LECTURE IN HONOUR OF LEOPOLD INFELD (EXTENDED OUTLINE ONLY) SPINORS IN GENERAL RELATIVITY*

Sir Roger Penrose

Mathematical Institute
24–29 St Giles, Oxford OX1 3LB, U.K.

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This article is an extended outline of the lecture delivered at the Infeld Centenial Meeting. In the lecture a review was given of the development of the theory of spinors and related objects in special and general relativity, with some emphasis on the twistor theory and its implications. The lecture was not intended as a detailed account of the subject, but it rather was a series of comments on the relevance of various spinor-type objects and their relation to some features of space-time structure. The present article is also a guide, with its author's personal preferences, to the extensive bibliography of the subject.

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It is a great honour for me to have this opportunity to pay my respects to Leopold Infeld. His seminal work showing how spinor calculus may be applied in general curved space-times has been extremely influential, and it has profoundly affected my own researches, these having been very greatly concerned with the relationship between spinor theory and Einstein's general relativity.

1. Preliminaries: flat-space spinors

Spinors were first found by Élie Cartan (1913; *cf.* also Cartan 1966). His spinor spaces constitute the representation spaces of 2-valued representations of orthogonal groups. He did not use the name "spinor" at that time, this term having been apparently introduced later by Ehrenfest (as I was recently informed by Andrzej Trautman), following Dirac's (1928) rediscovery of spinors and discovery of their application to the spin of the

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relativistic electron (and also Pauli's earlier work, in 1927, concerning the *non*-relativistic electron's spin). However many spinor-related ideas were known much earlier, particularly in relation to the 2-valued ("spin") representations of particular rotation groups. (There are *quaternions*, found by William Rowan Hamilton in 1837 and their explicit representation of rotation matrices given by the Cayley–Klein parameters, illustrating the local isomorphism $\mathrm{SU}(2) \to \mathrm{O}(3)$, and its complexification $\mathrm{SL}(2, \boldsymbol{C}) \to \mathrm{O}(3, \boldsymbol{C})$, and there is also the local isomorphism $\mathrm{SU}(4) \to \mathrm{O}(6)$ and its complexification $\mathrm{SL}(4, \boldsymbol{C}) \to \mathrm{O}(6, \boldsymbol{C})$, known to Sophus Lie in about 1872. For information on the history of these matters, see Crowe 1967 and van der Waerden 1985.) The general discussion of spin representations of rotation groups stems from the algebras of William Kingdon Clifford (1878), these arising as the common generalization of quaternions and Grassman algebras (*cf.* Grassman 1844, describing work done in 1825). An elegant and thorough discussion of spinors for general orthogonal groups is to be found in Brauer and Weyl (1935), based on Clifford algebras. (See also the appendix to Penrose and Rindler 1986; Budinich and Trautman 1988.) Chevalley (1954) showed how to develop a theory of spinors applicable to fields of finite characteristic (including the awkward case of characteristic 2).

The applications of spinors to physics stemmed initially from the work of Pauli and Dirac as cited above, where the spinors were brought in specifically for the treatment of particles of spin 1/2. Whereas "Pauli spinors" are the 2-component spinors for the (non-relativistic) rotation group O(3), "Dirac spinors" are 4-component entities, being spinors for the Lorentz group O(1,3). These 4-spinors split down into pairs of 2-component entities, sometimes referred to "half-spinors" (or "Weyl spinors"), which were considered by Weyl (1929) (and earlier, apparently, by Dirac himself, *cf.* Dirac 1982) to have relevance to a relativistic (reflection-non-invariant) wave equation. This equation is now often used to describe the neutrino. It is a general feature of spinors for an *even*-dimensional space that the spinors indeed break down into such "half-spinors" — or *reduced* spinors — in this way. (This is "reduction" under the narrowing down of the rotation group to its non-reflective subgroup.) For a $2n$-dimensional space, the reduced spinors are 2^{n-1}-dimensional, so the unreduced spinors are 2^n-dimensional; for a $(2n+1)$-dimensional space, the spinors are 2^n-dimensional.

In the case of a 4-dimensional space-time, subject to local Lorentz group symmetry, the reduced spinors are 2-component entities, usually referred to as 2-spinors. Each of the two 2-dimensional spaces of reduced spinors is a complex space; these two spaces go into each other under space-reflection, time-reflection, or complex conjugation. According to the 2-spinor calculus introduced by van der Waerden (1929), as notationally slightly modified in

Penrose (1960) and Penrose and Rindler (1984), the elements of these two spin-spaces are labelled by kernel symbols having indices which are, respectively, capital Roman letters without primes $(A, B, C, \ldots, A_0, B_0, \ldots, A_1, \ldots)$ and Roman letters with primes $(A', B', C', \ldots, A'_0, B'_0, \ldots, A'_1, \ldots)$. There are invariant skew-symmetrical quantities ε_{AB}, $\varepsilon_{A'B'}$, ε^{AB}, $\varepsilon^{A'B'}$, that can be used for raising and lowering indices (care having to be exercised to keep the signs consistent). The tensor calculus may be thought of as being embedded in the 2-spinor calculus where an unprimed and a primed spinor index taken together counts as a tensor index.

In this van der Waerden (1929) 2-spinor formalism, the Dirac equation for the electron becomes a pair of coupled linear differential equations relating an unprimed spinor α_A to a primed spinor $\beta_{A'}$. This formalism was subsequently used by Laporte and Uhlenbeck (1931) to represent, among other things, the Maxwell equations for the electromagnetic field in an elegant way. Then Dirac (1936) showed how to write down field equations for higher spin, generally, in a very neat way and this was followed up significantly by Fierz (1938). (In later work, Rarita, Schwinger, Duffin, Kemmer, and others formulated higher-spin equations, in special cases, but not using such a powerful general formalism; see Corson 1953 for a comprehensive account of all this.)

Lorentzian 2-spinors have a very neat geometrical description in terms of the null cone. A non-zero 2-spinor determines a future-pointing null vector and a null half-plane containing the direction of this vector. These are the "flagpole" and "flag plane", respectively, of this *null flag* interpretation of the spinor; see Payne (1952), Penrose and Rindler (1984). This geometrical description determines the 2-spinor completely up to a sign. (To interpret the sign, a non-local description is needed; see Penrose and Rindler 1984. Rotation of a spinor through 2π changes its sign, whereas rotation through 4π restores it to its original value.) The null *direction* of the flagpole describes the 2-spinor completely up to proportionality. A symmetrical n-valent spinor (an n-index quantity $\xi_{AB\ldots L}$, with $\xi_{AB\ldots L} = \xi_{(AB\ldots L)}$ describes an entity of spin $n/2$. It has a *canonical decomposition*, according to which it can be represented (uniquely up to scalings and ordering) as a symmetrized product of single-index 2-spinors $\xi_{AB\ldots L} = \alpha_{(A}\beta_B \ldots \lambda_{L)}$ and, therefore, up to proportionality, it corresponds uniquely to a symmetrical set of n null directions. This description gives the Majorana (1932) geometrical description of the general spin state of a spin $n/2$ particle (see Penrose 1989).

2. Spinors for curved space-time

The first treatment of spinors in *curved* space-time was that of Infeld and van der Waerden (1933), using the van der Waerden 2-spinor formalism. The translation between spinors and tensors is then achieved in local component form by use of the *Infeld–van der Waerden symbols* (in the terminology of Penrose and Rindler 1984). In this paper, the ingenious suggestion was made that the *spinor phase* should be the gauge quantity that generates electromagnetism. However, this idea has not stood the test of time (at least not in its original form) because it appears to imply a direct relation between the spin and the charge of a particle. (The spin/charge value for the neutron would appear to be in conflict, the neutron having been discovered in 1932, at about the same time as this paper was written.)

Although the initial use of the Infeld-van der Waerden formalism was for the description of particles with spin within the curved space-time framework of Einstein's general relativity, this formalism can also be used to study Einstein's theory itself. See Veblen and von Neumann (1936), Bergmann (1957), Witten (1959), Penrose (1960), Penrose and Rindler (1984, 1986). For example, the Weyl conformal tensor corresponds to a totally symmetric spinor Ψ_{ABCD} which resembles a spin 2 massless field, in the sense (referred to above) of Dirac–Fierz. The canonical decomposition applied to this spinor gives a neat classification scheme for vacuum space-times (*cf.* Penrose 1960, Penrose and Rindler 1986).

When all tensor as well as spinor components are referred to a choice of *spin-frame* (a basis in the local spin space at each space-time point), then the formalism of *spin-coefficients* is obtained (*cf.* Jordan, Ehlers, and Sachs 1961, Newman and Penrose 1962). This turns out to have a great calculational utility; see, for example, Chandrasekhar (1983). In certain situations, it turns out that because of the geometry of a problem, it may be convenient to specify merely a pair of null directions at each point (the flagpole directions of the two spinor basis elements) rather than an entire spin-frame. Then the more streamlined *compacted* spin-coefficient formalism can be very convenient to use (see Newman and Penrose 1966, Geroch, Held, and Penrose 1973, Penrose and Rindler 1984).

Most of the foregoing remarks, in this section, have been concerned with spinors only *locally* in curved space-time. In fact there are important global restrictions and ambiguities, for spinor fields to make *global* sense on a space-time manifold. These have to do with what are called *Stieffel–Whitney classes* on the manifold. The essential issue is the fact that the *sign* of a spinor does not have a local geometrical interpretation (something that was touched upon in the previous section), so the consistency of this sign globally depends upon global properties of the space-time. For results of relevance to this issue, see Milnor (1963), Lichnerowicz (1968), Geroch (1968, 1970), Penrose and Rindler (1984). (1984).

3. Twistor theory

It is possible to regard spinors as being, in some sense, more primitive than tensors. One may take the view that the light-cone structure and metric scaling are *determined* by the spinor structure of the space-time (*cf.* Bergmann 1957, Penrose and Rindler 1984). Perhaps, even the particular dimensionality and signature of the space-time metric may, in some sense, be regarded as *derived* concepts. However, so long as the space-time manifold itself must be given beforehand, it is hard to hold to such a view in a serious way. The manifold's dimension is determined by its topology, and the definition of tensors requires only its differentiable structure. However, the theory of *twistors* (Penrose 1967) provides the possibility of a more radical view. According to this scheme of ideas, the space-time manifold itself, not just its light cone structure, is indeed regarded as *derived* from a more primitive spinor-type space.

This is not the place to give a detailed account of twistor theory. (Such accounts can be found in Huggett and Tod 1985, Penrose and Rindler 1986, Ward and Wells 1989.) Only some brief comments of relevance will be given here. For flat space-time $\boldsymbol{M}$, which is to be a *Minkowski* 4-*space*, the translation between the *conformal compactification* $\boldsymbol{M}^{\#}$ of $\boldsymbol{M}$ and the twistor space $\boldsymbol{T}$ is very direct. The space $\boldsymbol{T}$ is a 4-complex-dimensional vector space with a Hermitian quadratic form Σ of signature $(++--)$. The parts of $\boldsymbol{T}$ on which $\Sigma > 0$, $\Sigma < 0$, and $\Sigma = 0$ are denoted by $\boldsymbol{T}^+$, $\boldsymbol{T}^-$, and $\boldsymbol{N}$, respectively. The elements of $\boldsymbol{N}$ are called *null* twistors. Any point $x \in \boldsymbol{M}^{\#}$ is interpreted as a 2-*dimensional linear subspace* $\boldsymbol{x}$ of $\boldsymbol{M}$, where $\boldsymbol{x} \subset \boldsymbol{N}$. It is often convenient to think in terms of the *projective* twistor space $\boldsymbol{PT}$, which is the complex projective 3-space whose points are the 1-dimensional linear subspaces of $\boldsymbol{T}$. Thus, there is 5-real-dimensional subspace $\boldsymbol{PN}$, of $\boldsymbol{PT}$, consisting of the projective null twistors. The projective version $\boldsymbol{P}x$, of $\boldsymbol{x} \subset \boldsymbol{N}$, is a *projective straight line* in $\boldsymbol{PN}$. It turns out that a projective null twistor represents a *light ray* (null geodesic) in $\boldsymbol{M}^{\#}$ and the generators of the light cone of a point $x \in \boldsymbol{M}^{\#}$ are thereby represented, in $\boldsymbol{PT}$, by the points of the line $\boldsymbol{Px}$. Two points x and y of $\boldsymbol{M}^{\#}$ are *null separated* iff the corresponding lines $\boldsymbol{P}x$ and $\boldsymbol{P}y$ intersect.

This gives the basic geometry of the correspondence between the (compactified) space-time $\boldsymbol{M}^{\#}$ and the (projective) twistor space $\boldsymbol{PT}$. In fact, this geometry is really a manifestation of the structure of Sophus Lie's local isomorphism $\mathrm{SL}(4,\boldsymbol{C}) \to \mathrm{O}(6,\boldsymbol{C})$, in its real form $\mathrm{SU}(2,2) \to \mathrm{O}(2,4)$. The relevance of $\mathrm{O}(2,4)$ here is that it is locally isomorphic to the 15-parameter *conformal group* of (compactified) Minkowski 4-space $\boldsymbol{M}^{\#}$. The group $\mathrm{SU}(2,2)$ acts on the spin space $\boldsymbol{T}$ and preserves the $(2,2)$-Hermitian-quadratic form Σ. The local isomorphism $\mathrm{SU}(2,2) \to \mathrm{O}(2,4)$ expresses the fact that twistors (elements of $\boldsymbol{T}$) are in fact reduced spinors for the pseudo-orthogonal group O(2,4) that describes the conformal symmetries of compactified space-time $\boldsymbol{M}^{\#}$. The other space of reduced spinors (the analogue of Lorentzian "primed spinors") turns out to be the dual space $\boldsymbol{T}^*$ of $\boldsymbol{T}$.

The underlying philosophy of twistor theory is that the complex space $\boldsymbol{T}$ is to be regarded as being, in some sense, more primitive than the space-time itself. (The main motivations for this twistor view come from a desire to bring together the basic, but incompatible, principles of quantum mechanics and Einstein's general relativity without trying to *impose* one upon the other. The complex-number structure of quantum mechanics and quantum non-locality find manifestations in space-time geometry via the twistor correspondence.) Over the years, it has been found that many of the basic physical notions, particularly those which have to do with massless particles and fields, indeed have elegant interpretations in twistor-space terms; also, twistor theory has had many applications within areas of pure mathematics. (See Hughston 1975, Huggett and Tod 1985, Penrose and Rindler 1986, Ward and Wells 1989, Bailey and Baston 1990, Mason and Woodhouse 1996, Huggett, Mason, Tod, Tsou, and Woodhouse 1998 for details).

Yet, that *crucial* physical field, which any attempt at unifying quantum mechanics with space-time structure must profoundly come to terms with, is Einstein's general relativity. How does twistor theory fare in this respect? At first sight (in fact a "first sight" which lasted for some 10 years or more), it had seemed that twistor theory is really just a scheme of things that applies only to (conformally) flat space-time. However, it eventually turned out (Penrose 1976) that by deforming (portions of) twistor space it becomes possible to encode the Einstein vacuum equations in the case of *anti-self-dual Weyl curvature* into the structure of the deformed twistor space. Anti-self-dual Weyl curvature corresponds, in the weak-field limit, to the *left-handed* graviton field, when we are thinking of the complex graviton wave-function as a weak-field deformation of complexified Minkowski space $\boldsymbol{CM}$. In fact, twistor theory is a profoundly *chiral* theory, and it has the curious feature that the left-handed and right-handed components of a massless quantum field are treated on a quite different footing. In the case of the Einstein gravitational field, it has turned out (using the standard twistor conventions) that the left-handed part of the graviton finds a twistor interpretation far more readily than does the right-handed part.

What is the present status of the problem of encoding the right-handed graviton also into the framework of twistor theory, over twenty years after the successful left-handed construction? For a number of years recently (*cf.* Penrose 1992), I had been pinning my hopes on the striking fact that massless fields of helicity 3/2 seem to provide an intermediary between the

Einstein vacuum equations and twistor theory. The consistency condition for helicity 3/2 fields is precisely the vacuum equations, whereas, in $\boldsymbol{M}$, the space of *charges* for these fields is actually *twistor space* $\boldsymbol{T}$. In the meantime, however, another idea has emerged (Penrose 1999) which does indeed encode the outstanding *right-handed* part of the gravitational field. It remains to be seen whether this all links together (perhaps via the agency of fields of helicity 3/2) to provide a full twistor description of Einstein's general relativity. If it does, then a new approach to the unification of quantum theory with space-time structure could well be provided.

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