\{F, G\}|S = 0. \tag{101}$$

If S is specified by equations $F_A = 0$, where F_A are m independent functions on T^*Q, then

$$\{F_A, F_B\}|S = 0. \tag{102}$$

There are three categories of Lagrangian submanifolds of cotangent bundles generated by increasingly complex objects.

3.1. Lagrangian submanifolds generated by functions

Let U be a function on Q. The image S of the differential $\mathrm{d}U\colon Q \to \mathsf{T}^*Q$ is a Lagrangian submanifold of $(\mathsf{T}^*Q, \omega_Q)$ since $\dim(S) = m$ and

$$(\mathrm{d}U)^*\omega_Q = (\mathrm{d}U)^*\mathrm{d}\vartheta_Q = \mathrm{d}(\mathrm{d}U)^*\vartheta_Q = \mathrm{d}\mathrm{d}U = 0. \tag{103}$$

The submanifold S is said to be *generated* by U. In terms of coordinates (q^κ, f_λ) the set S is described by equations

$$f_\lambda = \partial_\lambda U(q^\kappa), \tag{104}$$

equivalent to the simple version of the *principle of virtual work*

$$f_\lambda \delta q^\lambda = \delta U(q^\kappa) = \partial_\lambda U(q^\kappa)\delta q^\lambda, \tag{105}$$

where the *virtual displacements* δq^λ are coordinates of a vector $v \in \mathsf{T}Q$.

Let $S = \mathrm{im}(\sigma) \subset \mathsf{T}^*Q$ be the image of a 1-form interpreted as a section $\sigma\colon Q \to \mathsf{T}^*Q$ of the cotangent fibration. From

$$(\sigma)^*\omega_Q = (\sigma)^*\mathrm{d}\vartheta_Q = \mathrm{d}(\sigma)^*\vartheta_Q = \mathrm{d}\sigma \tag{106}$$

it follows that if S is a Lagrangian submanifold, then for each element $f_0 \in S$ there is a neighbourhood $W \subset \mathsf{T}^*Q$ of f_0 and a function U on Q such that $S \cap W = \mathrm{im}(\mathrm{d}U) \cap W$. This is a version of the Poincaré lemma.

3.2. Lagrangian submanifolds generated by constrained functions

Let $C \subset Q$ be a submanifold of dimension k and let $U\colon C \to \mathbb{R}$ be a differentiable function. The set

$$S = \left\{ f \in \mathsf{T}^*Q;\ q = \pi_Q(f) \in C, \forall_{v \in \mathsf{T}_qC \subset \mathsf{T}_qQ} \langle f, v\rangle = \langle \mathrm{d}U, v\rangle \right\} \tag{107}$$

is an affine subbundle of the cotangent bundle T^*Q restricted to C. At each point $q \in C$ the fibre $S_q = S \cap \mathsf{T}^*_qQ$ is an affine subspace of T^*_qQ modeled on the vector subspace $\mathsf{T}^\circ_qC \subset \mathsf{T}^*_qQ$ of dimension $m-k$. It follows that S is a submanifold of T^*Q of dimension m. We choose a function $\overline{U}\colon Q \to \mathbb{R}$ such that $\overline{U}|C = U$ and define functions $\widetilde{\overline{U}} = \overline{U} \circ \pi_Q$ on T^*Q and $\widetilde{U} = \widetilde{\overline{U}}|S$ on S. The function $\widetilde{U}$ does not depend on the choice of the function $\overline{U}$, it can be defined directly by $\widetilde{U}(f) = U(\pi_Q(f))$ for each $f \in S$. If $w \in \mathsf{T}S$, then $\mathsf{T}\pi_Q(w) \in \mathsf{T}C$ since $\pi_Q(S) = C$. From

$$\begin{aligned} \langle \vartheta_Q, w\rangle &= \langle \tau_{\mathsf{T}^*Q}(w), \mathsf{T}\pi_Q(w)\rangle \\ &= \langle \mathrm{d}U, \mathsf{T}\pi_Q(w)\rangle \\ &= \langle \mathrm{d}\overline{U}, \mathsf{T}\pi_Q(w)\rangle \\ &= \langle \mathrm{d}\widetilde{\overline{U}}, w\rangle \\ &= \langle \mathrm{d}\widetilde{U}, w\rangle \end{aligned} \tag{108}$$

it follows that

$$\vartheta_Q|S = \mathrm{d}\widetilde{U} \tag{109}$$

and

$$\omega_Q|S = \mathrm{d}\vartheta_Q|S = \mathrm{d}(\vartheta_Q|S) = \mathrm{d}\mathrm{d}\widetilde{U} = 0. \tag{110}$$

Hence, S is a Lagrangian submanifold of $(\mathsf{T}^*Q, \omega_Q)$.

Given a function $\overline{U}(q^\kappa)$ and $m-k$ independent functions $F_A(q^\kappa)$ such that the set C is described by the equations $F_A(q^\kappa) = 0$ we write the principle of virtual work

$$\begin{aligned} F_A(q^\kappa) &= 0 \\ f_\lambda \delta q^\lambda &= \partial_\lambda \overline{U}(q^\kappa)\delta q^\lambda \\ \partial_\lambda F_A(q^\kappa)\delta q^\lambda &= 0 \end{aligned} \tag{111}$$

for the set S. Coordinates (q^κ, f_λ) of elements of S satisfy the variational principle with arbitrary virtual displacements δq^λ satisfying the last equality. This last equality indicate that the virtual displacements are coordinates of

vectors tangent to C. Using Lagrange multipliers λ^A we write the equations for S in the form

$$\begin{aligned} F_A(q^\kappa) &= 0 \\ f_\lambda &= \partial_\lambda \overline{U}(q^\kappa) + \partial_\lambda F_A(q^\kappa)\lambda^A. \end{aligned} \tag{112}$$

Let (t^i) be the coordinates in C and let $q^\kappa = \sigma^\kappa(t^i)$ be the coordinate expression of the canonical injection of C in Q. If $U(t^i)$ is the internal energy, then S is represented by

$$\begin{aligned} q^\kappa &= \sigma^\kappa(t^i) \\ f_\lambda \partial_j \sigma^\lambda(t^i) &= \partial_j U(t^i). \end{aligned} \tag{113}$$

Let $C \subset Q$ be a submanifold and let S be an affine subbundle of the cotangent bundle T^*Q restricted to C modeled on the vector subbundle $\mathsf{T}^\circ C$ of T^*Q restricted to C. If S is a Lagrangian submanifold of $(\mathsf{T}^*Q, \omega_Q)$, then $\vartheta_Q|S$ is closed. Let f_0 be an element of S and let $W \subset \mathsf{T}^*Q$ be a neighbourhood of f_0 and $\widetilde{U}$ a function on $S \cap W$ such that $\vartheta_Q|S \cap W = \mathrm{d}\widetilde{U}$. We choose the neighbourhood W to have a connected intersection $W_q = S_q \cap W$ with the fibre $S_q = S \cap \mathsf{T}^*_q Q$ for each q in $V = \pi_Q(S \cap W)$. The restriction of ϑ_Q to the fibre $\mathsf{T}^*_q Q$ is the zero form since $\langle \vartheta_Q, w \rangle = 0$ if $\mathsf{T}\pi_Q(w) = 0$. Consequently,

$$\mathrm{d}\widetilde{U}|W_q = \vartheta_Q|W_q = 0 \tag{114}$$

and the function $\widetilde{U}$ is constant on the connected set W_q. This permits the introduction of a function U on C such that $\widetilde{U}(f) = U(\pi_Q(f))$ for each $f \in S \cap W$. The set

$$\left\{ f \in \mathsf{T}^*Q;\ q = \pi_Q(f) \in C, \forall_{v \in \mathsf{T}_q C \subset \mathsf{T}_q Q} \langle f, v \rangle = \langle \mathrm{d}U, v \rangle \right\} \tag{115}$$

intersected with W is the intersection of S with W. We have obtained an extension of the Poincaré lemma to constrained Lagrangian submanifolds.

3.3. Lagrangian submanifolds generated by Morse families

Let $\eta: Y \to Q$ be a differential fibration with coordinates (q^κ, y^A) adapted in the sense that

$$(q^\kappa) \circ \eta = (q^\kappa), \tag{116}$$

where the coordinates (q^κ) on the right hand side are coordinates in Y. Let $U: Y \to \mathbb{R}$ be a function interpreted as a family of functions defined on fibres

of the fibration η. The family is called a *Morse family* if the $k \times (m+k)$ matrix

$$\left(\frac{\partial^2 U}{\partial y^A \partial y^B} \quad \frac{\partial^2 U}{\partial y^A \partial q^\kappa}\right) \tag{117}$$

is of maximal rank. A Morse family generates a set

$$S = \left\{ f \in \mathsf{T}^*Q;\ \exists_{y \in Y_{\pi_q(f)}} \forall_{z \in \mathsf{T}_y Y} \langle f, \mathsf{T}\eta(z)\rangle = \langle \mathrm{d}U, z\rangle \right\}. \tag{118}$$

The *critical set*

$$\mathrm{Cr}(U,\eta) = \left\{ y \in Y;\ \forall_{w \in \mathsf{T}_y Y} \mathsf{T}\eta(w) = 0 \Rightarrow \langle \mathrm{d}U, w\rangle = 0 \right\} \tag{119}$$

of the Morse family is a submanifold of Y of dimension m. A mapping

$$\kappa\colon \mathrm{Cr}(U,\eta) \to \mathsf{T}^*Q \tag{120}$$

such that $\pi_Q(\kappa(y)) = \eta(y)$ is defined by

$$\langle \kappa(y), v\rangle = \langle \mathrm{d}U, w\rangle, \tag{121}$$

where v is any vector in $\mathsf{T}_{\eta(y)}$ and $w \in \mathsf{T}_y Y$ such that $\mathsf{T}\eta(w) = v$. This mapping is an immersion and $S = \mathrm{im}(\kappa)$. Let $y \in \mathrm{Cr}(U,\eta)$ and $w \in \mathsf{T}_y \mathrm{Cr}(U,\eta)$. From

$$\begin{aligned} \langle \kappa^* \vartheta_Q, w\rangle &= \langle \vartheta_Q, \mathsf{T}\kappa(w)\rangle \\ &= \langle \tau_{\mathsf{T}^*Q}(\mathsf{T}\kappa(w)), \mathsf{T}\pi_Q(\mathsf{T}\kappa(w))\rangle \\ &= \langle \kappa(y), \mathsf{T}\eta(w)\rangle \\ &= \langle \mathrm{d}U, w\rangle \end{aligned} \tag{122}$$

it follows that

$$\kappa^* \vartheta_Q = \mathrm{d}U|\,\mathrm{Cr}(U,\eta). \tag{123}$$

The set S is an immersed Lagrangian submanifold of $(\mathsf{T}^*Q, \omega_Q)$ since

$$\kappa^* \omega_Q = 0 \tag{124}$$

and $\dim(S) = \dim(\mathrm{Cr}(U,\eta)) = m$.

It follows from a theorem of Hörmander [4,7] that for each element f_0 of a Lagrangian submanifold of $(\mathsf{T}^*Q, \omega_Q)$ there is a neighbourhood $W \subset \mathsf{T}^*Q$ and a Morse family $U\colon Y \to \mathbb{R}$ of functions on fibres of a fibration $\eta\colon Y \to Q$ such that S and the Lagrangian submanifold generated by U coincide in W. This is an extension of the Poincaré lemma.

The coordinates (q^κ, f_λ) of elements of S satisfy equations

$$\begin{aligned} f_\lambda &= \partial_\lambda U(q^\kappa, y^A) \\ 0 &= \partial_B U(q^\kappa, y^A) \end{aligned} \tag{125}$$

derived from the variational principle of virtual work

$$f_\lambda \delta q^\lambda = \delta U(q^\kappa, y^A) = \partial_\lambda U(q^\kappa, y^A)\delta q^\lambda + \partial_B U(q^\kappa, y^A)\delta y^A \tag{126}$$

with some values of the variables (y^A) and all variations $(\delta q^\lambda, \delta y^B)$. The symbol ∂_A stands for the partial derivative with respect to y^A. Equations (125) imply the equalities

$$\begin{aligned} \mathrm{d}f_\lambda &= \partial_\mu\partial_\lambda U(q^\kappa, y^A)\mathrm{d}q^\mu + \partial_B\partial_\lambda U(q^\kappa, y^A)\mathrm{d}y^B \\ 0 &= \partial_\lambda\partial_B U(q^\kappa, y^A). \end{aligned} \tag{127}$$

Consequently,

$$\omega_Q|S = \mathrm{d}f_\lambda \wedge \mathrm{d}q^\lambda|S = \partial_\mu\partial_\lambda U(q^\kappa, y^A)\mathrm{d}q^\mu \wedge \mathrm{d}q^\lambda = 0. \tag{128}$$

It follows from the maximality of the rank of the matrix (117) that $\dim(S) = m$.

Note that the affine subbundle (107) is generated by the Morse family

$$U(q^\kappa, y^A) = \overline{U}(q^\kappa) + F_A(q^\kappa)y^A. \tag{129}$$

The rank of the matrix

$$\left(\frac{\partial^2 U}{\partial y^A \partial y^B} \quad \frac{\partial^2 U}{\partial y^A \partial q^\kappa}\right) = \left(0 \quad \frac{\partial F_A}{\partial q^\kappa}\right) \tag{130}$$

is maximal due to independence of the functions $F_A(q^\kappa)$. The function (129) depends linearly on the unrestricted variables (y^A). This is the characteristic feature of a Morse family equivalent to a constrained generating function. There is little difference between the variables (y^A) and the Lagrange multipliers (λ^A).

A Morse family generating a Lagrangian submanifold is not unique. It is frequently possible to reduce the dimension of the fibration η. Reductions are based on the following observation. Let S be a Lagrangian submanifold of $(\mathsf{T}^*Q, \omega_Q)$ generated by a Morse family $U\colon Y \to \mathbb{R}$ of functions defined on fibres of a fibration $\eta\colon Y \to Q$. If the critical set $\mathrm{Cr}(U, \eta)$ is the image of a section $\sigma\colon Q \to Y$ of η, then S is generated by the function $U \circ \sigma\colon Q \to \mathbb{R}$. If $f \in S$ and $q = \pi_Q(f)$, then $f = \kappa(\sigma(q))$ and

$$\langle f, v\rangle = \langle \kappa(\sigma(q)), v\rangle = \langle \mathrm{d}U, \mathsf{T}\sigma(v)\rangle = \langle \mathrm{d}(U \circ \sigma), v\rangle \tag{131}$$

for each $v \in \mathsf{T}_q Q$. Hence, $f = \mathrm{d}(U \circ \sigma)(q)$. This shows that $S \subset \mathrm{im}(U \circ \sigma)$. If $f = \mathrm{d}(U \circ \sigma)(q)$, then

$$\langle f, v\rangle = \langle \mathrm{d}(U \circ \sigma), v\rangle = \langle \mathrm{d}U, \mathsf{T}\sigma(v)\rangle = \langle \kappa(\sigma(q)), v\rangle \tag{132}$$

for each $v \in \mathsf{T}_q Q$. Hence, $f = \kappa(\sigma(q))$. It follows that $\mathrm{im}(U \circ \sigma) \subset S$. It may happen that the fibration η is the composition $\eta'' \circ \eta'$ of fibrations $\eta' : Y \to Y'$ and $\eta'' : Y' \to Q$ and that the critical set $\mathrm{Cr}(U, \eta')$ is the image of a section $\sigma : Y' \to Y$ of η'. In this case the Lagrangian submanifold S is generated by the Morse family $U \circ \sigma : Y' \to \mathbb{R}$ of functions on fibres of η''.

4. Statics of mechanical systems

Let Q be the *configuration space* of a static mechanical system. Elements of the cotangent bundle T^*Q are the *generalized forces* applied to the system. The *constitutive set* of a static system is subset S (usually a submanifold) of the cotangent bundle. An element $f \in S$ is the generalized force which when applied by an external controlling device will maintain the system in equilibrium at the configuration $q = \pi_Q(f)$. The constitutive set provides a complete characterization of the response of the static system to external control represented by generalized forces applied to it. The knowledge of equilibrium configurations of an isolated system does not characterize the system completely. Two systems may have the same equilibrium configurations and yet respond differently to external control.

The system is said to be *reciprocal* if $\omega_Q | S = 0$. Let w_1 and w_2 be vectors tangent to S such that $\tau_{\mathsf{T}^*Q}(w_2) = \tau_{\mathsf{T}^*Q}(w_1)$. Let $\delta_1 q^\kappa = \delta q^\kappa(w_1)$, $\delta_1 f_\kappa = \delta f_\kappa(w_1)$, $\delta_2 q^\kappa = \delta q^\kappa(w_2)$, and $\delta_2 f_\kappa = \delta f_\kappa(w_2)$. The equality

$$\delta_1 f_\kappa \delta_2 q^\kappa = \delta_2 f_\kappa \delta_1 q^\kappa \tag{133}$$

derived from $\langle \omega_Q, w_1 \wedge w_2\rangle = 0$ is a reciprocity relation. The system is said to be *potential* if S is a Lagrangian submanifold generated globally by a generating function, a constrained function or a Morse family. The generating function is interpreted as the *internal energy* of the system. A potential system is reciprocal.

In the following three examples the configuration space is an affine Euclidean plane with Cartesian coordinates (x, y). Coordinates (x, y, f, g) are used in T^*Q.

EXAMPLE 1. The function

$$U(x, y) = \frac{k}{2}(x^2 + y^2) \tag{134}$$

is the internal energy of an elastically suspended material point. The constitutive set S is the Lagrangian submanifold generated by U. It is described by equations

$$f = kx, \quad g = ky \tag{135}$$

derived from the principle of virtual work

$$f\delta x + g\delta y = \partial_x U(x,y)\delta x + \partial_y U(x,y)\delta y. \tag{136}$$

▲

EXAMPLE 2. Let $C \subset Q$ be the circle

$$x^2 + y^2 = a^2. \tag{137}$$

Let

$$\overline{U}(x,y) = ky \tag{138}$$

represent the internal energy of a material point constrained to the circle

$$x^2 + y^2 = a^2. \tag{139}$$

From the variational principle

$$\begin{aligned} x^2 + y^2 &= 0 \\ f\delta x + g\delta y &= k\delta y \\ x\delta x + y\delta y &= 0 \end{aligned} \tag{140}$$

we derive equations

$$\begin{aligned} x^2 + y^2 &= 0 \\ f &= \lambda x \\ g &= k + \lambda y \end{aligned} \tag{141}$$

for the constitutive set S with a Lagrange multiplier λ. With the parametric representation

$$\begin{aligned} x &= a\cos\vartheta \\ y &= a\sin\vartheta \end{aligned} \tag{142}$$

we obtain the expression $U(\vartheta) = ka\sin\vartheta$ for the internal energy and the variational principle

$$\begin{aligned} x &= a\cos\vartheta \\ y &= a\sin\vartheta \\ -fa\sin\vartheta\delta\vartheta + ga\cos\vartheta\delta\vartheta &= ka\cos\vartheta\delta\vartheta \end{aligned} \tag{143}$$

equivalent to (140). The constitutive set is generated by the Morse family

$$U(x, y, \lambda) = ky + \frac{\lambda}{2}(x^2 + y^2 - a^2). \tag{144}$$

▲

EXAMPLE 3. The function

$$U(x, y, \vartheta) = \frac{k}{2}((x - a\cos\vartheta)^2 + (y - a\sin\vartheta)^2) \tag{145}$$

is the internal energy of a material point tied elastically to a point left to move freely on the circle

$$x = a\cos\vartheta, \quad y = a\sin\vartheta. \tag{146}$$

The function U is a Morse family of functions of the variable ϑ since the rank of the 1×3 matrix

$$\left(\frac{\partial^2 U}{\partial\vartheta\partial\vartheta} \quad \frac{\partial^2 U}{\partial\vartheta\partial x} \quad \frac{\partial^2 U}{\partial\vartheta\partial y}\right) = (ka(x\cos\vartheta + y\sin\vartheta),\ ka\sin\vartheta,\ -ka\cos\vartheta) \tag{147}$$

is 1. From the principle of virtual work

$$\begin{aligned} f\delta x + g\delta y &= \delta U(x, y, \vartheta) \\ &= k(x - a\cos\vartheta)\delta x + k(y - a)\sin\vartheta)\delta y + ka(x\sin\vartheta - y\cos\vartheta)\delta\vartheta \end{aligned} \tag{148}$$

we obtain equations

$$\begin{aligned} f &= k(x - a\cos\vartheta) \\ g &= k(y - a\sin\vartheta) \\ 0 &= ka(x\sin\vartheta - y\cos\vartheta) \end{aligned} \tag{149}$$

for the constitutive set S. Equations

$$\begin{aligned} x &= \rho\cos\vartheta \\ y &= \rho\sin\vartheta \\ f &= k(\rho - a)\cos\vartheta \\ g &= k(\rho - a)\sin\vartheta \end{aligned} \tag{150}$$

represent a mapping σ from $\mathbb{R}^2$ to T^*Q. The set S is the image of this mapping. The matrix

$$\begin{pmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial y}{\partial \rho} & \dfrac{\partial f}{\partial \rho} & \dfrac{\partial g}{\partial \rho} \\ \dfrac{\partial x}{\partial \vartheta} & \dfrac{\partial y}{\partial \vartheta} & \dfrac{\partial f}{\partial \vartheta} & \dfrac{\partial g}{\partial \vartheta} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\vartheta & \sin\vartheta & k\cos\vartheta & k\sin\vartheta \\ -\rho\sin\vartheta & \rho\cos\vartheta & -k(\rho-a)\sin\vartheta & k(\rho-a)\cos\vartheta \end{pmatrix} \tag{151}$$

is of rank 2. This indicates that S is an immersed submanifold. With the exclusion of points corresponding to $\rho = 0$ the set S is the union of images of two sections of π_Q corresponding to the two different signs in the formulae

$$\begin{aligned} f &= \frac{kx}{\sqrt{x^2+y^2}}\left(\sqrt{x^2+y^2} \pm a\right) \\ g &= \frac{ky}{\sqrt{x^2+y^2}}\left(\sqrt{x^2+y^2} \pm a\right). \end{aligned} \tag{152}$$

With the exclusion of points corresponding to $x^2 + y^2 \geq a^2$, S is the set of points satisfying equations

$$\begin{aligned} F^0_x(x,y,f,g) &= x - \frac{f}{k\sqrt{f^2+g^2}}\left(\sqrt{f^2+g^2} - ka\right) = 0 \\ F^0_y(x,y,f,g) &= y - \frac{g}{k\sqrt{f^2+g^2}}\left(\sqrt{f^2+g^2} - ka\right) = 0. \end{aligned} \tag{153}$$

The functions F^0_x and F^0_y are obviously independent. It follows that S is an embedded submanifold. The rank of the Jacobian

$$\begin{pmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial y}{\partial \rho} \\ \dfrac{\partial x}{\partial \rho} & \dfrac{\partial y}{\partial \rho} \end{pmatrix} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\rho\sin\vartheta & \rho\cos\vartheta \end{pmatrix} \tag{154}$$

of the mapping $\pi_Q \circ \sigma$ represented by

$$\begin{aligned} x &= \rho\cos\vartheta \\ y &= \rho\sin\vartheta \end{aligned} \tag{155}$$

changes from 2 to 1 at $\rho = 0$. This indicates the presence of a *Lagrangian singularity* above the point with coordinates $(x, y) = (0, 0)$. ▲

5. Differential equations

The tangent fibration $\tau_q\colon \mathsf{T}Q \to Q$ was introduced in Section 2. Elements of the tangent bundle $\mathsf{T}Q$ were interpreted as virtual displacements. We return to the topic of tangent vectors this time interpreted as velocities. Coordinates $(q^\kappa, \delta q^\lambda)\colon \mathsf{T}Q \to \mathbb{R}^{2m}$ introduced in Section 2 will be now denoted by $(q^\kappa, \dot{q}^\lambda)$.

A *first order differential equation* in Q is a submanifold $D \subset \mathsf{T}Q$. A curve $\gamma\colon I \to Q$ defined on an open interval $I \subset \mathbb{R}$ is said to be a *solution* of D if for each $t \in I$ the vector $\mathsf{t}\gamma(t)$ tangent to γ at $\gamma(t)$ is an element of D. A differential equation D is said to be *integrable* if for each $v \in D$ there is a solution $\gamma\colon I \to Q$ of D such that $v = \mathsf{t}\gamma(t_0)$ for some $t_0 \in I$.

Not all differential equations are integrable. Let $D \subset \mathsf{T}Q$ be a differential equation and let C be the set $\tau_Q(D)$. If $v \in D$ and D is integrable, then there is a solution $\gamma\colon I \to Q$ of D such that $v = \mathsf{t}\gamma(t_0)$ for some $t_0 \in I$. Since $\mathsf{t}\gamma(t) \in D$ for each $t \in I$, it follows that $\gamma(t) \in C$ for each $t \in I$. Consequently $\mathsf{t}\gamma(t) \in \mathsf{T}C$ for each $t \in I$ and $v = \mathsf{t}\gamma(t_0) \in \mathsf{T}C$. We have shown that the condition $D \subset \mathsf{T}C$ is necessary for integrability of the equation D. This condition is sufficient for a class of differential equations described below.

The image $D = \mathrm{im}(X)$ of a vector field $X\colon Q \to \mathsf{T}Q$ is an integrable differential equation. Let $C \subset Q$ be a submanifold and let D be the union

$$\bigcup_{\alpha\in\mathsf{A}} \{\mathrm{im}(X_\alpha|C)\} \tag{156}$$

of a family of vector fields

$$X_\alpha\colon Q \to \mathsf{T}Q \tag{157}$$

restricted to C. If $D \subset \mathsf{T}C$, then each field X_α induces a vector field

$$\begin{aligned} \overline{X}_\alpha \ &: \ C \to \mathsf{T}C \\ &: \ q \mapsto X_\alpha(q) \end{aligned} \tag{158}$$

since $\mathrm{im}(X_\alpha|C) \subset D \subset \mathsf{T}C$. The differential equation $\mathrm{im}(\overline{X}_\alpha)$ is integrable for each $\alpha \in \mathsf{A}$ and

$$D = \bigcup_{\alpha\in\mathsf{A}} \{\mathrm{im}(\overline{X}_\alpha)\}. \tag{159}$$

Hence, D is integrable.

If the necessary condition $D \subset \mathsf{T}\tau_Q(D)$ is not satisfied, then the reduced equation $D \cap \mathsf{T}\tau_Q(D)$ is closer to being integrable although the condition $D \cap \mathsf{T}\tau_Q(D) \subset \mathsf{T}\tau_Q(D \cap \mathsf{T}\tau_Q(D))$ is not necessarily satisfied. This observation suggests the following algorithm for extracting the integrable part of a differential equation. We consider the sequence of sets

$$\overline{C}^0 = \tau_Q(D), \overline{C}^1 = \tau_Q(D \cap \mathsf{T}\overline{C}^0), \ldots, \overline{C}^k = \tau_Q(D \cap \mathsf{T}\overline{C}^{k-1}), \ldots \tag{160}$$

and the sequence of differential equations

$$\overline{D}^0 = D, \overline{D}^1 = D \cap \mathsf{T}\overline{C}^0, \ldots, \overline{D}^k = D \cap \mathsf{T}\overline{C}^{k-1}, \ldots \tag{161}$$

It may happen that after a finite number of steps the sets in the sequence (160) are all equal to a set $\overline{C}$. This set satisfies the equality

$$\overline{C} = \tau_Q(D \cap \mathsf{T}\overline{C}). \tag{162}$$

If the differential equation $\overline{D} = D \cap \mathsf{T}\overline{C}$ is integrable, then it is the integrable part of D.

In Section 12 we give an example of a version of the above algorithm applied to a Hamiltonian system.

Other algorithms for extracting the integrable part of a differential equations have been designed. They require the use of higher order tangent vectors.

The *second tangent bundle* of a manifold Q is the set T^2Q of equivalence classes of curves in Q. Two curves $\gamma\colon \mathbb{R} \to Q$ and $\gamma'\colon \mathbb{R} \to Q$ are equivalent if $\gamma'(0) = \gamma(0)$, $\mathrm{D}(f \circ \gamma')(0) = \mathrm{D}(f \circ \gamma)(0)$, and $\mathrm{D}^2(f \circ \gamma')(0) = \mathrm{D}^2(f \circ \gamma)(0)$ for each function $f\colon Q \to \mathbb{R}$. We use coordinates

$$(q^\kappa, \dot{q}^\lambda, \ddot{q}^\mu)\colon \mathsf{T}^2Q \to \mathbb{R}^{3m} \tag{163}$$

in T^2Q. If γ is a representative of a second tangent vector $a \in \mathsf{T}^2Q$, then $q^\kappa(a) = q^\kappa(\gamma(0))$, $\dot{q}^\lambda(a) = \mathrm{D}(q^\lambda \circ \gamma)(0)$, and $\ddot{q}^\mu(a) = \mathrm{D}^2(q^\mu \circ \gamma)(0)$. The equivalence class of a curve $\gamma\colon \mathbb{R} \to Q$ will be denoted by $\mathsf{t}^2\gamma(0)$. Each curve $\gamma\colon \mathbb{R} \to Q$ has a *second tangent prolongation*

$$\begin{aligned} \mathsf{t}^2\gamma \ &: \ \mathbb{R} \to \mathsf{T}^2Q \\ &: \ t \mapsto \mathsf{t}^2\gamma(\cdot + t)(0). \end{aligned} \tag{164}$$

The coordinate description of the prolongation is given by

$$(q^\kappa, \dot{q}^\lambda, \ddot{q}^\mu) \circ \mathsf{t}^2\gamma = (q^\kappa \circ \gamma, \mathrm{D}(q^\lambda \circ \gamma), \mathrm{D}^2(q^\mu \circ \gamma)). \tag{165}$$

The *second tangent fibration* is the mapping

$$\begin{aligned} \tau_{2\,Q} \ &: \ \mathsf{T}^2Q \to Q \\ &: \ \mathsf{t}\gamma(0) \mapsto \gamma(0). \end{aligned} \tag{166}$$

There is also the fibration

$$\begin{aligned} {\tau^1}_{2\,Q} \ &: \ \mathsf{T}^2Q \to \mathsf{T}Q \\ &: \ \mathsf{t}^2\gamma(0) \mapsto \mathsf{t}\gamma(0). \end{aligned} \tag{167}$$

A *second order differential equation* in Q is a submanifold $E \subset \mathsf{T}^2Q$. A *solution* is a curve $\gamma\colon I \to Q$ such that $\mathsf{t}^2\gamma(t) \in E$ for each t in the open interval $I \subset \mathbb{R}$. The concept of integrability is easily extended to second order equations. The image $\mathrm{im}(X)$ of a section

$$X\colon \mathsf{T}Q \to \mathsf{T}^2Q \tag{168}$$

of the fibration $\tau^1{}_{2\,Q}$ is an integrable differential equation.

Elements of the iterated tangent bundle $\mathsf{TT}Q$ are equivalence classes of curves in $\mathsf{T}Q$. Coordinates

$$(q^\kappa, \dot{q}^\lambda, q'^\mu, \dot{q}'^\nu)\colon \mathsf{T}Q \to \mathbb{R}^{4m} \tag{169}$$

will be used. These coordinates are related to coordinates $(q^\kappa, \dot{q}^\lambda)$ as coordinates $(q^\kappa, \dot{q}^\lambda)$ are related to coordinates (q^κ). We have fibrations

$$\tau_{\mathsf{TQ}}\colon \mathsf{TT}Q \to \mathsf{T}Q \tag{170}$$

and

$$\mathsf{T}\tau_Q\colon \mathsf{TT}Q \to \mathsf{T}Q \tag{171}$$

with coordinate representations

$$(q^\kappa, \dot{q}^\lambda) \circ \tau_{\mathsf{T}Q} = (q^\kappa, \dot{q}^\lambda) \tag{172}$$

and

$$(q^\kappa, \dot{q}^\lambda) \circ \mathsf{T}\tau_Q = (q^\kappa, q'^\lambda). \tag{173}$$

There is an useful immersion λ_Q of T^2Q in $\mathsf{TT}Q$. This immersion associates with a second tangent vector $a = \mathsf{t}^2\gamma(0)$ the vector $w = \mathsf{tt}\gamma(0)$ tangent to the prolongation $\mathsf{t}\gamma$ of the curve γ at $\mathsf{t}\gamma(0)$. The formal definition is expressed in

$$\begin{aligned} \lambda_Q \;&:\; \mathsf{T}^2Q \to \mathsf{TT}Q \\ &:\; \mathsf{t}^2\gamma(0) \mapsto \mathsf{tt}\gamma(0). \end{aligned} \tag{174}$$

From $\tau_{\mathsf{TQ}}(\mathsf{tt}\gamma(0)) = \mathsf{t}\gamma(0)$, $\mathsf{T}\tau_Q(\mathsf{tt}\gamma(0)) = \mathsf{t}\gamma(0)$, and $\tau^1{}_{2\,Q}(\mathsf{t}^2\gamma(0)) = \mathsf{t}\gamma(0)$ it follows that

$$\tau_{\mathsf{TQ}} \circ \lambda_Q = \mathsf{T}\tau_Q \circ \lambda_Q = \tau^1{}_{2\,Q}. \tag{175}$$

Let $D \subset \mathsf{T}Q$ be a differential equation. The set

$$\mathsf{P}D = \lambda_Q^{-1}(\mathsf{T}D) \subset \mathsf{T}^2Q \tag{176}$$

is a second order differential equation called the *prolongation* of D. If the differential equation is given in the form

$$D = \{v \in \mathsf{T}Q;\ \forall_i f_i(v) = 0\}\,, \tag{177}$$

where f_i are functions on $\mathsf{T}Q$, then $\mathsf{T}D$ is the set

$$\begin{aligned}&\{w \in \mathsf{TT}Q;\ \forall_i f_i(\tau_{\mathsf{T}Q}(w)) = 0,\\ &\partial_\mu f_i(\tau_{\mathsf{T}Q}(w))q'^\mu(a) + \partial_{\dot\mu} f_i(\tau_{\mathsf{T}Q}(w))\dot q'^\mu(a) = 0\}\end{aligned} \tag{178}$$

and $\mathsf{P}D$ is the set

$$\begin{aligned}&\{a \in \mathsf{T}^2Q;\ \forall_i f_i({\tau^1}_{2\,Q}(a)) = 0,\\ &\partial_\mu f_i({\tau^1}_{2\,Q}(a))\dot q^\mu(a) + \partial_{\dot\mu} f_i({\tau^1}_{2\,Q}(a))\ddot q^\mu(a) = 0\}.\end{aligned} \tag{179}$$

The symbol $\partial_{\dot\mu}$ stands for the partial derivative with respect to $\dot q^\mu$.

The inclusion

$${\tau^1}_{2\,Q}(\mathsf{P}D) \subset D \tag{180}$$

follows from

$${\tau^1}_{2\,Q}(\mathsf{P}D) = \tau_{\mathsf{T}Q}(\lambda_Q(\mathsf{P}D)) = \tau_{\mathsf{T}Q}(\lambda_Q(\lambda_Q^{-1}(\mathsf{T}D))) \subset \tau_{\mathsf{T}Q}(\mathsf{T}D) = D. \tag{181}$$

Let $D \subset \mathsf{T}Q$ be an integrable equation. If $v \in D$, then there is a solution $\gamma\colon I \to Q$ of D such that $\mathsf{t}\gamma(0) = v$. We have $\lambda_Q(\mathsf{t}^2\gamma(0)) = \mathsf{tt}\gamma(0) \in \mathsf{T}D$ since $\mathsf{t}\gamma(t) \in D$ for each $t \in I$. It follows that $v = \mathsf{t}\gamma(0) = {\tau^1}_{2\,Q}(\mathsf{t}^2\gamma(0)) \in {\tau^1}_{2\,Q}(\mathsf{P}D)$. Hence,

$$D \subset {\tau^1}_{2\,Q}(\mathsf{P}D) \tag{182}$$

if D is integrable. We have established a necessary condition

$${\tau^1}_{2\,Q}(\mathsf{P}D) = D \tag{183}$$

for integrability of a differential equation $D \subset \mathsf{T}Q$. If this condition is not satisfied, then the integrable part of D is a subset of the set ${\tau^1}_{2\,Q}(\mathsf{P}D)$. The set ${\tau^1}_{2\,Q}(\mathsf{P}D)$ is a subset of D closer to the integrable part without being necessarily integrable. These observations suggest a new algorithm for extracting the integrable part of a differential equation. We introduce the sequence of differential equations

$$\widetilde D^0 = D, \widetilde D^1 = {\tau^1}_{2\,Q}(\mathsf{P}\widetilde D^0), \dots, \widetilde D^k = {\tau^1}_{2\,Q}(\mathsf{P}\widetilde D^{k-1}), \dots \tag{184}$$

It may happen that after a finite number of steps the sets in the sequence (184) are all equal to a set $\widetilde D$. It may happen that $\widetilde D$ is the integrable part of D.

6. The iterated tangent bundle

We have already introduced the iterated tangent bundle $\mathsf{TT}Q$ and the coordinates

$$(q^\kappa, \dot{q}^\lambda, q'^\mu, \dot{q}'^\nu)\colon \mathsf{TT}Q \to \mathbb{R}^{4m}. \tag{185}$$

The fibration

$$\tau_{\mathsf{T}Q}\colon \mathsf{TT}Q \to \mathsf{T}Q \tag{186}$$

is a vector fibration. We have the operations

$$+\colon \mathsf{TT}Q \underset{(\tau_{\mathsf{T}Q}, \tau_{\mathsf{T}Q})}{\times} \mathsf{TT}Q \to \mathsf{TT}Q \tag{187}$$

and

$$\cdot\colon \mathbb{R} \times \mathsf{TT}Q \to \mathsf{TT}Q \tag{188}$$

with coordinate representations

$$\begin{aligned}&(q^\kappa, \dot{q}^\lambda, q'^\mu, \dot{q}'^\nu)(w_1 + w_2)\\ &= (q^\kappa(w_1), \dot{q}^\lambda(w_1), q'^\mu(w_1) + q'^\mu(w_2), \dot{q}'^\nu(w_1) + \dot{q}'^\nu(w_2))\end{aligned} \tag{189}$$

and

$$(q^\kappa, \dot{q}^\lambda, q'^\mu, \dot{q}'^\nu)(k \cdot w) = (q^\kappa(w), \dot{q}^\lambda(w), kq'^\mu(w), k\dot{q}'^\nu(w)). \tag{190}$$

The diagram

$$\begin{array}{ccc} \mathsf{TT}Q & \xrightarrow{\mathsf{T}\tau_Q} & \mathsf{T}Q \\ \Big\downarrow{\scriptstyle \tau_{\mathsf{T}Q}} & & \Big\downarrow{\scriptstyle \tau_Q} \\ \mathsf{T}Q & \xrightarrow{\tau_Q} & Q \end{array} \tag{191}$$

is a vector fibration morphism.

We show that the mapping

$$\mathsf{T}\tau_Q\colon \mathsf{TT}Q \to \mathsf{T}Q \tag{192}$$

is a vector fibration by constructing operations

$$\tilde{+}\colon \mathsf{TT}Q \underset{(\mathsf{T}\tau_Q, \mathsf{T}\tau_Q)}{\times} \mathsf{TT}Q \to \mathsf{TT}Q \tag{193}$$

and

$$\tilde{\cdot}\colon \mathbb{R} \times \mathsf{TT}Q \to \mathsf{TT}Q. \tag{194}$$

Let w_1 and w_2 be elements of $\mathsf{TT}Q$ such that $\mathsf{T}\tau_Q(w_2) = \mathsf{T}\tau_Q(w_1)$. It is possible to choose curves $\xi_1 \colon \mathbb{R} \to \mathsf{T}Q$ and $\xi_2 \colon \mathbb{R} \to \mathsf{T}Q$ such that $w_1 = \mathsf{t}\xi_1(0)$, $w_2 = \mathsf{t}\xi_2(0)$ and $\tau_Q \circ \xi_2 = \tau_Q \circ \xi_1$. The coordinate constructions

$$(q^\kappa, \dot{q}^\lambda) \circ \xi_1 = (q^\kappa(w_1) + q'^\kappa(w_1)s, \dot{q}^\lambda(w_1) + \dot{q}'^\lambda(w_1))s \tag{195}$$

and

$$(q^\kappa, \dot{q}^\lambda) \circ \xi_2 = (q^\kappa(w_2) + q'^\kappa(w_2)s, \dot{q}^\lambda(w_2) + \dot{q}'^\lambda(w_2))s \tag{196}$$

provide an example. The operation $\tilde{+}$ is defined by

$$w_1 \mathbin{\tilde{+}} w_2 = \mathsf{t}(\xi_1 + \xi_2)(0). \tag{197}$$

The operation $\tilde{\cdot}$ is defined by

$$k \mathbin{\tilde{\cdot}} \mathsf{t}\xi(0) = \mathsf{t}(k\xi)(0). \tag{198}$$

Coordinate representations of these operations are given by

$$\begin{aligned}&(q^\kappa, \dot{q}^\lambda, q'^\mu, \dot{q}'^\nu)(w_1 \mathbin{\tilde{+}} w_2)\\ &= (q^\kappa(w_1), \dot{q}^\lambda(w_1) + \dot{q}^\lambda(w_2), q'^\mu(w_1), \dot{q}'^\nu(w_1) + \dot{q}'^\nu(w_2))\end{aligned} \tag{199}$$

and

$$(q^\kappa, \dot{q}^\lambda, q'^\mu, \dot{q}'^\nu)(k \mathbin{\tilde{\cdot}} w) = (q^\kappa(w), k\dot{q}^\lambda(w), q'^\mu(w), k\dot{q}'^\nu(w)). \tag{200}$$

The diagram

$$\begin{array}{ccc} \mathsf{TT}Q & \xrightarrow{\mathsf{T}\tau_Q} & \mathsf{T}Q \\ {\scriptstyle \mathsf{T}\tau_Q}\downarrow & & \downarrow{\scriptstyle \tau_Q} \\ \mathsf{T}Q & \xrightarrow{\tau_Q} & Q \end{array} \tag{201}$$

is a vector fibration morphism.

Elements of the iterated bundle $\mathsf{TT}Q$ are equivalence classes of curves in a set of equivalence classes of curves in Q. A simpler representation of these elements is needed. Let $\chi \colon \mathbb{R}^2 \to Q$ be a differentiable mapping. For each $s \in \mathbb{R}$ we denote by $\mathsf{t}^{(0,1)}\chi(s,0)$ the vector $\mathsf{t}\chi(s,\cdot)(0) \in \mathsf{T}Q$. For each $t \in \mathbb{R}$ we denote by $\mathsf{t}^{(1,0)}\chi(0,t)$ the vector $\mathsf{t}\chi(\cdot,t)(0) \in \mathsf{T}Q$. We have curves

$$\mathsf{t}^{(0,1)}\chi(\cdot,0) \colon \mathbb{R} \to \mathsf{T}Q \tag{202}$$

and

$$\mathsf{t}^{(1,0)}\chi(0,\cdot) \colon \mathbb{R} \to \mathsf{T}Q. \tag{203}$$

Vectors $\mathsf{tt}^{(0,1)}\chi(\cdot,0)(0) \in \mathsf{TT}Q$ and $\mathsf{tt}^{(1,0)}\chi(0,\cdot)(0) \in \mathsf{TT}Q$ will be denoted by $\mathsf{tt}^{(0,1)}\chi(0,0)$ and $\mathsf{tt}^{(1,0)}\chi(0,0)$ respectively. For each $w \in \mathsf{TT}Q$ there is a mapping $\chi\colon \mathbb{R}^2 \to Q$ such that $w = \mathsf{tt}^{(0,1)}\chi(0,0)$. The mapping specified by coordinate relations

$$(q^\kappa \circ \chi)(s,t) = (q^\kappa(w) + \dot{q}^\kappa(w)t + q'^\kappa(w)s + \dot{q}'^\kappa(w)st) \tag{204}$$

has the required property. We consider mappings $\chi\colon \mathbb{R}^2 \to Q$ and $\chi'\colon \mathbb{R}^2 \to Q$ equivalent if

$$\mathsf{tt}^{(0,1)}\chi'(0,0) = \mathsf{tt}^{(0,1)}\chi(0,0). \tag{205}$$

These mappings are equivalent if

$$\chi'(0,0) = \chi(0,0), \tag{206}$$

$$\mathrm{D}^{(1,0)}\chi'(0,0) = \mathrm{D}^{(1,0)}\chi(0,0), \tag{207}$$

$$\mathrm{D}^{(0,1)}\chi'(0,0) = \mathrm{D}^{(0,1)}\chi(0,0), \tag{208}$$

and

$$\mathrm{D}^{(1,1)}\chi'(0,0) = \mathrm{D}^{(1,1)}\chi(0,0). \tag{209}$$

We have obtained an efficient representation of elements of $\mathsf{TT}Q$. In terms of this representation we define the *canonical involution*

$$\begin{aligned} \kappa_Q &: \mathsf{TT}Q \to \mathsf{TT}Q \\ &: \mathsf{tt}^{(0,1)}\chi(0,0) \mapsto \mathsf{tt}^{(1,0)}\chi(0,0) = \mathsf{tt}^{(0,1)}\widetilde{\chi}(0,0), \end{aligned} \tag{210}$$

with

$$\begin{aligned} \widetilde{\chi} &: \mathbb{R}^2 \to Q \\ &: (s,t) \mapsto \chi(t,s). \end{aligned} \tag{211}$$

The coordinate expression of this involution is given by

$$(q^\kappa, \dot{q}^\lambda, q'^\mu, \dot{q}'^\nu) \circ \kappa_Q = (q^\kappa, q'^\lambda, \dot{q}^\mu, \dot{q}'^\nu). \tag{212}$$

The commutative diagram

$$\begin{array}{ccc} \mathsf{TT}Q & \xrightarrow{\kappa_Q} & \mathsf{TT}Q \\ {\scriptstyle \mathsf{T}\tau_Q}\downarrow & & \downarrow{\scriptstyle \tau_{\mathsf{T}Q}} \\ \mathsf{T}Q & = & \mathsf{T}Q \end{array} \tag{213}$$

is a vector fibration isomorphism. The diagram

$$\begin{array}{ccc} \mathsf{TT}Q & \xrightarrow{\kappa_Q} & \mathsf{TT}Q \\ {\scriptstyle \tau_{\mathsf{T}Q}}\downarrow & & \downarrow{\scriptstyle \mathsf{T}\tau_Q} \\ \mathsf{T}Q & = & \mathsf{T}Q \end{array} \tag{214}$$

is the inverse isomorphism. For a differentiable mapping $\alpha\colon Q \to P$ we have

$$\mathsf{TT}\alpha(\mathsf{tt}^{(0,1)}\chi(0,0)) = \mathsf{tt}^{(0,1)}(\alpha \circ \chi)(0,0) \tag{215}$$

and

$$\kappa_P \circ \mathsf{TT}\alpha = \mathsf{TT}\alpha \circ \kappa_Q. \tag{216}$$

Let A be a 1-form on Q. A 0-form $\mathrm{i}_T A$ on $\mathsf{T}Q$ is defined as the function

$$\mathrm{i}_T A(v) = \langle A, v\rangle. \tag{217}$$

Let B be a 2-form on Q. If $w \in \mathsf{TT}Q$, then $(\tau_{\mathsf{T}Q}(w), \mathsf{T}\tau_Q(w)) \in \mathsf{T}Q \underset{(\tau_Q,\tau_Q)}{\times} \mathsf{T}Q$ since $\tau_Q \circ \mathsf{T}\tau_Q = \tau_Q \circ \tau_{\mathsf{T}Q}$. A 1-form $\mathrm{i}_T B$ on $\mathsf{T}Q$ is defined by

$$\langle \mathrm{i}_T B, w\rangle = \langle B, \tau_{\mathsf{T}Q}(w) \wedge \mathsf{T}\tau_Q(w)\rangle. \tag{218}$$

Let F, A, and $B = \mathrm{d}A$ be a 0-form, a 1-form, and an exact 2-form on Q respectively. We define a 0-form $\mathrm{d}_T F$, a 1-form $\mathrm{d}_T A$, and a 2-form $\mathrm{d}_T B$ on $\mathsf{T}Q$ by

$$\mathrm{d}_T F = \mathrm{i}_T \mathrm{d}F, \tag{219}$$

$$\mathrm{d}_T A = \mathrm{i}_T \mathrm{d}A + \mathrm{d}\mathrm{i}_T A, \tag{220}$$

and

$$\mathrm{d}_T B = \mathrm{d}\mathrm{i}_T B = \mathrm{d}\mathrm{i}_T \mathrm{d}A = \mathrm{d}\mathrm{d}_T A. \tag{221}$$

The coordinate expression of the function $\mathrm{d}_T F$ is

$$\mathrm{d}_T F(q^\kappa, \dot{q}^\lambda) = \partial_\lambda F(q^\kappa)\dot{q}^\lambda. \tag{222}$$

If

$$A = A_\kappa(q^\mu, p_\nu)\mathrm{d}q^\kappa \tag{223}$$

and

$$B = \frac{1}{2} B_{\kappa\lambda}(q^\mu, p_\nu)\mathrm{d}q^\kappa \wedge \mathrm{d}q^\lambda, \tag{224}$$

then

$$\mathrm{i}_T A = A_\kappa(q^\mu, p_\nu)\dot{q}^\kappa, \tag{225}$$
$$\mathrm{d}_T A = \partial_\lambda A_\kappa \dot{q}^\lambda \mathrm{d}q^\kappa, \tag{226}$$
$$\mathrm{i}_T B = B_{\kappa\lambda}(q^\mu, p_\nu)\dot{q}^\kappa \mathrm{d}q^\lambda, \tag{227}$$

and

$$\mathrm{di}_T \mathrm{d}A = \frac{1}{2}\partial_\mu(\partial\kappa A_\lambda - \partial\lambda A_\kappa)\dot{q}^\mu \mathrm{d}q^\kappa \wedge \mathrm{d}q^\lambda + (\partial\kappa A_\lambda - \partial\lambda A_\kappa)\mathrm{d}\dot{q}^\kappa \wedge \mathrm{d}q^\lambda. \tag{228}$$

Each 1-form on Q can be expressed as a sum of products $F\mathrm{d}G$ and from

$$\langle \mathrm{d}_T(F\mathrm{d}G), \mathsf{tt}^{(0,1)}\chi(0,0)\rangle = \frac{\partial}{\partial t}\left(F(\chi(0,t))\frac{\partial}{\partial s}G(\chi(s,t))\right)_{|s=0,t=0} \tag{229}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle F\mathrm{d}G, \mathsf{t}\chi(\cdot,t)(0)\rangle_{|t=0} = \frac{\partial}{\partial t}\left(F(\chi(0,t))\frac{\partial}{\partial s}G(\chi(s,t))\right)_{|s=0,t=0} \tag{230}$$

it follows that

$$\langle \mathrm{d}_T(F\mathrm{d}G), \mathsf{tt}^{(0,1)}\chi(0,0)\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle F\mathrm{d}G, \mathsf{t}\chi(\cdot,t)(0)\rangle_{|t=0}. \tag{231}$$

Hence,

$$\langle \mathrm{d}_T A, \mathsf{tt}^{(0,1)}\chi(0,0)\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle A, \mathsf{t}\chi(\cdot,t)(0)\rangle_{|t=0} \tag{232}$$

for each 1-form A.

7. A geometric framework for analytical mechanics

Let Q be a manifold of dimension m. We have already described the geometry of the tangent bundle $\mathsf{T}Q$, the cotangent bundle T^*Q and the tangent bundle TT^*Q of the cotangent bundle T^*Q. The present section is devoted to the study of the canonical symplectic structure of the bundle TT^*Q. We will use coordinates

$$(q^\kappa, \dot{q}^\lambda) \ : \ \mathsf{T}Q \to \mathbb{R}^{2m}, \tag{233}$$
$$(q^\kappa, p_\lambda) \ : \ \mathsf{T}^*Q \to \mathbb{R}^{2m}, \tag{234}$$

and

$$(q^\kappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu) \colon \mathsf{TT}^*Q \to \mathbb{R}^{4m} \tag{235}$$

in the manifolds $\mathsf{T}Q$, T^*Q and TT^*Q. The mappings τ_{T^*Q} and $\mathsf{T}\pi_Q$ have now the coordinate expressions

$$(q^\kappa, p_\lambda) \circ \tau_{\mathsf{T}^*Q} = (q^\kappa, p_\lambda) \tag{236}$$

and

$$(q^\kappa, \dot{q}^\lambda) \circ \mathsf{T}\pi_Q = (q^\kappa, \dot{q}^\lambda). \tag{237}$$

We introduce the exact 2-form

$$\mathrm{d}_T\omega_Q = \mathrm{d}\mathrm{i}_T\omega_Q. \tag{238}$$

It will be shown that this 2-form is non degenerate. The manifold TT^*Q with the form $\mathrm{d}_T\omega_Q$ form a symplectic manifold $(\mathsf{TT}^*Q, \mathrm{d}_T\omega_Q)$. We believe that the symplectic form $\mathrm{d}_T\omega_Q$ is the only natural symplectic form in TT^*Q. The discovery of a second symplectic structure in TT^*Q was announced in a recent Springer-Verlag publication [9]. We have not been able to identify the second symplectic structure. We strongly suspect that this announcement is false. The formula

$$\mathrm{d}\vartheta_\iota = \mathrm{d}\dot{q}^\kappa \wedge \mathrm{d}q^\kappa + \mathrm{d}\dot{p}_\kappa \wedge \mathrm{d}p_\kappa \tag{239}$$

for the Marsden–Ratiu symplectic form does not seem to have an intrinsic meaning since elementary rules of tensor calculus have been violated. We have the coordinate expressions

$$\mathrm{d}_T\vartheta_Q = \dot{p}_\kappa \mathrm{d}q^\kappa + p_\kappa \mathrm{d}\dot{q}^\kappa \tag{240}$$

and

$$\mathrm{d}_T\omega_Q = \mathrm{d}\dot{p}_\kappa \wedge \mathrm{d}q^\kappa + \mathrm{d}p_\kappa \wedge \mathrm{d}\dot{q}^\kappa. \tag{241}$$

The fibration $\tau_{\mathsf{T}^*Q}\colon \mathsf{TT}^*Q \to \mathsf{T}^*Q$ is a vector fibration. We will construct a vector fibration structure for the fibration $\mathsf{T}\pi_Q\colon \mathsf{TT}^*Q \to \mathsf{T}Q$. For two vectors $z_1 \in \mathsf{TT}^*Q$ and $z_2 \in \mathsf{TT}^*Q$ such that $\mathsf{T}\pi_Q(z_2) = \mathsf{T}\pi_Q(z_1)$ it is possible to choose representatives $\zeta_1\colon \mathbb{R} \to \mathsf{T}^*Q$ and $\zeta_2\colon \mathbb{R} \to \mathsf{T}^*Q$ such that $z_1 = \mathsf{t}\zeta_1(0)$, $z_2 = \mathsf{t}\zeta_2(0)$ and $\pi_Q \circ \zeta_2 = \pi_Q \circ \zeta_1$. An example is provided by the coordinate constructions

$$(q^\kappa, p_\lambda) \circ \zeta_1 = (q^\kappa(z_1) + \dot{q}^\kappa(z_1)s, p_\lambda(z_1) + \dot{p}_\lambda(z_1)s) \tag{242}$$

and

$$(q^\kappa, p_\lambda) \circ \zeta_2 = (q^\kappa(z_2) + \dot{q}^\kappa(z_2)s, p_\lambda(z_2) + \dot{p}_\lambda(z_2)s). \tag{243}$$

An operation

$$\tilde{+}\colon \mathsf{TT}^*Q \underset{(\mathsf{T}\pi_Q, \mathsf{T}\pi_Q)}{\times} \mathsf{TT}^*Q \to \mathsf{TT}^*Q \tag{244}$$

is defined by

$$z_1 \tilde{+} z_2 = \mathsf{t}(\zeta_1 + \zeta_2)(0). \tag{245}$$

An operation

$$\tilde{\cdot}\colon \mathbb{R} \times \mathsf{TT}Q \to \mathsf{TT}Q \tag{246}$$

is defined by

$$k \tilde{\cdot}\, \mathsf{t}\zeta(0) = \mathsf{t}(k\zeta)(0). \tag{247}$$

Coordinate representations of these operations are given by

$$(q^\kappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu)(z_1 \tilde{+} z_2) = (q^\kappa(z_1), p_\lambda(z_1) + p_\lambda(z_2), \dot{q}^\mu(z_1), \dot{p}_\nu(z_1) + \dot{p}_\nu(z_2)) \tag{248}$$

and

$$(q^\kappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu)(k \tilde{\cdot}\, z) = (q^\kappa(z), kp_\lambda(z), \dot{q}^\mu(z), k\dot{p}_\nu(z)). \tag{249}$$

The diagram

$$\begin{array}{ccc} \mathsf{TT}^*Q & \xrightarrow{\tau_{\mathsf{T}^*Q}} & \mathsf{T}^*Q \\ {\scriptstyle \mathsf{T}\pi_Q}\downarrow & & \downarrow{\scriptstyle \pi_Q} \\ \mathsf{T}Q & \xrightarrow{\tau_Q} & Q \end{array} \tag{250}$$

is a vector fibration morphism. The vector fibration $\mathsf{T}\pi_Q\colon \mathsf{TT}^*Q \to \mathsf{T}Q$ is dual to the vector fibration $\mathsf{T}\tau_Q\colon \mathsf{TT}Q \to \mathsf{T}Q$. The pairing

$$\langle\, ,\, \rangle^\sim\colon \mathsf{TT}^*Q \underset{(\mathsf{T}\pi_Q, \mathsf{T}\tau_Q)}{\times} \mathsf{TT}Q \to \mathbb{R} \tag{251}$$

is defined by

$$\langle z, w\rangle^\sim = \frac{\mathrm{d}}{\mathrm{d}s}\langle \zeta(s), \xi(s)\rangle_{|s=0}\,, \tag{252}$$

where $\zeta\colon \mathbb{R} \to \mathsf{T}^*Q$ and $\xi\colon \mathbb{R} \to \mathsf{T}Q$ are curves such that $z = \mathsf{t}\zeta(0)$, $w = \mathsf{t}\xi(0)$ and $\pi_Q \circ \zeta = \tau_Q \circ \xi$. Such curves are provided by the coordinate constructions

$$(q^\kappa, p_\lambda) \circ \zeta = (q^\kappa(z) + \dot{q}^\kappa(z)s, p_\lambda(z) + \dot{p}_\lambda(z)s) \tag{253}$$

and

$$(q^\kappa, \dot{q}^\lambda) \circ \xi = (q^\kappa(w) + q'^\kappa(w)s, \dot{q}^\lambda(w) + \dot{q}'^\lambda(w)s). \tag{254}$$

The coordinate expression of the pairing is

$$\langle z, w\rangle^\sim = p_\kappa(z)\dot{q}'^\kappa(w) + \dot{p}_\kappa(z)\dot{q}^\kappa(w). \tag{255}$$

A mapping

$$\psi_Q\colon \mathsf{TT}^*Q \underset{(\pi_Q \circ \tau_{\mathsf{T}^*Q}, \pi_Q)}{\times} \mathsf{T}^*Q \to \mathsf{TT}^*Q \tag{256}$$

is defined by

$$\psi_Q(w, f) = w - \mathsf{t}\eta(0) \tag{257}$$

with $\eta\colon \mathbb{R} \to \mathsf{T}^*Q$ defined by $\eta(s) = \tau_{\mathsf{T}^*Q}(w) + sf$. The coordinate expression of the mapping in terms of coordinates $(q^\kappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu)$ in TT^*Q and coordinates (q^κ, f_λ) in T^*Q is given by

$$(q^\kappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu) \circ \psi_Q = (q^\kappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu - f_\nu). \tag{258}$$

In the cotangent bundle $\mathsf{T}^*\mathsf{T}Q$ we use coordinates

$$(q^\kappa, \dot{q}^\lambda, a_\mu, b_\nu)\colon \mathsf{T}^*\mathsf{T}Q \to \mathbb{R}^{4m} \tag{259}$$

induced by coordinates $(q^\kappa, \dot{q}^\lambda)$ in $\mathsf{T}Q$. The Liouville form is the 1-form

$$\vartheta_{\mathsf{T}Q} = a_\kappa \mathrm{d}q^\kappa + b_\kappa \mathrm{d}\dot{q}^\kappa. \tag{260}$$

The 2-form

$$\omega_{\mathsf{T}Q} = \mathrm{d}a_\kappa \wedge \mathrm{d}q^\kappa + \mathrm{d}b_\kappa \wedge \mathrm{d}\dot{q}^\kappa \tag{261}$$

is the symplectic form on $\mathsf{T}^*\mathsf{T}Q$. A vector fibration isomorphism

$$\begin{array}{ccc} \mathsf{TT}^*Q & \xrightarrow{\alpha_Q} & \mathsf{T}^*\mathsf{T}Q \\ {\scriptstyle \mathsf{T}\pi_Q}\downarrow & & \downarrow{\scriptstyle \pi_{\mathsf{T}Q}} \\ \mathsf{T}Q & = & \mathsf{T}Q \end{array} \tag{262}$$

is defined as dual to the vector fibration isomorphism

$$\begin{array}{ccc} \mathsf{TT}Q & \xrightarrow{\kappa_Q} & \mathsf{TT}Q \\ {\scriptstyle \mathsf{T}\tau_Q}\downarrow & & \downarrow{\scriptstyle \tau_{\mathsf{T}Q}} \\ \mathsf{T}Q & = & \mathsf{T}Q \end{array} \tag{263}$$

in the sense that

$$\langle \alpha_Q(z), w \rangle = \langle z, \kappa_Q(w) \rangle^\sim \tag{264}$$

for $z \in \mathsf{TT}^*Q$ and $w \in \mathsf{TT}Q$ such that $\mathsf{T}\pi_Q(z) = \tau_{\mathsf{T}Q}(w)$. We have the coordinate characterization

$$(q^\kappa, \dot{q}^\lambda, a_\mu, b_\nu) \circ \alpha_Q = (q^\kappa, \dot{q}^\lambda, \dot{p}_\mu, p_\nu) \tag{265}$$

of the mapping α_Q.

For a vector $z = \mathsf{tt}^{(0,1)}\chi(0,0) \in \mathsf{TTT}^*Q$ represented by a mapping $\chi\colon \mathbb{R}^2 \to \mathsf{T}^*Q$ we have

$$\begin{aligned}
\langle \alpha_Q^* \vartheta_{\mathsf{T}Q}, z\rangle &= \langle \vartheta_{\mathsf{T}Q}, \mathsf{T}\alpha_Q(z)\rangle \\
&= \langle \tau_{\mathsf{T}^*\mathsf{T}Q}(\mathsf{T}\alpha_Q(z)), \mathsf{T}\pi_{\mathsf{T}Q}(\mathsf{T}\alpha_Q(z))\rangle \\
&= \langle \alpha_Q(\tau_{\mathsf{TT}^*Q}(z)), \mathsf{TT}\pi_Q(z)\rangle \\
&= \langle \tau_{\mathsf{TT}^*Q}(z), \kappa_Q(\mathsf{TT}\pi_Q(z))\rangle^\sim \\
&= \langle \tau_{\mathsf{TT}^*Q}(\mathsf{tt}^{(0,1)}\chi(0,0)), \kappa_Q(\mathsf{TT}\pi_Q(\mathsf{tt}^{(0,1)}\chi(0,0)))\rangle^\sim \\
&= \langle \mathsf{t}^{(0,1)}\chi(0,0), \kappa_Q(\mathsf{tt}^{(0,1)}(\pi_Q\circ\chi)(0,0))\rangle^\sim \\
&= \langle \mathsf{t}\chi(0,\cdot)(0), \mathsf{tt}^{(1,0)}(\pi_Q\circ\chi)(0,0)\rangle^\sim \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle \chi(0,t), \mathsf{t}^{(1,0)}(\pi_Q\circ\chi)(0,t)\rangle_{|t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle \chi(0,t), \mathsf{t}(\pi_Q\circ\chi)(\cdot,t)(0)\rangle_{|t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle \tau_{\mathsf{T}^*Q}(\mathsf{t}\chi(\cdot,t)(0)), \mathsf{T}\pi_Q(\mathsf{t}\chi(\cdot,t)(0))\rangle_{|t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle \vartheta_Q, \mathsf{t}\chi(\cdot,t)(0)\rangle_{|t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle \vartheta_Q, \mathsf{t}^{(1,0)}\chi(0,t)\rangle_{|t=0} \\
&= \langle \mathrm{d}_T\vartheta_Q, \mathsf{tt}^{(0,1)}\chi(0,0)\rangle \\
&= \langle \mathrm{d}_T\vartheta_Q, z\rangle. \qquad (266)
\end{aligned}$$

We have used the formula (232) and relations

$$\tau_{\mathsf{T}^*\mathsf{T}Q}\circ\mathsf{T}\alpha_Q = \alpha_Q\circ\tau_{\mathsf{TT}^*Q} \qquad (267)$$

and

$$\mathsf{T}\pi_{\mathsf{T}Q}\circ\mathsf{T}\alpha_Q = \mathsf{TT}\pi_Q \qquad (268)$$

derived from

$$\pi_{\mathsf{T}Q}\circ\alpha_Q = \mathsf{T}\pi_Q. \qquad (269)$$

We have shown that

$$\alpha_Q^*\vartheta_{\mathsf{T}Q} = \mathrm{d}_T\vartheta_Q. \qquad (270)$$

It follows that the 2-form $\mathrm{d}_T\omega_Q$ is non degenerate and that the mapping $\alpha_Q\colon \mathsf{TT}^*Q \to \mathsf{T}^*\mathsf{T}Q$ is a symplectomorphism from $(\mathsf{TT}^*Q, \mathrm{d}_T\omega_Q)$ to $(\mathsf{T}^*\mathsf{T}Q, \omega_{\mathsf{T}Q})$ since

$$\mathrm{d}_T\omega_Q = \mathrm{d}_T\mathrm{d}\vartheta_Q = \mathrm{d}\mathrm{d}_T\vartheta_Q = \mathrm{d}\alpha_Q^*\vartheta_{\mathsf{T}Q} = \alpha_Q^*\mathrm{d}\vartheta_{\mathsf{T}Q} = \alpha_Q^*\omega_{\mathsf{T}Q}. \qquad (271)$$

These results are confirmed by the coordinate calculations

$$\alpha_Q^*\vartheta_{\mathsf{T}Q} = \dot{p}_\kappa \mathrm{d}q^\kappa + p_\kappa \mathrm{d}\dot{q}^\kappa = \mathrm{d}_T\vartheta_Q \tag{272}$$

and

$$\alpha_Q^*\omega_{\mathsf{T}Q} = \mathrm{d}\dot{p}_\kappa \wedge \mathrm{d}q^\kappa + \mathrm{d}p_\kappa \wedge \mathrm{d}\dot{q}^\kappa = \mathrm{d}_T\omega_Q. \tag{273}$$

In the cotangent bundle $\mathsf{T}^*\mathsf{T}^*Q$ we use coordinates

$$(q^\kappa, p_\lambda, u_\mu, v^\nu)\colon \mathsf{T}^*\mathsf{T}^*Q \to \mathbb{R}^{4m} \tag{274}$$

induced by coordinates (q^κ, p_λ) in T^*Q. We have the Liouville form

$$\vartheta_{\mathsf{T}^*Q} = u_\kappa \mathrm{d}q^\kappa + v^\kappa \mathrm{d}p_\kappa. \tag{275}$$

and the symplectic 2-form

$$\omega_{\mathsf{T}^*Q} = \mathrm{d}u_\kappa \wedge \mathrm{d}q^\kappa + \mathrm{d}v^\kappa \wedge \mathrm{d}p_\kappa. \tag{276}$$

on $\mathsf{T}^*\mathsf{T}^*Q$. We have already introduced the mapping

$$\beta_{(\mathsf{T}^*Q,\omega_Q)}\colon \mathsf{T}\mathsf{T}^*Q \to \mathsf{T}^*\mathsf{T}^*Q \tag{277}$$

characterized by the equality

$$\langle \beta_{(\mathsf{T}^*Q,\omega_Q)}(u), v\rangle = \langle \omega_Q, u \wedge v\rangle \tag{278}$$

for vectors $u \in \mathsf{T}\mathsf{T}^*Q$ and $v \in \mathsf{T}\mathsf{T}^*Q$ such that $\tau_{\mathsf{T}^*Q}(v) = \tau_{\mathsf{T}^*Q}(u)$. The diagram

$$\begin{array}{ccc} \mathsf{T}\mathsf{T}^*Q & \xrightarrow{\beta_{(\mathsf{T}^*Q,\omega_Q)}} & \mathsf{T}^*\mathsf{T}^*Q \\ {\scriptstyle \tau_{\mathsf{T}^*Q}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathsf{T}^*Q}} \\ \mathsf{T}^*Q & = & \mathsf{T}^*Q \end{array} \tag{279}$$

is a vector fibration isomorphism. For each $z \in \mathsf{T}\mathsf{T}^*\mathsf{T}^*Q$ we have

$$\begin{aligned} \langle \beta^*_{(\mathsf{T}^*Q,\omega_Q)}\vartheta_{\mathsf{T}^*Q}, z\rangle &= \langle \vartheta_{\mathsf{T}^*Q}, \mathsf{T}\beta_{(\mathsf{T}^*Q,\omega_Q)}(z)\rangle \\ &= \langle \tau_{\mathsf{T}^*\mathsf{T}^*Q}(\mathsf{T}\beta_{(\mathsf{T}^*Q,\omega_Q)}(z)), \mathsf{T}\pi_{\mathsf{T}^*Q}(\mathsf{T}\beta_{(\mathsf{T}^*Q,\omega_Q)}(z))\rangle \\ &= \langle \beta_{(\mathsf{T}^*Q,\omega_Q)}(\tau_{\mathsf{T}\mathsf{T}^*Q}(z)), \mathsf{T}\tau_{\mathsf{T}^*Q}(z)\rangle \\ &= \langle \omega_Q, \tau_{\mathsf{T}\mathsf{T}^*Q}(z) \wedge \mathsf{T}\tau_{\mathsf{T}^*Q}(z)\rangle \\ &= \langle \mathrm{i}_T\omega_Q, z\rangle. \end{aligned} \tag{280}$$

The formula (218) and relations

$$\tau_{\mathsf{T}^*\mathsf{T}^*Q} \circ \mathsf{T}\beta_{(\mathsf{T}^*Q,\omega_Q)} = \beta_{(\mathsf{T}^*Q,\omega_Q)} \circ \tau_{\mathsf{T}\mathsf{T}^*Q} \tag{281}$$

and

$$\mathsf{T}\pi_{\mathsf{T}^*Q} \circ \mathsf{T}\beta_{(\mathsf{T}^*Q,\omega_Q)} = \mathsf{T}\tau_{\mathsf{T}^*Q} \tag{282}$$

derived from

$$\pi_{\mathsf{T}^*Q} \circ \beta_{(\mathsf{T}^*Q,\omega_Q)} = \tau_{\mathsf{T}^*Q} \tag{283}$$

were used. We have shown that

$$\beta^*_{(\mathsf{T}^*Q,\omega_Q)}\vartheta_{\mathsf{T}^*Q} = \mathrm{i}_T\omega_Q. \tag{284}$$

It follows that the mapping $\beta_{(\mathsf{T}^*Q,\omega_Q)}\colon \mathsf{T}\mathsf{T}^*Q \to \mathsf{T}^*\mathsf{T}^*Q$ is a symplectomorphism from $(\mathsf{T}\mathsf{T}^*Q, \mathrm{d}_T\omega_Q)$ to $(\mathsf{T}^*\mathsf{T}^*Q, \omega_{\mathsf{T}^*Q})$ since

$$\mathrm{d}_T\omega_Q = \mathrm{d}\mathrm{i}_T\omega_Q = \mathrm{d}\beta^*_{(\mathsf{T}^*Q,\omega_Q)}\vartheta_{\mathsf{T}^*Q} = \beta^*_{(\mathsf{T}^*Q,\omega_Q)}\mathrm{d}\vartheta_{\mathsf{T}^*Q} = \beta^*_{(\mathsf{T}^*Q,\omega_Q)}\omega_{\mathsf{T}^*Q}. \tag{285}$$

Coordinate calculations

$$\beta^*_{(\mathsf{T}^*Q,\omega_Q)}\vartheta_{\mathsf{T}^*Q} = \dot{p}_\kappa \mathrm{d}q^\kappa - \dot{q}^\kappa \mathrm{d}p_\kappa = \mathrm{i}_T\omega_Q \tag{286}$$

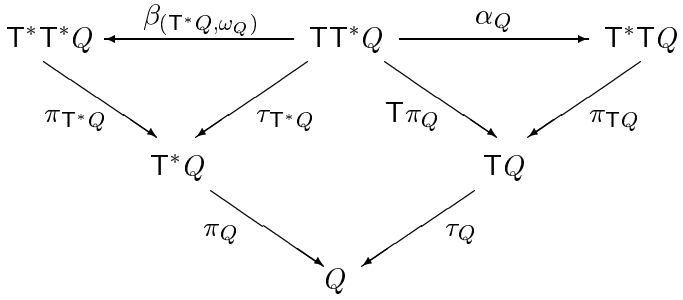
and

$$\beta^*_{(\mathsf{T}^*Q,\omega_Q)}\omega_{\mathsf{T}^*Q} = \mathrm{d}\dot{p}_\kappa \wedge \mathrm{d}q^\kappa + \mathrm{d}p_\kappa \wedge \mathrm{d}\dot{q}^\kappa = \mathrm{d}_T\omega_Q. \tag{287}$$

confirm these results.

8. Dynamics of mechanical systems

Let Q be the *configuration manifold* of a mechanical system. The cotangent bundle T^*Q is the *phase space* of the system. Elements of the phase space are *momenta*. The commutative diagram



(288)

contains the geometric structures used to formulate the dynamics of the system. The dynamics is a differential equation $D \subset \mathsf{TT}^*Q$. A solution $\pi\colon I \to \mathsf{T}^*Q$ of this equation is a phase space trajectory of the system. External forces have to be included in a complete description of dynamics. The dynamics of the system with external forces is the differential equation

$$D_f = \psi_Q^{-1}(D) \subset \mathsf{TT}^*Q \underset{(\pi_Q \circ \tau_{\mathsf{T}^*Q}, \pi_Q)}{\times} \mathsf{T}^*Q. \tag{289}$$

A solution is a curve $(\pi, \phi)\colon I \to \mathsf{T}^*Q \underset{(\pi_Q, \pi_Q)}{\times} \mathsf{T}^*Q$. The values of this curve represent the momenta and external forces. The differential equation D_f is of first order for the momentum component π and of zero order for the force component ϕ. This treatment of external forces is suitable for non relativistic systems. Relativistic systems described by homogeneous Lagrangians require a modification of the concept of force. We will deal with dynamics without external forces.

Trajectories of the system in the configuration manifold Q are solutions of the second order Euler–Lagrange equation

$$E = \mathsf{T}^2\pi_Q(\mathsf{P}D). \tag{290}$$

We have recognized the presence of a canonical symplectic structure in TT^*Q with the symplectic form $\mathrm{d}_T\omega_Q$. In most cases of interest in relativistic physics the dynamics is a Lagrangian submanifold of $(\mathsf{TT}^*Q, \mathrm{d}_T\omega_Q)$. Morphisms α_Q and $\beta_{(\mathsf{T}^*Q,\omega_Q)}$ are canonical symplectomorphisms from $(\mathsf{TT}^*Q, \mathrm{d}_T\omega_Q)$ to $(\mathsf{T}^*\mathsf{T}Q, \omega_{\mathsf{T}Q})$ and to $(\mathsf{T}^*\mathsf{T}^*Q, \omega_{\mathsf{T}^*Q})$. These symplectomorphisms with cotangent bundles create the possibility of generating the dynamics from (generalized) Lagrangians associated with $\mathsf{T}Q$ or (generalized) Hamiltonians associated with T^*Q.

We will present a number of examples of mechanical systems in Lagrangian and Hamiltonian formulations. We will perform the Legendre transformations and test the integrability criteria for these systems.

9. Lagrangian systems

Let

$$L\colon \mathsf{T}Q \to \mathbb{R} \tag{291}$$

be the Lagrangian of a mechanical system with configuration space Q. The Lagrangian may be defined on all of $\mathsf{T}Q$ or on an open subset. The image $N = \mathrm{im}(\mathrm{d}L)$ of the mapping

$$\mathrm{d}L\colon \mathsf{T}Q \to \mathsf{T}^*\mathsf{T}Q \tag{292}$$

is a Lagrangian submanifold of $(\mathsf{T}^*\mathsf{T}Q, \omega_{\mathsf{T}Q})$ and the set $D = \alpha_Q^{-1}(N) \subset \mathsf{TT}^*Q$ is a Lagrangian submanifold of $(\mathsf{TT}^*Q, \mathrm{d}_T\omega_Q)$. Coordinates $(q^\kappa, \dot{q}^\lambda, a_\mu, b_\nu)$ of N satisfy equations

$$\begin{aligned} a_\mu &= \partial_\mu L(q^\kappa, \dot{q}^\lambda) \\ b_\mu &= \partial_{\dot{\mu}} L(q^\kappa, \dot{q}^\lambda) \end{aligned} \tag{293}$$

and coordinates $(q^\kappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu)$ of elements of D satisfy equations

$$\begin{aligned} \dot{p}_\mu &= \partial_\mu L(q^\kappa, \dot{q}^\lambda) \\ p_\mu &= \partial_{\dot{\mu}} L(q^\kappa, \dot{q}^\lambda) \end{aligned} \tag{294}$$

derived from the variational principle

$$\dot{p}_\kappa \delta q^\kappa + p_\kappa \delta \dot{q}^\kappa = \delta L(q^\kappa, \dot{q}^\lambda) = \partial_\mu L(q^\kappa, \dot{q}^\lambda)\delta q^\mu + \partial_{\dot{\mu}} L(q^\kappa, \dot{q}^\lambda)\delta \dot{q}^\mu. \tag{295}$$

Substituting the equalities

$$\mathrm{d}\dot{p}_\mu = \partial_\nu \partial_\mu L \mathrm{d}q^\nu + \partial_{\dot{\nu}} \partial_\mu L \mathrm{d}\dot{q}^\nu \tag{296}$$

and

$$\mathrm{d}p_\mu = \partial_\nu \partial_{\dot{\mu}} L \mathrm{d}q^\nu + \partial_{\dot{\nu}} \partial_{\dot{\mu}} L \mathrm{d}\dot{q}^\nu \tag{297}$$

in

$$\mathrm{d}_T\omega_Q = \mathrm{d}\dot{p}_\mu \wedge \mathrm{d}q^\mu + \mathrm{d}p_\mu \wedge \mathrm{d}\dot{q}^\mu \tag{298}$$

we obtain the equality

$$\begin{aligned} \mathrm{d}_T\omega_Q|D &= \partial_\nu \partial_\mu L \mathrm{d}q^\nu \wedge \mathrm{d}q^\mu + \partial_{\dot{\nu}} \partial_\mu L \mathrm{d}\dot{q}^\nu \wedge \mathrm{d}q^\mu \\ &\quad + \partial_\nu \partial_{\dot{\mu}} L \mathrm{d}q^\nu \wedge \mathrm{d}\dot{q}^\mu + \partial_{\dot{\nu}} \partial_{\dot{\mu}} L \mathrm{d}\dot{q}^\nu \wedge \mathrm{d}\dot{q}^\mu = 0. \end{aligned} \tag{299}$$

This equality together with $\dim(D) = 2m = 1/2 \dim(\mathsf{TT}^*Q)$ confirms that D is a Lagrangian submanifold of $(\mathsf{TT}^*Q, \mathrm{d}_T\omega_Q)$. The set N is a Lagrangian submanifold since it is the image of the differential of a function and D is a Lagrangian submanifold since it is obtained from N by applying the symplectomorphism α_Q. We have confirmed this fact by direct calculation.

The set $D \subset \mathsf{TT}^*Q$ is a differential equation. A solution is a curve $\gamma\colon \mathbb{R} \to \mathsf{T}^*Q$ such that vectors tangent to γ are in D. Equations

$$\begin{aligned} \dot{\gamma}_\mu &= \partial_\mu L(\gamma^\kappa, \dot{\gamma}^\lambda) \\ \gamma_\nu &= \partial_{\dot{\nu}} L(\gamma^\kappa, \dot{\gamma}^\lambda) \end{aligned} \tag{300}$$

are differential equations for the coordinate expression

$$(\gamma^\kappa, \gamma_\lambda) = (q^\kappa, p_\lambda) \circ \gamma \tag{301}$$

of a curve γ derived from equations (294). Dots indicate derivatives. The differential equation D represents dynamics in the sense that solution curves are phase space trajectories of the mechanical system. We say that D is a *Lagrangian system* since it was obtained from a Lagrangian function (291). Equations (294) and equations (300) will be called the *Lagrange equations.*

The second order equations

$$\begin{aligned}
\dot{p}_\mu &= \partial_\mu L(q^\kappa, \dot{q}^\lambda) \\
p_\mu &= \partial_{\dot{\mu}} L(q^\kappa, \dot{q}^\lambda) \\
\ddot{p}_\mu &= \partial_\nu \partial_\mu L(q^\kappa, \dot{q}^\lambda)\dot{q}^\nu + \partial_{\dot{\nu}} \partial_\mu L(q^\kappa, \dot{q}^\lambda)\ddot{q}^\nu \\
\dot{p}_\mu &= \partial_\nu \partial_{\dot{\mu}} L(q^\kappa, \dot{q}^\lambda)\dot{q}^\nu + \partial_{\dot{\nu}} \partial_{\dot{\mu}} L(q^\kappa, \dot{q}^\lambda)\ddot{q}^\nu
\end{aligned} \tag{302}$$

represent the prolongation $\mathsf{P}D$ of the Lagrange equations. The equations

$$\partial_\nu \partial_{\dot{\mu}} L(q^\kappa, \dot{q}^\lambda)\dot{q}^\nu + \partial_{\dot{\nu}} \partial_{\dot{\mu}} L(q^\kappa, \dot{q}^\lambda)\ddot{q}^\nu - \partial_\mu(q^\kappa, \dot{q}^\lambda) = 0 \tag{303}$$

are the Euler–Lagrange equation $\mathsf{T}^2\pi_Q(\mathsf{P}D)$ in coordinate form.

Example 4. Let Q be a manifold of dimension 3 with coordinates $(q^i) = (q^1, q^2, q^3)$. There is a Riemannian metric tensor g_{ij}, a function ϕ, and a 1-form $A = A_i \mathrm{d}q^i$ on Q. The function

$$L(q^i, \dot{q}^j) = \frac{m}{2} g_{ij}\dot{q}^i\dot{q}^j - e\phi + eA_i\dot{q}^i \tag{304}$$

is the Lagrangian of a particle of mass m and charge e in an electric field

$$E = E_i \mathrm{d}q^i = -\mathrm{d}\phi = -\partial_i \phi \mathrm{d}q^i \tag{305}$$

and a magnetic field (induction)

$$\begin{aligned}
B &= \frac{1}{2} B_{ij} \mathrm{d}q^i \wedge \mathrm{d}q^j = \mathrm{d}A = \mathrm{d}A_j \wedge \mathrm{d}q^j \\
&= \partial_i A_j \mathrm{d}q^i \wedge \mathrm{d}q^j = \frac{1}{2}(\partial_i A_j - \partial_j A_i)\mathrm{d}q^i \wedge \mathrm{d}q^j.
\end{aligned} \tag{306}$$

The dynamics D of the particle is described by equations

$$\begin{aligned}
\dot{p}_i &= \frac{m}{2}\partial_i g_{jk}\dot{q}^j\dot{q}^k - e\partial_i\phi + e\partial_i A_j \dot{q}^j \\
p_i &= m g_{ij}\dot{q}^j + eA_i.
\end{aligned} \tag{307}$$

Gauge independent covariant second order Euler–Lagrange differential equations

$$m g_{ij}(\ddot{q}^j + \Gamma^j_{kl}\dot{q}^k\dot{q}^l) = eE_i + B_{ij}\dot{q}^j \tag{308}$$

are easily derived. The symbol Γ^j_{kl} is the Christoffel symbol

$$\Gamma^j_{kl} = \frac{1}{2} g^{ji} \left(\partial_k g_{li} + \partial_l g_{ki} - \partial_i g_{kl} \right). \tag{309}$$

Solution curves of (308) are motions in the configuration space Q. These equations provide a partial description of dynamics. The complete description of dynamics is obtained by complementing these equations with the gauge dependent velocity-momentum relation

$$p_i = m g_{ij} \dot{q}^j + e A_i. \tag{310}$$

A gauge transformation

$$A_i \mapsto A_i + \partial_i \psi \tag{311}$$

will not modify equations (308) but will change the velocity-momentum relation. ▲

EXAMPLE 5. A gauge independent formulation of dynamics of a charged particle is obtained by extending the configuration space Q to a manifold $\overline{Q}$ of four dimensions with coordinates (q, q^i). A *gauge transformation* is a coordinate transformation $(q, q^i) \mapsto (q + \psi(q^k), q^i)$. The dynamics is derived from the gauge independent Lagrangian

$$L(q, q^i, \dot{q}, \dot{q}^j) = \frac{m}{2} g_{ij} \dot{q}^i \dot{q}^j - e\phi + e A_i \dot{q}^i + e\dot{q}. \tag{312}$$

The equations

$$\begin{aligned} \dot{p} &= 0 \\ \dot{p}_i &= \frac{m}{2} \partial_i g_{jk} \dot{q}^j \dot{q}^k - e \partial_i \phi + e \partial_i A_j \dot{q}^j \\ p &= e \\ p_i &= m g_{ij} \dot{q}^j + e A_i \end{aligned} \tag{313}$$

provide a description of dynamics in terms of coordinates $(q, q^i, p, p_j, \dot{q}, \dot{q}^k, \dot{p}, \dot{p}_l)$ in $\mathsf{T}\mathsf{T}^*\overline{Q}$. These equations are gauge independent and can be given an explicitly covariant and gauge independent form

$$\begin{aligned} \dot{p} &= 0 \\ m g_{ij} (\ddot{q}^j + \Gamma^j_{kl} \dot{q}^k \dot{q}^l) &= e E_i + B_{ij} \dot{q}^j \\ p &= e \\ p_i - e A_i &= m g_{ij} \dot{q}^j. \end{aligned} \tag{314}$$

The gauge invariant quantity $(p_i - e A_i)$ is the momentum of the particle. ▲

EXAMPLE 6. Let Q be the space-time of general relativity with coordinates $(q^\kappa) = (q^0, q^1, q^2, q^3)$. The gravitational field is represented by a Minkowski metric $g_{\kappa\lambda}$ and the electromagnetic field is a 2-form

$$\begin{aligned} F &= \frac{1}{2}F_{\kappa\lambda}\mathrm{d}q^\kappa \wedge \mathrm{d}q^\lambda = -\mathrm{d}A = -\mathrm{d}A_\lambda \wedge \mathrm{d}q^\lambda \\ &= -\partial_\kappa A_\lambda \mathrm{d}q^\kappa \wedge \mathrm{d}q^\lambda = \frac{1}{2}(\partial_\lambda A_\kappa - \partial_\kappa A_\lambda)\mathrm{d}q^\kappa \wedge \mathrm{d}q^\lambda \end{aligned} \tag{315}$$

derived from a potential

$$A = A_\kappa \mathrm{d}q^\kappa. \tag{316}$$

The dynamics of a relativistic particle of mass m and charge e is derived from the Lagrangian

$$L(q^\kappa, \dot{q}^\lambda) = m\sqrt{g_{\kappa\lambda}\dot{q}^\kappa\dot{q}^\lambda} + eA_\kappa\dot{q}^\kappa \tag{317}$$

defined for time-like vectors — vectors satisfying $g_{\kappa\lambda}\dot{q}^\kappa\dot{q}^\lambda > 0$. Dynamics $D \subset \mathsf{T}\mathsf{T}^*Q$ is described by the Lagrange equations

$$\begin{aligned} \dot{p}_\kappa &= m\partial_\kappa g_{\lambda\mu}\frac{\dot{q}^\lambda\dot{q}^\mu}{2\|\dot{q}\|} + e\partial_\kappa A_\lambda\dot{q}^\lambda = mg_{\lambda\nu}\Gamma^\nu_{\kappa\mu}\frac{\dot{q}^\lambda\dot{q}^\mu}{\|\dot{q}\|} + e\partial_\kappa A_\lambda\dot{q}^\lambda \\ p_\kappa &= mg_{\kappa\lambda}\frac{\dot{q}^\lambda}{\|\dot{q}\|} + eA_\kappa \end{aligned} \tag{318}$$

with $\|\dot{q}\| = \sqrt{g_{\kappa\lambda}\dot{q}^\kappa\dot{q}^\lambda}$. Note that these equations are reparametrization invariant: if $\gamma\colon \mathbb{R} \to \mathsf{T}^*Q$ is a solution and $\sigma\colon \mathbb{R} \to \mathbb{R}$ is a diffeomorphism with positive derivative, then $\gamma \circ \sigma$ is a solution. One can say that solutions are one-dimensional oriented but not parametrized submanifolds of T^*Q.

The Euler–Lagrange equations

$$\frac{m}{\|\dot{q}\|}g_{\kappa\lambda}\left(\ddot{q}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{q}^\mu\dot{q}^\nu\right) + eF_{\kappa\lambda}\dot{q}^\lambda = \frac{m}{\|\dot{q}\|^3}g_{\kappa\lambda}\dot{q}^\lambda g_{\mu\nu}\left(\ddot{q}^\mu + \Gamma^\lambda_{\rho\sigma}\dot{q}^\rho\dot{q}^\sigma\right)\dot{q}^\nu \tag{319}$$

are reparametrization invariant and gauge invariant. If proper time is chosen as the parameter, then $\|\dot{q}\| = 1$ and the world line of the particle in space-time is a solution of simplified gauge invariant second order differential equations

$$m(\ddot{q}^\kappa + \Gamma^\kappa_{\lambda\mu}\dot{q}^\lambda\dot{q}^\mu) = -eg^{\kappa\lambda}F_{\lambda\mu}\dot{q}^\mu. \tag{320}$$

The complete dynamics is gauge dependent. ▲

EXAMPLE 7. Let Q be the space-time of general relativity with coordinates (q^κ), a Minkowski metric $g_{\kappa\lambda}$ and an electromagnetic potential $A = A_\kappa \mathrm{d}q^\kappa$.

Let $\overline{Q}$ be a manifold of dimension 5 with coordinates (q, q^κ). A *gauge transformation* is a coordinate transformation $(q, q^\kappa) \mapsto (q + \psi(q^\mu), q^\kappa)$. We use coordinates $(q, q^\kappa, p, p_\lambda)$ in $\mathsf{T}^*\overline{Q}$. Two gauge invariant quantities are derived from the 5-*momentum* (p, p_λ). These are the *charge* p and the *energy momentum* $(p_\lambda - pA_\lambda)$. There are two equivalent interpretations of the manifold $\overline{Q}$ [6]. This manifold is interpreted as a pseudoriemannian manifold (Kaluza [5]) with a metric tensor

$$\begin{pmatrix} 1 & A_\lambda \\ A_\kappa & g_{\kappa\lambda} + A_\kappa A_\lambda \end{pmatrix} \tag{321}$$

or as the total space of a principal fibration (Utiyama [14])

$$\chi\colon \overline{Q} \to Q \tag{322}$$

characterized by

$$(q^\kappa) \circ \chi = (q^\kappa) \tag{323}$$

The electromagnetic potential is used to introduce the connection form

$$\alpha = \mathrm{d}q + A_\kappa \mathrm{d}q^\kappa \tag{324}$$

in the principal bundle $\overline{Q}$. The curvature form

$$\begin{aligned} \beta = \mathrm{d}\alpha &= -\tfrac{1}{2} F_{\kappa\lambda} \mathrm{d}q^\kappa \mathrm{d}q^\lambda = \mathrm{d}A_\lambda \wedge \mathrm{d}q^\lambda \\ &= \partial_\kappa A_\lambda \mathrm{d}q^\kappa \wedge \mathrm{d}q^\lambda = -\tfrac{1}{2}(\partial_\lambda A_\kappa - \partial_\kappa A_\lambda) \mathrm{d}q^\kappa \wedge \mathrm{d}q^\lambda \end{aligned} \tag{325}$$

represents the electromagnetic field.

The Lagrangian of a particle with mass m and charge e is the gauge invariant function

$$L(q, q^\kappa, \dot{q}, \dot{q}^\lambda) = m\sqrt{g_{\kappa\lambda}\dot{q}^\kappa \dot{q}^\lambda} + eA_\kappa \dot{q}^\kappa + e\dot{q}. \tag{326}$$

Coordinates $(q, q^\kappa, p, p_\lambda, \dot{q}, \dot{q}^\mu, \dot{p}, \dot{p}_\nu)$ are used in $\mathsf{TT}^*\overline{Q}$. The Lagrange equations

$$\begin{aligned} \dot{p} &= 0 \\ \dot{p}_\kappa &= m\partial_\kappa g_{\lambda\mu} \frac{\dot{q}^\lambda \dot{q}^\mu}{2\|\dot{q}\|} + e\partial_\kappa A_\lambda \dot{q}^\lambda = m g_{\lambda\nu} \Gamma^\nu_{\kappa\mu} \frac{\dot{q}^\lambda \dot{q}^\mu}{\|\dot{q}\|} + e\partial_\kappa A_\lambda \dot{q}^\lambda \\ p &= e \\ p_\kappa &= m g_{\kappa\lambda} \frac{\dot{q}^\lambda}{\|\dot{q}\|} + eA_\kappa \end{aligned} \tag{327}$$

are equivalent to the explicitly covariant and gauge independent second order equations

$$\begin{aligned} \dot{p} &= 0 \\ m(\ddot{q}^\kappa + \Gamma^\kappa_{\lambda\mu}\dot{q}^\lambda\dot{q}^\mu) &= eg^{\kappa\lambda}F_{\lambda\mu}\dot{q}^\mu \\ p &= e \\ p_\kappa - eA_\kappa &= mg_{\kappa\lambda}\dot{q}^\lambda. \end{aligned} \tag{328}$$

These equations are obtained by adopting the simplifying condition $\|\dot{q}\| = 1$. Trajectories in $\overline{Q}$ satisfy the second order equations

$$m(\ddot{q}^\kappa + \Gamma^\kappa_{\lambda\mu}\dot{q}^\lambda\dot{q}^\mu) = eg^{\kappa\lambda}F_{\lambda\mu}\dot{q}^\mu \tag{329}$$

with no conditions on q. Compatibility with field equations requires that trajectories in $\overline{Q}$ be two dimensional. The dynamics of a charge particle has to be suitably modified for correct description of interaction with the electromagnetic field. ▲

Not all mechanical systems are Lagrangian systems derived from a Lagrangian defined on the tangent bundle $\mathsf{T}Q$. The dynamics could be generated by a Morse family of functions defined on fibres of a fibration

$$\eta\colon Y \to \mathsf{T}Q. \tag{330}$$

We have coordinates $(q^\kappa, \dot{q}^\lambda)$ in $\mathsf{T}Q$. In the space Y we use adapted coordinates

$$(q^\kappa, \dot{q}^\lambda, y^A)\colon Y \to \mathbb{R}^{2m+k} \tag{331}$$

such that

$$(q^\kappa, \dot{q}^\lambda) \circ \eta = (q^\kappa, \dot{q}^\lambda). \tag{332}$$

Let $L\colon Y \to \mathbb{R}$ be a Morse family of functions defined on fibres of η. The $k \times (2m + k)$ matrix

$$\left(\frac{\partial^2 L}{\partial y^A \partial y^B} \quad \frac{\partial^2 L}{\partial y^A \partial q^\kappa} \quad \frac{\partial^2 L}{\partial y^A \partial \dot{q}^\lambda} \right) \tag{333}$$

is of maximal rank. The Lagrangian submanifold $N \subset \mathsf{T}^*\mathsf{T}Q$ generated by the family is the set of elements of $\mathsf{T}^*\mathsf{T}Q$ with coordinates $(q^\kappa, \dot{q}^\lambda, a_\mu, b_\nu)$ satisfying equations

$$\begin{aligned} a_\mu &= \partial_\mu L(q^\kappa, \dot{q}^\lambda, y^A) \\ b_\nu &= \partial_{\dot{\nu}} L(q^\kappa, \dot{q}^\lambda, y^A) \\ 0 &= \partial_A L(q^\kappa, \dot{q}^\lambda, y^A), \end{aligned} \tag{334}$$

for some values of the variables y^A. The set $D = \alpha_Q^{-1}(N) \subset \mathsf{TT}^*Q$ is a Lagrangian submanifold of $(\mathsf{TT}^*Q, \mathrm{d}_T\omega_Q)$. It is the set of elements of TT^*Q with coordinates $(q^\kappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu)$ satisfying equations

$$\begin{aligned} \dot{p}_\mu &= \partial_\mu L(q^\kappa, \dot{q}^\lambda, y^A) \\ p_\nu &= \partial_{\dot{\nu}} L(q^\kappa, \dot{q}^\lambda, y^A) \\ 0 &= \partial_A L(q^\kappa, \dot{q}^\lambda, y^A) \end{aligned} \tag{335}$$

for some values of the variables y^A. These equations can be derived from a variational principle

$$\begin{aligned} \dot{p}_\kappa \delta q^\kappa + p_\kappa \delta \dot{q}^\kappa &= \delta L(q^\kappa, \dot{q}^\lambda, y^A) = \partial_\mu L(q^\kappa, \dot{q}^\lambda, y^A)\delta q^\mu \\ &\quad + \partial_{\dot{\mu}} L(q^\kappa, \dot{q}^\lambda, y^A)\delta \dot{q}^\mu + \partial_B L(q^\kappa, \dot{q}^\lambda, y^A)\delta y^B. \end{aligned} \tag{336}$$

Equations (335) present the set D parametrized by variables $(q^\kappa, \dot{q}^\lambda, y^A)$ subject to $\partial_A L(q^\kappa, \dot{q}^\lambda, y^A) = 0$. From

$$\mathrm{d}\dot{p}_\mu = \partial_\nu \partial_\mu L \mathrm{d}q^\nu + \partial_{\dot{\nu}} \partial_\mu L \mathrm{d}\dot{q}^\nu + \partial_A \partial_\mu L \mathrm{d}y^A, \tag{337}$$

$$\mathrm{d}p_\mu = \partial_\nu \partial_{\dot{\mu}} L \mathrm{d}q^\nu + \partial_{\dot{\nu}} \partial_{\dot{\mu}} L \mathrm{d}\dot{q}^\nu + \partial_A \partial_{\dot{\mu}} \mathrm{d}y^A, \tag{338}$$

$$\partial_A \partial_\mu L = 0, \tag{339}$$

$$\partial_A \partial_{\dot{\mu}} L = 0 \tag{340}$$

we obtain the equality

$$\mathrm{d}_T\omega_Q | D = 0. \tag{341}$$

Equations $\partial_A L(q^\kappa, \dot{q}^\lambda, y^A) = 0$ leave only $2m$ out of the $2m + k$ variables $(q^\kappa, \dot{q}^\lambda, y^A)$ independent. This is a consequence of maximality of the rank of the matrix (333). It follows that $\dim(D) = 2m$. We have thus confirmed that D is a Lagrangian submanifold of $(\mathsf{TT}^*Q, \mathrm{d}_T\omega_Q)$. The set D is a differential equation and may represent the dynamics of a mechanical system.

EXAMPLE 8. Let Q be the space-time of general relativity with coordinates $(q^\kappa) = (q^0, q^1, q^2, q^3)$ and a Minkowski metric $g_{\kappa\lambda}$. Let

$$L(q^\kappa, \dot{q}^\lambda, y) = \frac{1}{2y} g_{\kappa\lambda} \dot{q}^\kappa \dot{q}^\lambda \tag{342}$$

be a function on $\overset{\circ}{\mathsf{T}}Q \times \mathbb{R}_+$, where $\overset{\circ}{\mathsf{T}}Q$ is the tangent bundle with the zero vectors removed. This function is a Morse family of functions of the variable

y with the coordinates $(q^\kappa, \dot{q}^\lambda)$ treated as parameters. It represents the Lagrangian of a particle of mass zero. The dynamics of the particle is governed by the equations

$$\begin{aligned} \dot{p}_\kappa &= \frac{1}{2y}\partial_k g_{\lambda\mu}\dot{q}^\lambda\dot{q}^\mu \\ p_\kappa &= \frac{1}{y}g_{\kappa\lambda}\dot{q}^\lambda \\ 0 &= g_{\kappa\lambda}\dot{q}^\kappa\dot{q}^\lambda \end{aligned} \tag{343}$$

satisfied for some value of the variable y. The variable y can be eliminated from the equation for $\dot{p}_\kappa$. It follows from the resulting equation

$$\dot{p}_\kappa - \Gamma^\lambda_{\kappa\mu}p_\lambda\dot{q}^\mu = 0 \tag{344}$$

that the covector p_κ is covariant constant along the world line. If an affine parameter is chosen then y is constant and the dynamics satisfies equations

$$\begin{aligned} \ddot{q}^\lambda + \Gamma^\lambda_{\kappa\mu}\dot{q}^\kappa\dot{q}^\mu &= 0 \\ g_{\kappa\lambda}q^\kappa q^\lambda &= 0 \\ p_\kappa &= \frac{1}{y}g_{\kappa\lambda}q^\lambda \end{aligned} \tag{345}$$

for some constant $y > 0$. ▲

10. Hamiltonian systems

Let

$$H\colon \mathsf{T}^*Q \to \mathbb{R} \tag{346}$$

be the Hamiltonian of a mechanical system with configuration space Q. The mapping

$$-\mathrm{d}H\colon \mathsf{T}^*Q \to \mathsf{T}^*\mathsf{T}^*Q \tag{347}$$

is a section of the fibration π_{T^*Q}. Consequently

$$X = \beta^{-1}_{(\mathsf{T}^*Q,\omega_Q)} \circ (-\mathrm{d}H)\colon \mathsf{T}^*Q \to \mathsf{T}\mathsf{T}^*Q \tag{348}$$

is a vector field. The image $M = \mathrm{im}(-\mathrm{d}H)$ is a Lagrangian submanifold of $(\mathsf{T}^*\mathsf{T}^*Q, \omega_{\mathsf{T}^*Q})$ and $D = \mathrm{im}(X)$ is a Lagrangian submanifold of $(\mathsf{T}\mathsf{T}^*Q, \mathrm{d}_T\omega_Q)$. The equations describing D are the *Hamilton equations*

$$\begin{aligned} \dot{q}^\kappa &= \partial^\kappa H \\ \dot{p}_\kappa &= -\partial_\kappa H \end{aligned} \tag{349}$$

derived from the variational principle

$$\dot{p}_\mu \delta q^\mu - \dot{q}^\mu \delta p_\mu = \delta H(q^\kappa, p_\lambda) = -\partial_\mu H(q^\kappa, p_\lambda)\delta q^\mu - \partial^\mu H(q^\kappa, p_\lambda)\delta p_\kappa. \quad (350)$$

The symbol ∂^κ denotes the partial derivative $\frac{\partial}{\partial p_\kappa}$. The dimension of D is $2m$ and by using equalities

$$\mathrm{d}\dot{q}^\mu = \partial_\nu \partial^\mu H \mathrm{d}q^\nu + \partial^\nu \partial^\mu H \mathrm{d}p_\nu \quad (351)$$

and

$$\mathrm{d}\dot{p}_\mu = -\partial_\nu \partial_\mu H \mathrm{d}q^\nu - \partial^\nu \partial_\mu H \mathrm{d}p_\nu \quad (352)$$

in

$$\mathrm{d}_T \omega_Q = \mathrm{d}\dot{p}_\mu \wedge \mathrm{d}q^\mu + \mathrm{d}p_\mu \wedge \mathrm{d}\dot{q}^\mu \quad (353)$$

we obtain the equality

$$\begin{aligned} \mathrm{d}_T \omega_Q | D \;=\; & -\partial_\nu \partial_\mu H \mathrm{d}q^\nu \wedge \mathrm{d}q^\mu - \partial^\nu \partial_\mu H \mathrm{d}p_\nu \wedge \mathrm{d}q^\mu \\ & + \partial_\nu \partial^\mu H \mathrm{d}p_\mu \wedge \mathrm{d}q^\nu + \partial^\nu \partial_\mu H \mathrm{d}p_\mu \wedge \mathrm{d}p_\nu = 0. \end{aligned} \quad (354)$$

It follows that D is a Lagrangian submanifold of $(\mathsf{TT}^*Q, \mathrm{d}_T\omega_Q)$. The set D is a differential equation and may represent the dynamics of a mechanical system.

EXAMPLE 9. Equations (307) of Example 4 can be rewritten in the form

$$\begin{aligned} \dot{q}^i \;=\; & \frac{1}{m} g^{ij}(p_j - eA_j) \\ \dot{p}_i \;=\; & \frac{1}{2m} \partial_i g_{jk} g^{jl} g^{km} (p_l - eA_l)(p_m - eA_m) \\ & - e\partial_i \phi + \frac{e}{m} \partial_i A_j g^{jl} (p_l - eA_l). \end{aligned} \quad (355)$$

These equations describe a Hamiltonian vector field. They are the Hamilton equations for the Hamiltonian

$$H(q^i, p_j) = \frac{1}{2m} g^{ij}(p_i - eA_i)(p_j - eA_j) + e\phi. \quad (356)$$

▲

Gauge independent dynamics of charged particles and the dynamics of relativistic particles are not images of Hamiltonian vector fields. Dirac [1] introduced *generalized Hamiltonian systems* in order to be able to deal with similar cases. In the original construction of Dirac a generalized Hamiltonian system is a family of Hamiltonian vector fields on the phase space T^*Q

restricted to a *constraint set* $C \subset \mathsf{T}^*Q$. We have translated this construction in an equivalent construction of a differential equation $D \subset \mathsf{TT}^*Q$.

Let $C \subset \mathsf{T}^*Q$ be a submanifold and let

$$H\colon C \to \mathbb{R} \tag{357}$$

be a differentiable function. The set

$$M = \{b \in \mathsf{T}^*\mathsf{T}^*Q;\ a = \pi_{\mathsf{T}^*Q}(b) \in C, \forall_{u \in \mathsf{T}_aC \subset \mathsf{T}_a\mathsf{T}^*Q} \langle b, u\rangle = -\langle \mathrm{d}H, u\rangle\} \tag{358}$$

is a Lagrangian submanifold of $(\mathsf{T}^*\mathsf{T}^*Q, \omega_{\mathsf{T}^*Q})$ and

$$\begin{aligned} D &= \beta^{-1}_{(\mathsf{T}^*Q, \omega_Q)}(M) \\ &= \{w \in \mathsf{TT}^*Q;\ a = \tau_{\mathsf{T}^*Q}(w) \in C, \forall_{u \in \mathsf{T}_aC \subset \mathsf{T}_a\mathsf{T}^*Q} \langle \omega_Q, u \wedge w\rangle = \langle \mathrm{d}H, u\rangle\} \end{aligned} \tag{359}$$

is a Lagrangian submanifold of $(\mathsf{TT}^*Q, \mathrm{d}_T\omega_Q)$. If (Φ_A) is a set of k independent functions on T^*Q such that

$$C = \{a \in \mathsf{T}^*Q;\ \Phi_A(a) = 0 \text{ for } A = 1, \ldots, k\} \tag{360}$$

and $\overline{H}$ is a function on T^*Q such that $\overline{H}|C = H$, then coordinates $(q^\kappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu)$ of elements of D satisfy the equations

$$\begin{aligned} \Phi_A(q^\kappa, p_\lambda) &= 0 \\ \dot{q}^\kappa &= \partial^\kappa \overline{H} + v^A \partial^\kappa \Phi_A \\ \dot{p}_\kappa &= -\partial_\kappa \overline{H} + v^A \partial_\kappa \Phi_A \end{aligned} \tag{361}$$

derived from the variational principle

$$\begin{aligned} &\Phi_A(q^\kappa, p_\lambda) = 0 \\ &\dot{p}_\mu \delta q^\mu - \dot{q}^\mu \delta p_\mu = \delta \overline{H}(q^\kappa, p_\lambda) = -\partial_\mu \overline{H}(q^\kappa, p_\lambda)\delta q^\mu - \partial^\mu \overline{H}(q^\kappa, p_\lambda)\delta p_\kappa \end{aligned} \tag{362}$$

with variations $(\delta q^\kappa, \delta p_\lambda)$ satisfying

$$\partial_\kappa \Phi_A \delta q^\kappa + \partial^\kappa \Phi_A \delta p_\kappa = 0. \tag{363}$$

Lagrange multipliers ($v^A \in \mathbb{R}^k$) appear in these equations. The same equations are derived from the variational principle

$$\begin{aligned} \dot{p}_\mu \delta q^\mu &- \dot{q}^\mu \delta p_\mu = \delta \widetilde{H}(q^\kappa, p_\lambda, v^A) \\ &= -\partial_\mu H(q^\kappa, p_\lambda)\delta q^\mu - \partial^\mu H(q^\kappa, p_\lambda)\delta p_\kappa + \Phi_A(q^\kappa, p_\lambda)\delta v^A \end{aligned} \tag{364}$$

corresponding to the Morse family

$$\widetilde{H}(q^\kappa, p_\lambda, v^A) = \overline{H}(q^\kappa, p_\lambda) - v^A \Phi_A(q^\kappa, p_\lambda) \tag{365}$$

of functions of the variables ($v^A \in \mathbb{R}^k$). The set D is a Lagrangian submanifold since it is generated by a Morse family. At each point $a \in C$ the set $D_a = D \cap \mathsf{T}_a\mathsf{T}^*Q$ is an affine subspace of the vector space $\mathsf{T}_a\mathsf{T}^*Q$.

In Dirac's construction the dynamics is described by vector fields

$$X = \beta^{-1}_{(\mathsf{T}^*Q, \omega_Q)} \circ (v^A \mathrm{d}\Phi_A - \mathrm{d}\overline{H}) \colon \mathsf{T}^*Q \to \mathsf{T}\mathsf{T}^*Q \tag{366}$$

with arbitrary functions $v^A(q^\kappa, p_\lambda)$. These fields are restricted to the constraint set C.

EXAMPLE 10. The dynamics of the charged particle of Example 5 is a Dirac system. The Hamiltonian is the function

$$\overline{H}(q, q^i, p, p_j) = \frac{1}{2m} g^{ij}(p_i - eA_i)(p_j - eA_j) + e\phi \tag{367}$$

and the constraint is the set characterized by

$$\Phi(q, q^i, p, p_j) = p - e = 0. \tag{368}$$

The function

$$\widetilde{H}(q, q^i, p, p_j, v) = \frac{1}{2m} g^{ij}(p_i - eA_i)(p_j - eA_j) + e\phi - v(p - e) \tag{369}$$

is a Morse family of functions of $v \in \mathbb{R}$. Equations

$$\begin{aligned} p &= e \\ \dot{q} &= v \\ \dot{q}^i &= \frac{1}{m} g^{ij}(p_j - eA_j) \\ \dot{p} &= 0 \\ \dot{p}_i &= \frac{1}{2m} \partial_i g_{jk} g^{jl} g^{km} (p_l - eA_l)(p_m - eA_m) - e\partial_i \phi + \frac{e}{m} \partial_i A_j g^{jl} (p_l - eA_l). \end{aligned} \tag{370}$$

obtained with this Morse family are equivalent to equations (313). ▲

The dynamics of a non relativistic charged particle in the above example is the only Dirac system known to us. Hamiltonian formulations of relativistic dynamics require a higher level of complexity. Differential equations

generated by Lagrangians and by Lagrangian Morse families are Lagrangian submanifolds of $(\mathsf{TT}^*Q, \mathrm{d}_T\omega_Q)$. The same is true of Dirac systems. We define a *generalized Dirac system* as a differential equation $D \subset \mathsf{TT}^*Q$, which is a Lagrangian submanifold of $(\mathsf{TT}^*Q, \mathrm{d}_T\omega_Q)$. Existence of Morse families for open subsets of Lagrangian submanifolds is guaranteed by Hörmander's theorem. Known generalized Dirac systems are globally generated by Hamiltonian Morse families.

EXAMPLE 11. Lagrange equations (318) of Example 6 have an equivalent form

$$\begin{aligned} g^{\kappa\lambda}(p_\kappa - eA_\kappa)(p_\lambda - eA_\lambda) &= m^2 \\ \dot{q}^\kappa &= \frac{v}{m} g^{\kappa\lambda}(p_\lambda - eA_\lambda) \\ \dot{p}_\kappa &= -\frac{v}{2m}\partial_\kappa g^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu) \\ &\quad + \frac{ve}{m} g^{\mu\nu}\partial_\kappa A_\mu (p_\nu - eA_\nu) \end{aligned} \tag{371}$$

with arbitrary $v > 0$. These equations are obtained from the Morse family

$$H(q^\kappa, p_\lambda, v) = v\left(\sqrt{g^{\kappa\lambda}(p_\kappa - eA_\kappa)(p_\lambda - eA_\lambda)} - m\right) \tag{372}$$

of functions of the variable $v > 0$. ▲

EXAMPLE 12. The gauge independent dynamics in Example 7 is a generalized Dirac system. Equations

$$\begin{aligned} g^{\kappa\lambda}(p_\kappa - eA_\kappa)(p_\lambda - eA_\lambda) &= m^2 \\ p &= e \\ \dot{q} &= v^2 \\ \dot{q}^\kappa &= \frac{v^1}{m} g^{\kappa\lambda}(p_\lambda - eA_\lambda) \\ \dot{p} &= 0 \\ \dot{p}_\kappa &= -\frac{v^1}{2m}\partial_\kappa g^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu) \\ &\quad + \frac{v^1 e}{m} g^{\mu\nu}\partial_\kappa A_\mu (p_\nu - eA_\nu) \end{aligned} \tag{373}$$

with $v^1 > 0$ and arbitrary v^2 are equivalent to equations (328). These equations are generated by the Morse family

$$H(q^\kappa, p_\lambda, v^1, v^2) = v^1 \left(\sqrt{g^{\kappa\lambda}(p_\kappa - eA_\kappa)(p_\lambda - eA_\lambda)} - m \right) + v^2(p-e) \quad (374)$$

of functions of the variables $v^1 > 0$ and v^2. ▲

EXAMPLE 13. The generalized Dirac system of Example 8 is described by equations

$$\begin{aligned} g^{\kappa\lambda}p_\kappa p_\lambda &= 0 \\ \dot{q}^\kappa &= v g^{\kappa\lambda} p_\lambda \\ \dot{p}_\kappa &= -\frac{v}{2}\partial_\kappa g^{\mu\nu} p_\mu p_\nu \end{aligned} \quad (375)$$

derived from the Morse family

$$H(q^\kappa, p_\lambda, v) = \frac{v}{2} g^{\kappa\lambda} p_\kappa p_\lambda \quad (376)$$

with $v > 0$. ▲

11. The Legendre transformation

A Lagrangian $L(q^\kappa, \dot{q}^\lambda)$ is said to be *hyperregular* if the *Legendre mapping*

$$\lambda \colon \mathsf{T}Q \to \mathsf{T}^*Q \quad (377)$$

defined by

$$(q^\kappa, p_\lambda) \circ \lambda = (q^\kappa, \partial_{\dot{\lambda}} L(q^\rho, \dot{q}^\sigma)) \quad (378)$$

is a diffeomorphism. Let the mapping

$$\chi \colon \mathsf{T}^*Q \to \mathsf{T}Q \quad (379)$$

represented by

$$(q^\kappa, \dot{q}^\lambda) \circ \chi = (q^\kappa, \chi^\lambda(q^\rho, p_\sigma)) \quad (380)$$

be the inverse diffeomorphism. Relations

$$\partial_{\dot{\kappa}} L(q^\mu, \chi^\nu(q^\rho, p_\sigma)) = p_\kappa \quad (381)$$

$$\chi^\kappa(q^\mu, \partial\dot{\nu} L(q^\rho, \dot{q}^\sigma)) = \dot{q}^\kappa \quad (382)$$

hold. Using the diffeomorphism χ to eliminate the velocities $(\dot{q}^\kappa)$ from the *energy function*

$$E(q^\kappa, p_\lambda, \dot{q}^\mu) = p_\lambda \dot{q}^\lambda - L(q^\kappa, \dot{q}^\mu) \quad (383)$$

we obtain the Hamiltonian

$$H(q^\mu, p_\nu) = p_\lambda \chi^\lambda(q^\mu, p_\nu) - L(q^\mu, \chi^\nu(q^\rho, p_\sigma)). \tag{384}$$

The energy function is defined on $\mathsf{T}^*Q \underset{(\pi_Q, \tau_Q)}{\times} \mathsf{T}Q$. The passage from the Lagrangian to the above Hamiltonian is the *Legendre transformation* for a hyperregular Lagrangian.

Let H be the Hamiltonian (384) obtained from a hyperregular Lagrangian L. For the mappings

$$\mathrm{d}H \colon \mathsf{T}^*Q \to \mathsf{T}^*\mathsf{T}^*Q, \tag{385}$$

$$\beta^{-1}_{(\mathsf{T}^*Q, \omega_Q)} \colon \mathsf{T}^*\mathsf{T}^*Q \to \mathsf{T}\mathsf{T}^*Q, \tag{386}$$

and

$$\beta^{-1}_{(\mathsf{T}^*Q, \omega_Q)} \circ (-\mathrm{d}H) \colon \mathsf{T}^*Q \to \mathsf{T}\mathsf{T}^*Q \tag{387}$$

we have

$$(q^\kappa, p_\lambda, u_\mu, v^\nu) \circ \mathrm{d}H = (q^\kappa, p_\lambda, \partial_\mu H(q^\rho, p_\sigma), \partial^\nu H(q^\rho, p_\sigma)), \tag{388}$$

$$(q^\kappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu) \circ \beta^{-1}_{(\mathsf{T}^*Q, \omega_Q)} = (q^\kappa, p_\lambda, -v^\mu, u_\nu), \tag{389}$$

and

$$(q^\kappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu) \circ \beta^{-1}_{(\mathsf{T}^*Q, \omega_Q)} \circ (-\mathrm{d}H) = (q^\kappa, p_\lambda, \partial^\mu H(q^\rho, p_\sigma), -\partial_\nu H(q^\rho, p_\sigma)). \tag{390}$$

On the other hand, we have

$$(q^\kappa, \dot{q}^\lambda, a_\mu, b_\nu) \circ \mathrm{d}L = (q^\kappa, \dot{q}^\lambda, \partial_\mu L(q^\rho, \dot{q}^\sigma), \partial_\nu L(q^\rho, \dot{q}^\sigma)), \tag{391}$$

$$(q^\kappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu) \circ \alpha_Q^{-1} = (q^\kappa, b_\lambda, \dot{q}^\mu, a_\nu), \tag{392}$$

$$(q^\kappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu) \circ \alpha_Q^{-1} \circ \mathrm{d}L = (q^\kappa, \partial_\lambda L(q^\rho, \dot{q}^\sigma), \dot{q}^\mu, \partial_\nu L(q^\rho, \dot{q}^\sigma)), \tag{393}$$

and

$$\begin{aligned}&(q^\kappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu) \circ \alpha_Q^{-1} \circ \mathrm{d}L \circ \chi = \\ &(q^\kappa, \partial_\lambda L(q^\rho, \chi^\sigma(q^\omega, p_\tau)), \chi^\mu(q^\omega, p_\tau), \partial_\nu L(q^\rho, \chi^\sigma(q^\omega, p_\tau)))\end{aligned} \tag{394}$$

for the mappings

$$\mathrm{d}L \ : \ \mathsf{T}Q \to \mathsf{T}^*\mathsf{T}Q, \tag{395}$$

$$\alpha_Q^{-1} \ : \ \mathsf{T}^*\mathsf{T}Q \to \mathsf{T}\mathsf{T}^*Q, \tag{396}$$

$$\alpha_Q^{-1} \circ \mathrm{d}L \ : \ \mathsf{T}Q \to \mathsf{T}\mathsf{T}^*Q, \tag{397}$$

and

$$\alpha_Q^{-1} \circ \mathrm{d}L \circ \chi \colon \mathsf{T}^*Q \to \mathsf{TT}^*Q. \tag{398}$$

From

$$\begin{aligned} \partial_\kappa H(q^\mu, p_\nu) &= p_\mu \partial_\kappa \chi^\mu(q^\rho, p_\sigma) - \partial_\kappa L(q^\mu, \chi^\nu(q^\rho, p_\sigma)) \\ &\quad -\partial_{\dot\rho} L(q^\mu, \chi^\nu(q^\omega, p_\tau))\partial_\kappa \chi^\nu(q^\rho, p_\sigma) \\ &= -\partial_\kappa L(q^\mu, \chi^\nu(q^\rho, p_\sigma)) \end{aligned} \tag{399}$$

and

$$\begin{aligned} \partial^\kappa H(q^\mu, p_\nu) &= \chi^\kappa(q^\rho, p_\sigma) + p_\mu \partial^\kappa \chi^\mu(q^\rho, p_\sigma) \\ &\quad -\partial_{\dot\rho} L(q^\mu, \chi^\nu(q^\omega, p_\tau))\partial^\kappa \chi^\nu(q^\rho, p_\sigma) \\ &= \chi^\kappa(q^\rho, p_\sigma) \end{aligned} \tag{400}$$

it follows that

$$\alpha_Q^{-1} \circ \mathrm{d}L \circ \chi = \beta^{-1}_{(\mathsf{T}^*Q, \omega_Q)} \circ (-\mathrm{d}H). \tag{401}$$

We see that the Hamiltonian and the Lagrangian generate the same dynamics

$$D = \mathrm{im}(\beta^{-1}_{(\mathsf{T}^*Q, \omega_Q)} \circ (-\mathrm{d}H)) = \mathrm{im}(\alpha_Q^{-1} \circ \mathrm{d}L). \tag{402}$$

Example 14. The Lagrangian

$$L(q^i, \dot q^j) = \frac{m}{2} g_{ij} \dot q^i \dot q^j - e\phi + eA_i \dot q^i \tag{403}$$

of Example 4 is hyperregular. The Legendre mapping and its inverse are the mappings

$$(q^i, p_j) \circ \lambda = (q^i, m g_{jk} \dot q^k + eA_j) \tag{404}$$

and

$$(q^i, \dot q^j) \circ \chi = \left(q^i, \frac{1}{m} g^{ij}(p_j - eA_j) \right). \tag{405}$$

From the energy function

$$E(q^i, p_j, \dot q^k) = p_i \dot q^i - \frac{m}{2} g_{ij} \dot q^i \dot q^j + e\phi - eA_i \dot q^i \tag{406}$$

we derive the Hamiltonian

$$H(q^i, p_j) = \frac{1}{2m} g^{ij}(p_i - eA_i)(p_j - eA_j) + e\phi. \tag{407}$$

▲

If a Lagrangian is not hyperregular, then the Legendre mapping is not invertible. It may happen that the image of the Legendre mapping λ is a submanifold $C \subset \mathsf{T}^*Q$. Let $G \subset \mathsf{T}^*Q \underset{(\pi_Q,\tau_Q)}{\times} \mathsf{T}Q$ be the graph of λ (intersected with $\mathsf{T}^*Q \underset{(\pi_Q,\tau_Q)}{\times} \mathsf{T}Q \subset \mathsf{T}^*Q \times \mathsf{T}Q$). The energy function (383) restricted to the graph G does not depend on $\dot{q}^\kappa$ since

$$\partial_{\dot{\kappa}} E = p_\kappa - \partial_{\dot{\kappa}} L. \tag{408}$$

If fibres of the projection $pr_1 \colon \mathsf{T}^*Q \underset{(\pi_Q,\tau_Q)}{\times} \mathsf{T}Q \to \mathsf{T}^*Q$ are connected, then a function

$$H(q^\kappa, p_\lambda) = E(q^\kappa, p_\lambda, \dot{q}^\mu) \tag{409}$$

can be defined on C by substituting in E any values of the velocities $(\dot{q}^\mu)$ such that $p_\kappa - \partial_{\dot{\kappa}} L = 0$. This is the construction of the constrained Hamiltonian used by Dirac. This is also the construction of the *generalized Legendre transformation* introduced by Cendra, Holm, Hoyle and Marsden. The Dirac system generated by the constrained Hamiltonian is sometimes equal to the Lagrangian system generated by the original Lagrangian.

EXAMPLE 15. Applying the Cendra–Holm–Hoyle–Marsden generalized Legendre transformation to the Lagrangian

$$L(q, q^i, \dot{q}, \dot{q}^j) = \frac{m}{2} g_{ij} \dot{q}^i \dot{q}^j - e\phi + eA_i \dot{q}^i + e\dot{q}. \tag{410}$$

of Example 5 we obtain the constraint C described by

$$p = e \tag{411}$$

and the Hamiltonian

$$H(q, q^i, p_j) = \frac{1}{2m} g^{ij} (p_i - eA_i)(p_j - eA_j) + e\phi \tag{412}$$

defined on the constraint C with coordinates (q, q^i, p_j). This Hamiltonian generates the equations (370) of Example 10 equivalent to the Lagrange equations (313) of Example 5. The Cendra–Holm–Hoyle–Marsden version of the Legendre transformation gives correct results for this example of a mechanical system. ▲

The mechanical system in the above example is the only mechanical system known to us for which this version of the Legendre transformation functions correctly.

EXAMPLE 16. The Lagrangian

$$L(q^\kappa, \dot{q}^\lambda) = m\sqrt{g_{\kappa\lambda}\dot{q}^\kappa\dot{q}^\lambda} + eA_\kappa\dot{q}^\kappa \tag{413}$$

of Example 6 is singular. The image of the Legendre mapping

$$(q^\kappa, p_\lambda) \circ \lambda = (q^\kappa, mg_{\lambda\mu}\frac{\dot{q}^\mu}{\|\dot{q}\|} + eA_\lambda) \tag{414}$$

is the constraint set C described by

$$g^{\kappa\lambda}(p_\kappa - eA_\kappa)(p_\lambda - eA_\lambda) = m^2. \tag{415}$$

The energy function

$$E(q^\kappa, p_\lambda, \dot{q}^\mu) = p_\lambda\dot{q}^\lambda - m\sqrt{g_{\kappa\lambda}\dot{q}^\kappa\dot{q}^\lambda} - eA_\kappa\dot{q}^\kappa \tag{416}$$

vanishes on the graph G of the Legendre mapping and the Dirac Hamiltonian is zero. Differential equations derived from this constrained Hamiltonian are the equations

$$\begin{aligned} &g^{\kappa\lambda}(p_\kappa - eA_\kappa)(p_\lambda - eA_\lambda) = m^2 \\ &\dot{q}^\kappa = \frac{v}{m}g^{\kappa\lambda}(p_\lambda - eA_\lambda) \\ &\dot{p}_\kappa = -\frac{v}{2m}\partial_\kappa g^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu) + \frac{ve}{m}g^{\mu\nu}\partial_\kappa A_\mu(p_\nu - eA_\nu) \end{aligned} \tag{417}$$

with arbitrary values of the Lagrange multiplier v. The Dirac system represented by these equations is the union $D_+ \cup D_0 \cup D_-$ of three sets corresponding to $v > 0$, $v = 0$, and $v < 0$ respectively. The set D_+ is the generalized Dirac system D of Example 11 equivalent to the Lagrangian system of Example 6. The set D_0 described by

$$\begin{aligned} g^{\kappa\lambda}(p_\kappa - eA_\kappa)(p_\lambda - eA_\lambda) &= m^2 \\ \dot{q}^\kappa &= 0 \\ \dot{p}_\kappa &= 0 \end{aligned} \tag{418}$$

must be excluded since the velocities $\dot{q}\kappa$ evaluated on vectors tangent to world lines are never zero. The set D_- is a generalized Dirac system obtained from the Lagrangian

$$L_-(q^\kappa, \dot{q}^\lambda) = -m\sqrt{g_{\kappa\lambda}\dot{q}^\kappa\dot{q}^\lambda} + eA_\kappa\dot{q}^\kappa. \tag{419}$$

Setting $v = 1$ in equations (417) we obtain equations

$$\begin{aligned} p_\kappa &= mg_{\kappa\lambda}\dot{q}^\lambda + eA_\kappa \\ m(\ddot{q}^\kappa + \Gamma^\kappa_{\lambda\mu}\dot{q}^\lambda\dot{q}^\mu) &= -eg^{\kappa\lambda}F_{\lambda\mu}\dot{q}^\mu \end{aligned} \tag{420}$$

correctly describing the dynamics of charged particles. Solutions of these equations are curves using proper time as the parameter. With $y = -1$ equations (417) result in the equation

$$p_\kappa = -mg_{\kappa\lambda}\dot{q}^\lambda + eA_\kappa \tag{421}$$

and the second order system

$$m(\ddot{q}^\kappa + \Gamma^\kappa_{\lambda\mu}\dot{q}^\lambda\dot{q}^\mu) = eg^{\kappa\lambda}F_{\lambda\mu}\dot{q}^\mu. \tag{422}$$

Solutions of the second order equations are world lines of particles with mass m and charge $-e$ or particles with mass $-m$ and charge e. The equation (421) suggests that we are dealing with particles with negative mass $-m$. The principle that world lines of particles with positive energy should be oriented towards the future and that world lines of particles with negative energy (antiparticles) should be oriented towards the past (Stueckelberg [11], Feynman [3]) is violated. We conclude that the Cendra–Holm–Hoyle–Marsden version of the Legendre transformation is too fast to provide correct results for this important example of a mechanical system. ▲

The Dirac system generated by a constrained Hamiltonian

$$H\colon C \to \mathbb{R} \tag{423}$$

is characterized by the variational relation

$$\dot{p}_\kappa\delta q^\kappa - \dot{q}^\kappa\delta p_\kappa = -\partial_z kH\delta q^\kappa - \partial^\kappa H\delta p_\kappa \tag{424}$$

on C. A constrained Hamiltonian derived from a singular Lagrangian can be considered a function on the graph G of the Legendre mapping defined by

$$H(q^\kappa, p_\lambda) = E(q^\kappa, p_\lambda, \dot{q}^\mu) = p_\lambda\dot{q}^\lambda - (q^\kappa, \dot{q}^\mu). \tag{425}$$

The variational relation

$$\dot{p}_\kappa\delta q^\kappa - \dot{q}^\kappa\delta p_\kappa = -\partial_z kE\delta q^\kappa - \partial^\kappa E\delta p_\kappa - \partial_{\dot{\kappa}}E\delta\dot{q}^\kappa \tag{426}$$

on G is equivalent to the relation (424) for the Hamiltonian (425). The variations $(\delta q^\kappa, \delta p_\lambda, \delta\dot{q}^\mu)$ are components of a vector tangent to G. Hence

$$\delta p_\kappa = \partial_\mu\partial_{\dot{\kappa}}\delta q^\mu + \partial_{\dot{\mu}}\partial_{\dot{\kappa}}\delta\dot{q}^\mu. \tag{427}$$

If Lagrange equations

$$\begin{aligned} \dot{p}_\mu &= \partial_\mu L(q^\kappa, \dot{q}^\lambda) \\ p_\nu &= \partial_{\dot{\nu}} L(q^\kappa, \dot{q}^\lambda) \end{aligned} \tag{428}$$

are satisfied, then

$$\dot{p}_\kappa \delta q^\kappa - \dot{q}^\kappa \delta p_\kappa = \partial_\kappa L \delta q^\kappa - \dot{q}^\kappa \delta p_\kappa \tag{429}$$

and

$$\begin{aligned} -\partial_z k E \delta q^\kappa - \partial^\kappa E \delta p_\kappa - \partial_{\dot{\kappa}} E \delta \dot{q}^\kappa &= \partial_\kappa L \delta q^\kappa - \dot{q}^\kappa \delta p_\kappa - (p_\kappa - \partial_{\dot{\kappa}} L) \delta \dot{q}^\kappa \\ &= \partial_\kappa L \delta q^\kappa - \dot{q}^\kappa \delta p_\kappa. \end{aligned} \tag{430}$$

This seems to imply that the Dirac system generated by the Hamiltonian (425) is equivalent to the Lagrangian system (428). This conclusion is not correct. It is true that the variational relation (424) follows from the Lagrange equations. The converse is not true since the same constrained Hamiltonian may be in the relation (425) with different energy functions constructed from different Lagrangians. The Lagrangians $L_+ = L$ and L_- of Example 16 generate different Lagrangian systems D_+ and D_- but lead to the same constrained Hamiltonian.

We observe that if $E(q^\kappa, p_\lambda, \dot{q}^\mu)$ is the energy function associated with a Lagrangian $L(q^\kappa, \dot{q}^\lambda)$, then $E(q^\kappa, p_\lambda, v^\mu)$ is a Morse family of the variables (v^μ). The rank of the matrix

$$\left(\frac{\partial^2 E}{\partial v^\kappa \partial v^\lambda} \quad \frac{\partial^2 E}{\partial \dot{v}^\kappa \partial q^\mu} \quad \frac{\partial^2 E}{\partial v^\kappa \partial p_\nu} \right) = \left(\frac{-\partial^2 L}{\partial v^\kappa \partial v^\lambda} \quad \frac{-\partial^2}{\partial \dot{v}^\kappa \partial q^\mu} \quad \delta^\nu_\kappa \right) \tag{431}$$

is maximal. No requirements are imposed on the Lagrangian. The generalized Dirac system generated by the Morse family $E(q^\kappa, p_\lambda, v^\mu)$ is obtained from the variational relation

$$\dot{p}_\kappa \delta q^\kappa - \dot{q}^\kappa \delta p_\kappa = -\partial_z k E \delta q^\kappa - \partial^\kappa E \delta p_\kappa + \partial_{\dot{\kappa}} E \delta v^\kappa \tag{432}$$

with arbitrary variations $(\delta q^\kappa, \delta p_\lambda, \delta v^\mu)$. With

$$E(q^\kappa, p_\lambda, v^\mu) = p_\lambda v^\lambda - L(q^\kappa, v^\mu) \tag{433}$$

the relation takes the form

$$\dot{p}_\kappa \delta q^\kappa - \dot{q}^\kappa \delta p_\kappa = \partial_\kappa L(q^\kappa, v^\mu) \delta q^\kappa - v^\kappa \delta p_\kappa - (p_\kappa - \partial_{\dot{\kappa}} L(q^\kappa, v^\mu)) \delta v^\kappa. \tag{434}$$

Equations

$$\begin{aligned} \dot{p}_\mu &= \partial_\mu L(q^\kappa, v^\lambda) \\ p_\nu &= \partial_{\dot{\nu}} L(q^\kappa, v^\lambda) \\ \dot{q}^\lambda &= v^\lambda \end{aligned} \tag{435}$$

obtained from this relation are equivalent to the Lagrange equations.

The passage from the Lagrangian $L(q^\kappa, \dot{q}^\lambda)$ to the global Hamiltonian Morse family $E(q^\kappa, p_\lambda, v^\mu)$ is the *slow and careful Legendre transformation* required to provide a correct Hamiltonian formulation of any Lagrangian system. The Morse family generating a Lagrangian submanifold is not unique. Modifications resulting in a reduction of the number of variables are usually possible.

EXAMPLE 17. The Hamiltonian Morse family

$$E(q, q^i, p, p_j, v, v^k) = pv + p_i v^i - \frac{m}{2} g_{ij} v^i v^j + e\phi - eA_i v^i - ev \tag{436}$$

for the charged particle of Example 5 reduces to the Hamiltonian Morse family

$$H(q, q^i, p, p_j, v) = \frac{1}{2m} g^{ij}(p_i - eA_i)(p_j - eA_j) + e\phi - v(p - e) \tag{437}$$

of Example 10. The reduction is obtained by using in (436) the equality

$$v^k = \frac{1}{m} g^{kj}(p_j - eA_j) \tag{438}$$

obtained from

$$\frac{\partial E}{\partial v^i} = p_i - m g_{ij} v^j - eA_i = 0. \tag{439}$$

▲

EXAMPLE 18.

$$E(q^\kappa, p_\lambda, v^\mu) = p_\kappa v^\kappa - m\sqrt{g_{\kappa\lambda} v^\kappa v^\lambda} - eA_\kappa v^\kappa \tag{440}$$

for the charged particle of Example 6 reduces to the Hamiltonian Morse family

$$H(q^\kappa, p_\lambda, v) = v\left(\sqrt{g^{\kappa\lambda}(p_\kappa - eA_\kappa)(p_\lambda - eA_\lambda)} - m\right) \tag{441}$$

of a single variable $v > 0$ introduced in Example 11. Variables $(q^\kappa, p_\lambda, v^\mu)$ are coordinates in the manifold $\mathsf{T}^*Q \underset{(\pi_Q, \tau_Q)}{\times} \mathsf{T}Q$. Only the open subset described by the inequality $g_{\kappa\lambda} v^\kappa v^\lambda > 0$ is considered. Variables (q^κ, p_λ, v) are coordinates in $\mathsf{T}^*Q \times \mathbb{R}_+$. The function $E(q^\kappa, p_\lambda, v^\mu)$ is a Morse family of functions on fibres of the fibration

$$\zeta : \mathsf{T}^*Q \underset{(\pi_Q, \tau_Q)}{\times} \mathsf{T}Q \to \mathsf{T}^*Q \tag{442}$$

characterized by

$$(q^\kappa, p_\lambda) \circ \zeta = (q^\kappa, p_\lambda). \tag{443}$$

This fibration can be interpreted as the fibration

$$\zeta' \colon \mathsf{T}^*Q \underset{(\pi_Q, \tau_Q)}{\times} \mathsf{T}Q \to \mathsf{T}^*Q \times \mathbb{R}_+ \tag{444}$$

characterized by

$$(q^\kappa, p_\lambda, v) \circ \zeta' = \left(q^\kappa, p_\lambda, \sqrt{g_{\mu\nu} v^\mu v^\nu}\right) \tag{445}$$

followed by the projection

$$pr_{\mathsf{T}^*Q} \colon \mathsf{T}^*Q \times \mathbb{R}_+ \to \mathsf{T}^*Q. \tag{446}$$

Fibres of ζ' are hyperboloids $g_{\mu\nu} v^\mu v^\nu = v^2$. From

$$\delta E(q^\kappa, p_\lambda, v^\mu) = \frac{\partial}{\partial v^\nu} \delta v^\nu = 0 \tag{447}$$

with variations δv^ν satisfying

$$\delta(g_{\mu\nu} v^\mu v^\nu) = 2 g_{\mu\nu} v^\mu \delta v^\nu = 0 \tag{448}$$

we obtain two critical points

$$v^\kappa = \frac{\pm v}{\sqrt{g_{\mu\nu} v^\mu v^\nu}} g^{\kappa\lambda} (p_\lambda - e A_\lambda) \tag{449}$$

in a fibre over the point with coordinates (q^κ, p_λ, v). Evaluating the function E at these points results in two Morse families

$$H \pm (q^\kappa, p_\lambda, v) = \pm v \left(\sqrt{g^{\kappa\lambda} (p_\kappa - e A_\kappa)(p_\lambda - e A_\lambda)} \mp m \right) \tag{450}$$

of functions of v. One of these is the reduced Morse family (441). The other generates an empty set since it has no critical points. The family (441) depends linearly on the variable v restricted to positive values. No further reduction is possible. ▲

EXAMPLE 19. The dynamics of the charged particle of Example 7 is generated by the Hamiltonian Morse family

$$E(q, q^\kappa, p, p_\lambda, v, v^\mu) = pv + p_\kappa v^\kappa - m\sqrt{g_{\kappa\lambda} v^\kappa v^\lambda} - e A_\kappa v^\kappa - ev \tag{451}$$

or by the Hamiltonian Morse family

$$H(q^\kappa, p_\lambda, v^1, v^2) = v^1\left(\sqrt{g^{\kappa\lambda}(p_\kappa - eA_\kappa)(p_\lambda - eA_\lambda)} - m\right) + v^2(p - e) \quad (452)$$

of functions of two variables $v^1 > 0$ and v^2 as in Example 12. ▲

The slow Legendre transformation can be extended to Lagrangian Morse families. If

$$L(q^\kappa, \dot{q}^\lambda, y^A) \quad (453)$$

is a Morse family, then the $k \times (k + 2m)$ matrix

$$\left(\frac{\partial^2 L}{\partial y^A \partial y^B} \quad \frac{\partial^2 L}{\partial y^A \partial v^\lambda} \quad \frac{\partial^2 L}{\partial y^A \partial p_\mu}\right) \quad (454)$$

is of maximal rank k. It follows that the function

$$E(q^\kappa, p_\lambda, y^A, v^\mu) = p_\kappa v^\kappa - L(q^\kappa, v^\lambda, y^A) \quad (455)$$

is a Morse family of functions of the variables (y^A, v^μ) since the $(k + m) \times (k + 3m)$ matrix

$$\left(\begin{array}{cccc} \dfrac{\partial^2 E}{\partial y^A \partial y^B} & \dfrac{\partial^2 E}{\partial y^A \partial v^\lambda} & \dfrac{\partial^2 E}{\partial y^A \partial q^\mu} & \dfrac{\partial^2 E}{\partial y^A \partial p_\nu} \\ \dfrac{\partial^2 E}{\partial v^\kappa \partial y^B} & \dfrac{\partial^2 E}{\partial v^\kappa \partial v^\lambda} & \dfrac{\partial^2 E}{\partial v^\kappa \partial q^\mu} & \dfrac{\partial^2 E}{\partial v^\kappa \partial p_\nu} \end{array}\right) =$$

$$\left(\begin{array}{ccc} \dfrac{-\partial^2 L}{\partial y^A \partial y^B} & \dfrac{-\partial^2 L}{\partial y^A \partial v^\lambda} & \dfrac{-\partial^2 L}{\partial y^A \partial q^\mu} \quad 0 \\ \dfrac{-\partial^2 L}{\partial v^\kappa \partial y^B} & \dfrac{-\partial^2 L}{\partial v^\kappa \partial v^\lambda} & \dfrac{-\partial^2 L}{\partial v^\kappa \partial q^\mu} \quad \delta_\kappa^\nu \end{array}\right) \quad (456)$$

is of rank $k + m$.

EXAMPLE 20. The Hamiltonian Morse family

$$E(q^\kappa, p_\lambda, y, v^\mu) = p_\kappa v^\kappa - \frac{1}{2y} g_{\kappa\lambda} v^\kappa v^\lambda \quad (457)$$

generates the generalized Dirac system of Example 8. The equations

$$\partial_{\dot{\mu}} E(q^\kappa, p_\lambda, y, v^\mu) = p_\mu - \frac{1}{y} g_{\mu\lambda} v^\lambda \quad (458)$$

permits the elimination of v^μ from $E(q^\kappa, p_\lambda, y, v^\mu)$. The result is the Hamilton Morse family

$$H(q^\kappa, p_\lambda, y) = \frac{y}{2} g^{\kappa\lambda} p_\kappa p_\lambda \quad (459)$$

with $y > 0$. It is the Morse family of Example 13 with v replaced by y. ▲

12. Integrability

In Section 5 we have established integrability criteria for a class of differential equations which can be specified in terms of vector fields restricted to a submanifold. The dynamics of relativistic mechanical systems presented in our examples all admit the Dirac-style formulations in terms of Hamiltonian vector fields. The integrability criterion and the first integrability algorithm formulated in Section 5 can be adapted to this situation.

Let Q be the configuration manifold of a mechanical system and let the dynamics of the system be represented by the union

$$D = \bigcup_{\alpha\in\mathsf{A}} \{\mathrm{im}(X_\alpha|C)\} \tag{460}$$

of a family of Hamiltonian vector fields

$$X_\alpha\colon \mathsf{T}^*Q \to \mathsf{T}\mathsf{T}^*Q \tag{461}$$

generated by a family of Hamiltonians

$$H_\alpha\colon \mathsf{T}^*Q \to \mathbb{R} \tag{462}$$

and restricted to a submanifold $C \subset \mathsf{T}^*Q$ specified as

$$C = \{p \in \mathsf{T}^*Q;\ \forall_A \Phi_A(p) = 0\}\,, \tag{463}$$

where Φ_A are independent functions on T^*Q called *primary constraints*. The condition $D \subset \mathsf{T}C$ means that at points of C the vector fields X_α are tangent to C or that

$$\langle \mathrm{d}\Phi_A, X_\alpha\rangle|C = 0 \tag{464}$$

for each $\alpha \in \mathsf{A}$ and each function Φ_A. In view of

$$X_\alpha = \beta^{-1}_{(\mathsf{T}^*Q,\omega_Q)} \circ (-\mathrm{d}H_\alpha) \tag{465}$$

the integrability criterion for the dynamics D assumes the form

$$\{H_\alpha, \Phi_A\}|C = 0 \tag{466}$$

for each $\alpha \in \mathsf{A}$ and each function Φ_A.

Example 21. The dynamics of the system in Example 9 is integrable since it is the image of a Hamiltonian vector field. ▲

EXAMPLE 22. The dynamics in Example 10 is a Dirac system. It is the union of images of the family of Hamiltonian vector fields generated by the family

$$H_\alpha(q, q^i, p, p_j) = \frac{1}{2m} g^{ij}(p_i - eA_i)(p_j - eA_j) + e\phi - \alpha(p - e) \tag{467}$$

restricted to a constraint set C. The parameter α is an arbitrary function on T^*Q. There is one primary constraint. It is the function

$$\Phi(q, q^i, p, p_j) = p - e. \tag{468}$$

The system is integrable since

$$\{H_\alpha, \Phi\}|C = 0. \tag{469}$$

▲

EXAMPLE 23. The dynamics in Example 11 is a generalized Dirac system. It is described by the family of Hamiltonian vector fields generated by the family

$$H_\alpha(q^\kappa, p_\lambda) = \alpha\left(\sqrt{g^{\kappa\lambda}(p_\kappa - eA_\kappa)(p_\lambda - eA_\lambda)} - m\right) \tag{470}$$

of Hamiltonians parametrized by a function $\alpha > 0$ on T^*Q. There is one primary constraint

$$\Phi(q^\kappa, p_\lambda) = \sqrt{g^{\kappa\lambda}(p_\kappa - eA_\kappa)(p_\lambda - eA_\lambda)} - m. \tag{471}$$

The integrability criterion is satisfied.

▲

EXAMPLE 24. The generalized Dirac system of Example 12 is described by Hamiltonian vector fields generated by the family

$$H_{(\alpha^1,\alpha^2)}(q^\kappa, p_\lambda) = \alpha^1\left(\sqrt{g^{\kappa\lambda}(p_\kappa - eA_\kappa)(p_\lambda - eA_\lambda)} - m\right) + \alpha^2(p - e) \tag{472}$$

parametrized by functions $\alpha^1 > 0$ and α^2 on T^*Q. There are two primary constraints:

$$\Phi_1(q, q^\kappa, p, p_\lambda) = \sqrt{g^{\kappa\lambda}(p_\kappa - eA_\kappa)(p_\lambda - eA_\lambda)} - m \tag{473}$$

and

$$\Phi_2(q, q^\kappa, p, p_\lambda) = p - e. \tag{474}$$

Integrability criteria are again satisfied. ▲

EXAMPLE 25. For the generalized Dirac system in Example 13 we use the family of Hamiltonians

$$H_\alpha(q^\kappa, p_\lambda) = \frac{\alpha}{2} g^{\kappa\lambda} p_\kappa p_\lambda \tag{475}$$

depending on a function $\alpha > 0$. There is one primary constraint

$$\Phi(q^\kappa, p_\lambda) = g^{\kappa\lambda} p_\kappa p_\lambda. \tag{476}$$

The system is integrable. ▲

EXAMPLE 26. Let Q be the affine space-time of special relativity with Cartesian coordinates (q^κ) and a constant Minkowski metric tensor $(g_{\kappa\lambda})$. We analyse the dynamics of two interacting relativistic particles [15] [16]. The configuration space is the product $Q \times Q$ with coordinates $(q_1^\kappa, q_2^\lambda)$. Coordinates $(q_1^\kappa, q_2^\lambda, \dot{q}_1^\mu, \dot{q}_2^\lambda)$, $(q_1^\kappa, q_2^\lambda, p_\mu^1, p_\nu^2)$, and $(q_1^\kappa, q_2^\lambda, p_\mu^1, p_\nu^2, \dot{q}_1^\rho, \dot{q}_2^\sigma, \dot{p}_\tau^1, \dot{p}_\omega^2)$ will be used in $\mathsf{T}(Q \times Q)$, $\mathsf{T}^*(Q \times Q)$, and $\mathsf{TT}^*(Q \times Q)$ respectively. Masses of the particles are denoted by m_1 and m_2. The interaction potential is a function V of a real positive argument. Relations

$$\begin{aligned} g_{\kappa\lambda}\dot{q}_1^\kappa \dot{q}_1^\lambda &> 0 \\ g_{\kappa\lambda}\dot{q}_2^\kappa \dot{q}_2^\lambda &> 0 \\ g^{\kappa\lambda} p_\kappa^1 p_\lambda^1 &> 0 \\ g^{\kappa\lambda} p_\kappa^2 p_\lambda^2 &> 0 \\ g_{\kappa\lambda}(q_2^\kappa - q_1^\kappa)(q_2^\lambda - q_1^\lambda) &< 0 \end{aligned} \tag{477}$$

are assumed to be satisfied. Abbreviations

$$\begin{aligned} \|\dot{q}_1\| &= \sqrt{g_{\kappa\lambda}\dot{q}_1^\kappa \dot{q}_1^\lambda} \\ \|\dot{q}_2\| &= \sqrt{g_{\kappa\lambda}\dot{q}_2^\kappa \dot{q}_2^\lambda} \\ \|\dot{p}^1\| &= \sqrt{g^{\kappa\lambda} p_\kappa^1 p_\lambda^1} \\ \|\dot{p}^2\| &= \sqrt{g^{\kappa\lambda} p_\kappa^2 p_\lambda^2} \\ \|q_2 - q_1\| &= \sqrt{-g_{\kappa\lambda}(q_2^\kappa - q_1^\kappa)(q_2^\lambda - q_1^\lambda)} \\ \overline{m}_1 &= \sqrt{{m_1}^2 + V(\|x_2 - x_1\|)} \\ \overline{m}_2 &= \sqrt{{m_2}^2 + V(\|x_2 - x_1\|)} \end{aligned} \tag{478}$$

will be used.

The dynamics of the particles is a generalized Dirac system described by Lagrange equations

$$\begin{aligned}
\dot{p}^1_\kappa &= \frac{\mathrm{D}V(\|q_2-q_1\|)}{2\|q_2-q_1\|}\left(\frac{\|\dot{q}_1\|}{\overline{m}_1}+\frac{\|\dot{q}_2\|}{\overline{m}_2}\right)g_{\kappa\lambda}(q_2^\lambda-q_1^\lambda)\\
\dot{p}^2_\kappa &= \frac{\mathrm{D}V(\|q_2-q_1\|)}{2\|q_2-q_1\|}\left(\frac{\|\dot{q}_1\|}{\overline{m}_1}+\frac{\|\dot{q}_2\|}{\overline{m}_2}\right)g_{\kappa\lambda}(q_1^\lambda-q_2^\lambda)\\
p^1_\kappa &= \frac{\overline{m}_1}{\|\dot{q}_1\|}g_{\kappa\lambda}\dot{q}_1^\lambda\\
p^2_\kappa &= \frac{\overline{m}_2}{\|\dot{q}_2\|}g_{\kappa\lambda}\dot{q}_2^\lambda
\end{aligned} \tag{479}$$

derived from the Lagrangian

$$L(q_1^\kappa,q_2^\lambda,\dot{q}_1^\mu,\dot{q}_2^\nu)=\overline{m}_1\,\|\dot{q}_1\|+\overline{m}_2\,\|\dot{q}_2\|. \tag{480}$$

The Hamiltonian is the Morse family

$$H(q_1^\kappa,q_2^\lambda,p^1_\mu,p^2_\nu,v^1,v^2)=v^1\,(\|p_1\|-\overline{m}_1)+v^2\,(\|p_2\|-\overline{m}_2) \tag{481}$$

of functions of the variables $v^1>0$ and $v^2>0$. Hamilton equations

$$\begin{aligned}
\dot{p}^1_\kappa &= \frac{\mathrm{D}V(\|q_2-q_1\|)}{2\|q_2-q_1\|}\left(\frac{v^1}{\overline{m}_1}+\frac{v^2}{\overline{m}_2}\right)g_{\kappa\lambda}(q_2^\lambda-q_1^\lambda)\\
\dot{p}^2_\kappa &= \frac{\mathrm{D}V(\|q_2-q_1\|)}{2\|q_2-q_1\|}\left(\frac{v^1}{\overline{m}_1}+\frac{v^2}{\overline{m}_2}\right)g_{\kappa\lambda}(q_1^\lambda-q_2^\lambda)\\
\dot{q}_1^\kappa &= \frac{v^1}{\|p^1\|}g^{\kappa\lambda}p^1_\lambda\\
\dot{q}_2^\kappa &= \frac{v^2}{\|p^2\|}g^{\kappa\lambda}p^2_\lambda\\
\|p^1\| &= \overline{m}_1\\
\|p^2\| &= \overline{m}_2
\end{aligned} \tag{482}$$

are equivalent to the Lagrange equations (479).

The dynamics is the union of the family of Hamiltonian vector fields

$$X_{(\alpha^1,\alpha^2)}\colon \mathsf{T}^*(Q\times Q)\to\mathsf{TT}^*(Q\times Q) \tag{483}$$

generated by the family of Hamiltonians

$$H_{(\alpha^1,\alpha^2)}(q_1^\kappa,q_2^\lambda,p^1_\mu,p^2_\nu)=\alpha^1\,(\|p_1\|-\overline{m}_1)+\alpha^2\,(\|p_2\|-\overline{m}_2) \tag{484}$$

and restricted to a submanifold $C \subset \mathsf{T}^*(Q \times Q)$ described by two primary constraints:

$$\Phi_1(q_1^\kappa, q_2^\lambda, p_\mu^1, p_\nu^2) = \|p_1\| - \overline{m}_1 \tag{485}$$

and

$$\Phi_2(q_1^\kappa, q_2^\lambda, p_\mu^1, p_\nu^2) = \|p^2\| - \overline{m}_2. \tag{486}$$

The parameters (α^1, α^2) are positive functions on $\mathsf{T}^*(Q \times Q)$. The Poisson brackets

$$\{H_{(\alpha^1,\alpha^2)}, \Phi_1\}|C = \alpha^2 \frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\overline{m}_1\overline{m}_2\|q_2 - q_1\|}(p_\kappa^1 + p_\kappa^2)(q_1^\kappa - q_2^\kappa) \tag{487}$$

and

$$\{H_{(\alpha^1,\alpha^2)}, \Phi_2\}|C = \alpha^1 \frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\overline{m}_1\overline{m}_2\|q_2 - q_1\|}(p_\kappa^1 + p_\kappa^2)(q_2^\kappa - q_1^\kappa) \tag{488}$$

indicate that the dynamics is not integrable unless the potential V is constant. ▲

The generalized Dirac system in Example 26 is the only case of a non integrable dynamics known to us. We will extract the integrable part of this system by applying different versions of the extraction algorithm.

EXAMPLE 27. We apply the first algorithm of Section 5 to Hamilton equations (482) and constraint set C described by

$$\begin{aligned} \|p^1\| &= \overline{m}_1 \\ \|p^2\| &= \overline{m}_2. \end{aligned} \tag{489}$$

We obtain equations

$$\begin{aligned} \|p^1\| &= \overline{m}_1 \\ \|p^2\| &= \overline{m}_2 \\ g^{\kappa\lambda}p_\kappa^1\dot{p}_\lambda^1 &= \frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\|q_2 - q_1\|}g_{\kappa\lambda}(q_2^\kappa - q_1^\kappa)(\dot{q}_2^\lambda - \dot{q}_1^\lambda) \\ g^{\kappa\lambda}p_\kappa^2\dot{p}_\lambda^2 &= \frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\|q_2 - q_1\|}g_{\kappa\lambda}(q_2^\kappa - q_1^\kappa)(\dot{q}_2^\lambda - \dot{q}_1^\lambda) \end{aligned} \tag{490}$$

for $\mathsf{T}C$ and equations

$$\dot{p}_\kappa^1 = \frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\|q_2 - q_1\|}\left(\frac{v^1}{\overline{m}_1} + \frac{v^2}{\overline{m}_2}\right)g_{\kappa\lambda}(q_2^\lambda - q_1^\lambda)$$

$$\begin{aligned}
\dot{p}^2_\kappa &= \frac{\mathrm{D}V(\|q_2-q_1\|)}{2\|q_2-q_1\|}\left(\frac{v^1}{\overline{m}_1}+\frac{v^2}{\overline{m}_2}\right)g_{\kappa\lambda}(q_1^\lambda-q_2^\lambda)\\
\dot{q}^\kappa_1 &= \frac{v^1}{\|p^1\|}g^{\kappa\lambda}p^1_\lambda\\
\dot{q}^\kappa_2 &= \frac{v^2}{\|p^2\|}g^{\kappa\lambda}p^2_\lambda\\
\|p^1\| &= \overline{m}_1\\
\|p^2\| &= \overline{m}_2\\
0 &= (p^1_\kappa+p^2_\kappa)(q_2^\kappa-q_1^\kappa)\\
v^1 &> 0\\
v^2 &> 0
\end{aligned} \tag{491}$$

for the intersection $\overline{D}^1 = D \cap \mathsf{T}C$ if the potential V is not constant. For the new constraint set $\overline{C}^1 = \tau_Q(\overline{D}^1)$ we have equations

$$\begin{aligned}
\|p^1\| &= \overline{m}_1\\
\|p^2\| &= \overline{m}_2\\
0 &= (p^1_\kappa+p^2_\kappa)(q_2^\kappa-q_1^\kappa)
\end{aligned} \tag{492}$$

and equations

$$\begin{aligned}
\|p^1\| &= \overline{m}_1\\
\|p^2\| &= \overline{m}_2\\
0 &= (p^1_\kappa+p^2_\kappa)(q_2^\kappa-q_1^\kappa)\\
g^{\kappa\lambda}p^1_\kappa\dot{p}^1_\lambda &= \frac{\mathrm{D}V(\|q_2-q_1\|)}{2\|q_2-q_1\|}g_{\kappa\lambda}(q_2^\kappa-q_1^\kappa)(\dot{q}_2^\lambda-\dot{q}_1^\lambda)\\
g^{\kappa\lambda}p^2_\kappa\dot{p}^2_\lambda &= \frac{\mathrm{D}V(\|q_2-q_1\|)}{2\|q_2-q_1\|}g_{\kappa\lambda}(q_2^\kappa-q_1^\kappa)(\dot{q}_2^\lambda-\dot{q}_1^\lambda)\\
0 &= (\dot{p}^1_\kappa+\dot{p}^2_\kappa)(q_2^\kappa-q_1^\kappa)+(p^1_\kappa+p^2_\kappa)(\dot{q}_2^\kappa-\dot{q}_1^\kappa)
\end{aligned} \tag{493}$$

for $\mathsf{T}\overline{C}^1$. The equations

$$\begin{aligned}
\dot{p}^1_\kappa &= \frac{\mathrm{D}V(\|q_2-q_1\|)}{2\|q_2-q_1\|}\left(\frac{v^1}{\overline{m}_1}+\frac{v^2}{\overline{m}_2}\right)g_{\kappa\lambda}(q_2^\lambda-q_1^\lambda)\\
\dot{p}^2_\kappa &= \frac{\mathrm{D}V(\|q_2-q_1\|)}{2\|q_2-q_1\|}\left(\frac{v^1}{\overline{m}_1}+\frac{v^2}{\overline{m}_2}\right)g_{\kappa\lambda}(q_1^\lambda-q_2^\lambda)\\
\dot{q}^\kappa_1 &= \frac{v^1}{\|p^1\|}g^{\kappa\lambda}p^1_\lambda
\end{aligned}$$

$$
\begin{aligned}
\dot{q}_2^\kappa &= \frac{v^2}{\|p^2\|} g^{\kappa\lambda} p^2_\lambda \\
\|p^1\| &= \overline{m}_1 \\
\|p^2\| &= \overline{m}_2 \\
p^1_\kappa + p^2_\kappa &\neq 0 \\
0 &= (p^1_\kappa + p^2_\kappa)(q_2^\kappa - q_1^\kappa) \\
v^1 &> 0 \\
v^2 &> 0 \\
\frac{v^1}{\overline{m}_1} g^{\kappa\lambda}(p^1_\kappa + p^2_\kappa) p^1_\lambda &= \frac{v^2}{\overline{m}_2} g^{\kappa\lambda}(p^1_\kappa + p^2_\kappa) p^2_\lambda
\end{aligned} \tag{494}
$$

for $\overline{D}^2 = D \cap \mathsf{T}\overline{C}^1$ of D are the last step in the algorithm. We have excluded from $\overline{D}^2$ the case $p^1_\kappa + p^2_\kappa = 0$. The system $\overline{D} = \overline{D}^2$ will be shown to be integrable. ▲

EXAMPLE 28. We apply the second algorithm of Section 5 to the Lagrange equations (479). The prolongation of the Lagrange equations is the system of second order equations

$$
\begin{aligned}
\dot{p}^1_\kappa &= \frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\|q_2 - q_1\|} \left(\frac{\|\dot{q}_1\|}{\overline{m}_1} + \frac{\|\dot{q}_2\|}{\overline{m}_2} \right) g_{\kappa\lambda}(q_2^\lambda - q_1^\lambda) \\
\dot{p}^2_\kappa &= \frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\|q_2 - q_1\|} \left(\frac{\|\dot{q}_1\|}{\overline{m}_1} + \frac{\|\dot{q}_2\|}{\overline{m}_2} \right) g_{\kappa\lambda}(q_1^\lambda - q_2^\lambda) \\
p^1_\kappa &= \frac{\overline{m}_1}{\|\dot{q}_1\|} g_{\kappa\lambda} \dot{q}_1^\lambda \\
p^2_\kappa &= \frac{\overline{m}_2}{\|\dot{q}_2\|} g_{\kappa\lambda} \dot{q}_2^\lambda \\
\ddot{p}^1_\kappa &= f^1(q_1^\kappa, q_2^\lambda, \dot{q}_1^\mu, \dot{q}_2^\lambda, \ddot{q}_1^\rho, \ddot{q}_2^\sigma) \\
\ddot{p}^2_\kappa &= f^2(q_1^\kappa, q_2^\lambda, \dot{q}_1^\mu, \dot{q}_2^\lambda, \ddot{q}_1^\rho, \ddot{q}_2^\sigma) \\
\dot{p}^1_\kappa &= -\frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\overline{m}_1\|q_2 - q_1\|} g_{\mu\nu}(q_2^\mu - q_1^\mu)(\dot{q}_2^\nu - \dot{q}_1^\nu) g_{\kappa\lambda} \frac{\dot{q}_1^\lambda}{\|\dot{q}_1\|} \\
&\quad + \overline{m}_1 g_{\kappa\lambda} \left(\delta^\lambda_\mu - g_{\mu\nu} \frac{\dot{q}_1^\lambda \dot{q}_1^\nu}{\|\dot{q}_1\|} \right) \frac{\ddot{q}_1^\mu}{\|\dot{q}_1\|} \\
\dot{p}^2_\kappa &= -\frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\overline{m}_2\|q_2 - q_1\|} g_{\mu\nu}(q_2^\mu - q_1^\mu)(\dot{q}_2^\nu - \dot{q}_1^\nu) g_{\kappa\lambda} \frac{\dot{q}_2^\lambda}{\|\dot{q}_2\|} \\
&\quad + \overline{m}_1 g_{\kappa\lambda} \left(\delta^\lambda_\mu - g_{\mu\nu} \frac{\dot{q}_2^\lambda \dot{q}_2^\nu}{\|\dot{q}_2\|} \right) \frac{\ddot{q}_2^\mu}{\|\dot{q}_2\|}.
\end{aligned} \tag{495}
$$

The exact form of the functions $f^1(q_1^\kappa, q_2^\lambda, \dot{q}_1^\mu, \dot{q}_2^\lambda, \ddot{q}_1^\rho, \ddot{q}_2^\sigma)$ and

$f^2(q_1^\kappa, q_2^\lambda, \dot{q}_1^\mu, \dot{q}_2^\lambda, \ddot{q}_1^\rho, \ddot{q}_2^\sigma)$ is of no interest since equations for $\ddot{p}_\kappa^1$ and $\ddot{p}_\kappa^2$ can not impose restrictions on first derivatives. Projection in $\mathsf{T}(Q \times Q)$ eliminates these equations and also the components of the last two equations orthogonal to $(\dot{q}_1^\kappa)$ and $(\dot{q}_2^\kappa)$ respectively. The resulting first order equations

$$\begin{aligned}
\dot{p}_\kappa^1 &= \frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\|q_2 - q_1\|}\left(\frac{\|\dot{q}_1\|}{\overline{m}_1} + \frac{\|\dot{q}_2\|}{\overline{m}_2}\right) g_{\kappa\lambda}(q_2^\lambda - q_1^\lambda) \\
\dot{p}_\kappa^2 &= \frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\|q_2 - q_1\|}\left(\frac{\|\dot{q}_1\|}{\overline{m}_1} + \frac{\|\dot{q}_2\|}{\overline{m}_2}\right) g_{\kappa\lambda}(q_1^\lambda - q_2^\lambda) \\
p_\kappa^1 &= \frac{\overline{m}_1}{\|\dot{q}_1\|} g_{\kappa\lambda}\dot{q}_1^\lambda \\
p_\kappa^2 &= \frac{\overline{m}_2}{\|\dot{q}_2\|} g_{\kappa\lambda}\dot{q}_2^\lambda \\
\dot{p}_\kappa^1 \dot{q}_1^\kappa &= -\frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\|q_2 - q_1\|}\frac{\|\dot{q}_1\|}{\overline{m}_1} g_{\kappa\lambda}(q_2^\lambda - q_1^\lambda)\dot{q}_1^\kappa \\
\dot{p}_\kappa^2 \dot{q}_2^\kappa &= -\frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\|q_2 - q_1\|}\frac{\|\dot{q}_2\|}{\overline{m}_2} g_{\kappa\lambda}(q_1^\lambda - q_2^\lambda)\dot{q}_2^\kappa
\end{aligned} \tag{496}$$

are equivalent to the simplified equations

$$\begin{aligned}
\dot{p}_\kappa^1 &= \frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\|q_2 - q_1\|}\left(\frac{\|\dot{q}_1\|}{\overline{m}_1} + \frac{\|\dot{q}_2\|}{\overline{m}_2}\right) g_{\kappa\lambda}(q_2^\lambda - q_1^\lambda) \\
\dot{p}_\kappa^2 &= \frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\|q_2 - q_1\|}\left(\frac{\|\dot{q}_1\|}{\overline{m}_1} + \frac{\|\dot{q}_2\|}{\overline{m}_2}\right) g_{\kappa\lambda}(q_1^\lambda - q_2^\lambda) \\
p_\kappa^1 &= \frac{\overline{m}_1}{\|\dot{q}_1\|} g_{\kappa\lambda}\dot{q}_1^\lambda \\
p_\kappa^2 &= \frac{\overline{m}_2}{\|\dot{q}_2\|} g_{\kappa\lambda}\dot{q}_2^\lambda \\
0 &= (p_\kappa^1 + p_\kappa^2)(q_2^\kappa - q_1^\kappa).
\end{aligned} \tag{497}$$

These equations are not yet integrable. They were falsely declared the integrable part of the dynamics in [8]. The prolongation of equations (497) projected in $\mathsf{TT}^*(Q \times Q)$ results in equations

$$\begin{aligned}
\dot{p}_\kappa^1 &= \frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\|q_2 - q_1\|}\left(\frac{\|\dot{q}_1\|}{\overline{m}_1} + \frac{\|\dot{q}_2\|}{\overline{m}_2}\right) g_{\kappa\lambda}(q_2^\lambda - q_1^\lambda) \\
\dot{p}_\kappa^2 &= \frac{\mathrm{D}V(\|q_2 - q_1\|)}{2\|q_2 - q_1\|}\left(\frac{\|\dot{q}_1\|}{\overline{m}_1} + \frac{\|\dot{q}_2\|}{\overline{m}_2}\right) g_{\kappa\lambda}(q_1^\lambda - q_2^\lambda) \\
p_\kappa^1 &= \frac{\overline{m}_1}{\|\dot{q}_1\|} g_{\kappa\lambda}\dot{q}_1^\lambda
\end{aligned}$$

$$\begin{aligned} p_\kappa^2 &= \frac{\overline{m}_2}{\|\dot{q}_2\|} g_{\kappa\lambda}\dot{q}_2^\lambda \\ 0 &= (p_\kappa^1 + p_\kappa^2)(q_2^\kappa - q_1^\kappa) \\ 0 &= (p_\kappa^1 + p_\kappa^2)(\dot{q}_2^\kappa - \dot{q}_1^\kappa). \end{aligned} \tag{498}$$

From

$$(p_\kappa^1 + p_\kappa^2)(\dot{q}_2^\kappa - \dot{q}_1^\kappa) = \frac{\|\dot{q}_2\|}{\overline{m}_2} g^{\kappa\lambda}(p_\kappa^1 + p_\kappa^2)p_\lambda^2 - \frac{\|\dot{q}_1\|}{\overline{m}_1} g^{\kappa\lambda}(p_\kappa^1 + p_\kappa^2)p_\lambda^1 \tag{499}$$

we see that the last equation in (498) imposes a synchronization relation between parametrizations of the world lines of the particles unless $p_\kappa^1 + p_\kappa^2 = 0$. The case $p_\kappa^1 + p_\kappa^2 = 0$ seems to be too restrictive to be of interest. We will exclude this case. The integrability algorithm applied to equations (498) produces no further restrictions. ▲

EXAMPLE 29. Although the dynamics of two relativistic particles is not a Dirac system it admits the Dirac-style representation in terms of constrained Hamiltonian vector fields. This representation can be used to simplify the algorithm in Example 27 and to prove the integrability of the resulting equations. The simplified algorithm is an adaptation of Dirac's original algorithm [1]. Since the Poisson brackets (487) and (488) do not vanish on C we restrict the set C by adding the *secondary constraint*

$$\Psi(q_1^\kappa, q_2^\lambda, p_\mu^1, p_\nu^2) = (p_\kappa^1 + p_\kappa^2)(q_2^\kappa - q_1^\kappa). \tag{500}$$

The resulting system $\overline{D}^1$ is now the union of images of Hamiltonian vector fields generated by the family (484) restricted to the constraint set $\overline{C}^1 \subset \mathsf{T}^*(Q \times Q)$ satisfying the equations

$$\Phi_1 = 0, \quad \Phi_2 = 0, \quad \Psi = 0. \tag{501}$$

The system is not yet integrable. In the second step we require the vanishing of the Poisson bracket

$$\{H_{(\alpha^1,\alpha^2)}, \Psi\} = \frac{\alpha^1}{\|p^1\|} g^{\kappa\lambda}(p_\kappa^1 + p_\kappa^2)p_\lambda^1 - \frac{\alpha^2}{\|p^2\|} g^{\kappa\lambda}(p_\kappa^1 + p_\kappa^2)p_\lambda^2 \tag{502}$$

on the new constraint set. We will exclude the case $p_\kappa^1 + p_\kappa^2 = 0$ and impose the condition

$$\frac{\alpha^1}{\overline{m}_1} g^{\kappa\lambda}(p_\kappa^1 + p_\kappa^2)p_\lambda^1 - \frac{\alpha^2}{\overline{m}_2} g^{\kappa\lambda}(p_\kappa^1 + p_\kappa^2)p_\lambda^2 = 0 \tag{503}$$

on the functions (α^1, α^2). The resulting system $\overline{D}^2$ is the union of images of constrained Hamiltonian vector fields generated by the family (484) with the positive functions (α^1, α^2) satisfying the condition (503). The vector fields are restricted to the constraint set $\overline{C}^1$. The system is exactly the system described by equations (494) and (498). The system is integrable since

$$\{H_{(\alpha^1,\alpha^2)}, \Phi_1\}|\overline{C}^1 = 0, \tag{504}$$

$$\{H_{(\alpha^1,\alpha^2)}, \Phi_2\}|\overline{C}^1 = 0, \tag{505}$$

and

$$\{H_{(\alpha^1,\alpha^2)}, \Psi\}|\overline{C}^1 = 0\,. \tag{506}$$

▲

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