

OLD AND NEW TRICKS IN RELATIVISTIC TWO-BODY EQUATIONS*

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A brief review of relativistic two-body equations in QED and their non-relativistic reductions is presented, beginning with the atomic Dirac–Breit equation. The emphasis is on lepton-antilepton bound states (leptonium), with a look at possible extensions to quarkonium.

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1. Introduction

Theorists working on QCD tend to prefer heavier quarkonia over lighter ones, not only because of smaller confining potential effects, but also because of the approximate validity of the nonrelativistic two-body Schrödinger equation. The latter point may be unessential, however, in view of the quantitative successes of relativistic calculations for QED bound states. Modern atomic theory is largely based on the n -electron Dirac–Breit equation, in which the binding arises from a nucleus of charge Ze . As Z can reach 100 and $e^2 = \alpha = 1/137$ ($\hbar = c = 1$), the effective expansion parameter $Z\alpha$ can get close to 1, where nonrelativistic expansions break down.

As particle theorists we are more interested in simpler systems such as atomic hydrogen or “leptonium” (essentially e^-e^+ and $e^-\mu^+$), where an external nuclear Coulomb potential is absent, and the total leptonium momentum is conserved,

$$(\mathbf{p}_1 + \mathbf{p}_2)\psi = \mathbf{K}\psi, \quad \mathbf{p}_i = -\nabla_i. \quad (1.1)$$

Relativistic effects are small here but have been calculated and verified experimentally to reasonably high orders of α . In the hyperfine structure

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of muonium, the leading term of order α^8 is already experimentally relevant [17]. The corresponding QCD hyperfine structure is particularly interesting, because it is surprisingly large in light quarkonium.

The new tricks in the second half of this lecture require a symmetric treatment of the two fermions. Atomic hydrogen is presently excluded; the proton's large anomalous magnetic moment requires a special "anomalous Dirac equation", which is mentioned in many textbooks.

Whereas the calculation of relativistic two-body effects has enjoyed steady progress over the past 70 years, the use of relativistic two-fermion equations has had several drastic oscillations. One year after Dirac published his relativistic one-electron equation, Breit [4] published his equation for two relativistic electrons in the presence of the nuclear Coulomb potential. Breit hoped that his electron–electron operator would be exact, but already next year [5] he found that it gave too large effects in second-order perturbation theory. This defect was largely eliminated by the inclusion of "positive-energy projectors", which account for the fact that the negative-energy solutions of the Dirac equation turn into antielectrons (positrons) of positive energies upon second quantization. An early account of the arguments is found in the standard book of Bethe and Salpeter [2]. The problems created by the negative-energy states in higher-order perturbation theory were first pointed out by Brown [6] and went under the name "Brown's disease". Sucher [22] emphasized the "continuum dissolution", which is caused already by the Coulomb repulsion between the two electrons: One electron is excited into the positive-energy continuum, the other falls into the negative-energy continuum, without any change in the sum of the two energies. An exact form of the required projectors exists only for products of single-particle orbitals, where the electronic Coulomb repulsion is neglected. A good starting point for such cases is the "Furry picture" of QED, which is QED in an external static potential [12].

But relativistic effects are normally small even in atomic physics. For two electrons, the $4 \times 4 = 16$ -component wavefunction ψ of Dirac–Breit may then be reduced to a $2 \times 2 = 4$ -component Schrödinger wavefunction ψ_{Sch} , on which an effective Hamiltonian H_{eff} acts. Instead of the Dirac matrices γ , it contains only the Pauli matrices $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ of the two electrons. For a single electron, the reduction was written down by Pauli and then more systematically by Foldy and Wouthuysen [11]. For two fermions of arbitrary masses, it was done by Chraplyvy [8]. For example, the Dirac kinetic energies $\boldsymbol{\alpha}_i \mathbf{p}_i$ (see below) become $p_i^2/2m_i + p_i^4/8m_i^3$ to order α^4 in H_{eff} , and the commutator between $\boldsymbol{\alpha}_i \mathbf{p}_i$ and the external Coulomb potentials $V_{ei} = -Z\alpha/r_i$ produces the famous spin–orbit potentials,

$$V_{LSi} = \frac{dV_{ei}}{dr_i} \frac{\boldsymbol{\sigma}_i \mathbf{L}_i}{4r_i m_i^2}. \quad (1.2)$$

Important for leptonium is the nonrelativistic reduction of the Breit operator, because in the absence of V_{ei} ($Z = 0$), it is the only operator in addition to the Coulomb potential between the two leptons,

$$V(r) = \frac{q_1 q_2}{r}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2. \quad (1.3)$$

Among the reduced Breit operators, there are again two spin-orbit operators,

$$V_{LS} = \frac{\alpha}{r^3} \left[\boldsymbol{\sigma}_1 \left(\frac{\mathbf{r} \times \mathbf{p}_1}{4m_1^2} - \frac{\mathbf{r} \times \mathbf{p}_2}{2m_1 m_2} \right) - \boldsymbol{\sigma}_2 \left(\frac{\mathbf{r} \times \mathbf{p}_2}{4m_2^2} - \frac{\mathbf{r} \times \mathbf{p}_1}{2m_1 m_2} \right) \right], \quad (1.4)$$

and also a tensor operator which mixes S-states with D-states. So at least for leptonium, the first fully relativistic approach ended in nonrelativistic reductions for both particles.

The second fully relativistic approach was the Bethe-Salpeter (BS) equation [1], an integral equation for the 16-component ψ . Its applications in atomic physics were not important and are skipped here. For leptonium, the BS-equation is a covariant integral equation in the variables $\mathbf{r}$ and $\Delta t = t_1 - t_2$. Its claim of exactness was disproved by Wick [23], who discovered unphysical excited states in Δt in addition to the physical radial excitations (which have nodes in the radial wavefunction $R(r)$). The BS-equation is normally solved in momentum space, where the variable corresponding to Δt is the energy transfer $q^0 = K_1^0 - K_1^{0'} = -K_2^0 + K_2^{0'}$ (the total energy is conserved, $K^0 = K_1^0 + K_2^0 = K_1^{0'} + K_2^{0'}$). The Δt -excitations disappear when the kernel $\mathcal{K}$ of the integral equation is divided into an unperturbed Coulomb kernel $\mathcal{K}_V$ and various perturbations $\delta\mathcal{K}$. The q^0 -integration must be done explicitly for $\mathcal{K}_V$ in the cms, where the system's total momentum $\mathbf{K}$ (1.1) vanishes. The q^0 -integration of $\delta\mathcal{K}$ must be done in a perturbative series. The value of K^0 in the cms will be called E ; it is simply the leptonium mass:

$$K_{\text{cms}}^\mu = (K^0, \mathbf{K})_{\text{cms}} = (E, \mathbf{0}), \quad \mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{p}. \quad (1.5)$$

Thus the covariance of the BS-equation is deceiving: The equation must be solved in the cms (1.5) and may then be Lorentz transformed to other values of K^μ . In practice of course, the same procedure is applied to the Schrödinger equation. In the cms, the kinetic energy operators to order α^2 combine into

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{p^2}{2\mu_{nr}}, \quad \mu_{nr} = \frac{m_1 m_2}{(m_1 + m_2)}. \quad (1.6)$$

The eigenvalues of $H_{\text{eff}}\psi_{\text{Sch}} = E\psi_{\text{Sch}}$ to order α^2 in the cms are the familiar

$$E_n = m_1 + m_2 - \frac{\alpha^2 \mu_{nr}}{2n^2}. \quad (1.7)$$

They may then be Galilei transformed to a lab system with $\mathbf{K} \neq \mathbf{0}$. The essential property of the cms for the BS-equation is the vanishing of q^0 for the mass-shell particles, which enter the first Born approximation for $\mathcal{K}$: The factor $(q_\mu q^\mu)^{-1} = (q^{02} - \mathbf{q}^2)^{-1}$ in the photon propagator is then reduced to $(-\mathbf{q}^2)^{-1}$; it is independent of the integration variable q^0 . In coordinate space, $-\alpha/\mathbf{q}^2$ provides the Coulomb potential $V(r)$.

An important two-photon exchange contribution to $\delta\mathcal{K}$ gives the so called Salpeter shift, which is of the order α^5/π and vanishes in the static limit $m_1 \ll m_2$. Calculations with the BS equation contain unexplained cancellations between different operators $\delta\mathcal{K}$, beginning at the order α^4 . A fairly recent review of the application of the BS-equation to leptonium is given by Sapirstein and Yennie [21].

Sooner or later in BS-calculations, a nonrelativistic expansion is used for at least one of the two particles. Beyond a certain order of α , one sets, for $m_2 > m_1$, $(E_2 + m_2)^{-1} \approx (2m_2)^{-1}$, implying that E_2 is near m_2 . This approximation is particularly useful for atomic hydrogen, with $m_2/m_1 \approx 2000$. Grotch and Yennie [13] found that this case could be treated more easily by an extension of the Dirac equation, of the form

$$(H_{1D} + H_{2,nr})\psi_{1D} = E_1\psi_{1D}, \quad E_1 = E - m_2, \quad (1.8)$$

where H_{1D} is the one-electron Dirac operator. $H_{2,nr}$ is approximately $p^2/2m_2$, better forms are given below. Braun [3] found a form for $H_{2,nr}$ that gives all corrections of order m_1^2/m_2 to the Dirac equation, including the corrections to the radiative corrections (in the case of the Salpeter shift, the exact mass dependence had already been calculated from the BS equation). For positronium, however, the Braun method is insufficient. Essential progress here came from Caswell and Lepage [7], who pushed the calculation of H_{eff} for ψ_{Sch} to the order α^6 . Their method is nowadays called nonrelativistic quantum electrodynamics (NRQED). It was extended by Pachucki [19] to the unequal mass case. In this manner, also the second fully relativistic approach ended in nonrelativistic reductions for both particles. But for $m_2 > m_1$, one may still use (1.8) for the highest orders in α , say α^7 and α^8 .

Several other fully relativistic methods have been proposed which reproduce the known results to order α^4 , but which are unable to calculate higher orders. Among these, the ‘‘constraint Hamiltonian’’ method may be mentioned, which was originally proposed by Dirac and elaborated by Crater

and Van Alstine [9]. It uses two different equations at two different times for one wavefunction $\psi(x_1^\mu, x_2^\mu)$, which exclude the excitations in Δt found by Wick for the BS-equation.

The new tricks promised in this lecture refer to a fairly recent fully relativistic two-fermion equation [14]. For the nS-states, it has been evaluated to the order α^8 [15]. At the order α^6 , it misses a hyperfine operator which was found by Pachucki from the exchange of three photons, and which has not yet been calculated by the new equation. The equation is analogous to Dirac–Breit in its differential form and to BS in its integral form. Its essential point is that the QED interaction is included only after the free equation has been reduced from 16 to 8 components.

2. The old tricks

In our units $\hbar = c = 1$, the Dirac equation is (for $q = -e$)

$$(iD_\mu\gamma^\mu - m)\psi = 0, \quad D_\mu = \partial_\mu - ieA_\mu. \quad (2.1)$$

An external potential V is part of A_0 . Multiplication of (2.1) by γ^0 produces the Hamiltonian form, $i\partial_0\psi = H_D\psi$, where ieA_0 is now included in the Dirac Hamiltonian H_D . For two particles of masses m_i one needs

$$H_{D,i} = m_i\gamma_i^0 + V_i(\mathbf{r}_i) + \mathbf{p}_i\gamma_i^0\gamma_i. \quad (2.2)$$

Dirac needed only $\gamma_i^0\gamma_i$ which he called α_i , and γ_i^0 which he called β_i (the γ -matrices came later and had to use the third letter in the alphabet). We shall use yet another nomenclature, calling

$$\gamma_i^0\gamma_i = \alpha_i = \gamma_i^5\sigma_i. \quad (2.3)$$

This will be necessary in order to decouple the algebra of the two sets of Pauli matrices, σ_1 and σ_2 , from the two sets of “proper” Dirac matrices, $\gamma_1^0, \gamma_2^0, \gamma_1^5, \gamma_2^5$. The Pauli matrices act only on spin components; they do not distinguish between the upper and lower Dirac components (or equivalently between the right- and left-handed Weyl or “chirality” components); the “proper” Dirac matrices do not act on spin components, they will be used here in the Weyl basis, where the γ_i^5 are diagonal:

$$\gamma_i^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_i, \quad \gamma_i^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_i. \quad (2.4)$$

In this sense, all our matrices are 2×2 , which is also typographically convenient. The Dirac–Breit equation has the form

$$H_{DB}\psi = E\psi, \quad H_{DB} = H_{D1} + H_{D2} + V(r) + H_B, \quad (2.5)$$

with H_B given below. As ψ has 16 components, every term of H_{DB} is strictly a 16×16 -matrix, including the Coulomb potential $V(r)$. For example, the operator $\alpha_1 \alpha_2$ is written mathematically correct as

$$\alpha_1 \alpha_2 = \gamma_1^5 \otimes \gamma_2^5 \otimes \sigma_1 \otimes \sigma_2. \quad (2.6)$$

Physicists normally suppress $\otimes$ symbols and unit matrices. With these conventions, the Breit operator reads

$$H_B = -\frac{1}{2} V \gamma_1^5 \gamma_2^5 (\sigma_1 \sigma_2 + \sigma_{1r} \sigma_{2r}), \quad \sigma_{ir} = \sigma_i \mathbf{r}/r. \quad (2.7)$$

Breit found it by postulating that its expectation values should be identical with the energy shifts obtained in second-order perturbation theory from the vacuum expectation values of the vector field operators $\mathbf{A}_1$ and $\mathbf{A}_2$ which occur in $\mathbf{D}_1$ and $\mathbf{D}_2$ in (2.1). This explains the absence of these operators in (2.5). The Lambshift has a similar origin, but is not easily replaced by an equivalent first-order operator. Breit used instead of the Weyl basis Dirac's "low-energy" basis, in which the two matrices of (2.4) are simply exchanged. This basis is also called the parity basis because it has γ_i^0 (which is part of the parity operator) diagonal. γ_i^5 is off-diagonal then, it connects the upper, large components with the lower, small ones. For positive-energy states, the relevant matrix elements are small; $\gamma_i^5 \sigma_i$ may be viewed as velocity operators. However, γ_1^5 and γ_2^5 also have large matrix elements, which ensure $(\gamma_i^5)^2 = 1$. In second-order perturbation theory for H_B , the γ_i^5 disappear, and the resulting energy shift is much too large. This large contribution is eliminated by the positive-energy projectors mentioned before.

The nonrelativistic reductions of Pauli, Foldy and Wouthuysen assume positive-energy states in their derivations. Projectors are then unnecessary. On the other hand, the resulting operators are so singular at $r = 0$, that they cannot be used beyond first-order perturbation theory: r^{-3} in V_{LS} , and $\delta(\mathbf{r})$, for example in the Fermi contact hyperfine interaction:

$$H_F = \frac{8}{3} \frac{\pi \alpha \sigma_1 \sigma_2 \delta(\mathbf{r})}{m_1 m_2}. \quad (2.8)$$

In NRQED, these operators are consistently regularized, such that also their squares and products exist. The final results contain only certain combinations in which the divergences cancel out. For the calculation of leptonium levels to the order α^6 , Pachucki [19] used

$$V = -(\alpha/r) \lim_{\lambda \rightarrow \infty} (1 - e^{-\lambda m \alpha r}), \quad (2.9)$$

such that $V(r \rightarrow 0) = -\alpha^2 \lambda m$ for finite λ . Correspondingly, the combination $4\pi \delta(\mathbf{r})$ in (2.8) is replaced by $\nabla^2 V$. A more elegant method is the

dimensional regularization [10]. The origin of the cancellations is of course a less singular structure of the relativistic operators, but these are not always known.

The Bethe–Salpeter equation will be discussed only briefly: It is a four-dimensional integral equation for the four-point function of fermion-fermion scattering:

$$G = S + \int d^4q S \mathcal{K} G, \quad (2.10)$$

where S is the product of two free fermion propagators:

$$S = (\not{p}_1 - m_1)^{-1} (\not{p}_2 - m_2)^{-1}, \quad \not{p}_i = p_{i,\mu} \gamma_i^\mu, \quad (2.11)$$

and $\mathcal{K}$ is the kernel. Despite the elegant notation, the q^0 integration must be performed in the cms, as explained in the Introduction. Moreover, S is a $4 \times 4 \times 4 \times 4$ matrix, and $\mathcal{K}$ is correspondingly large. This is connected with the cancellations mentioned in the Introduction. There exists an equivalent 8×8 -component version which should be free from cancellations [14]. The corresponding differential equation is discussed in the next section.

When one of the two fermions is nonrelativistic, the problem is equally well treated by differential equations. The total Hamiltonian is taken as the sum of the relativistic electron Hamiltonian H_{1D} and the nonrelativistic $H_{2,nr}$ as in (1.8). For a spinless particle 2, one has [13]

$$H_{2,nr} = \frac{\mathbf{p}_2^2}{2m_2} + H'_B, \quad H'_B = -\frac{1}{2} V \gamma_1^5 \boldsymbol{\sigma}_1 \frac{(\mathbf{p}_2 + \frac{\mathbf{r} p_{2r}}{r})}{m_2}, \quad (2.12)$$

with $p_{2r} = -i\partial/\partial r_2$. This form is obtained from (2.7) by replacing the “relativistic velocity” operator $\gamma_2^5 \boldsymbol{\sigma}_2$ by the nonrelativistic velocity $\mathbf{v}_2 = \mathbf{p}_2/m_2$, which is $-\mathbf{p}/m_2$ in the cms. The combination of a relativistic form with a nonrelativistic one is only tested to first order in $1/m_2$. It is probably not possible to just add corrections such as $p_2^4/8m_2^3$. A more precise form of $H_{2,nr}$ is [3]

$$H_{2,nr} = \frac{(\mathbf{p}_2 - e\mathbf{A}(\mathbf{r}_2))^2}{2m_2}. \quad (2.13)$$

It produces all first-order recoil corrections to the Dirac energy levels, including the Salpeter shift and the recoil corrections to the radiative corrections [20].

3. The new tricks

Nonrelativistically, the cms $\mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{p}$ allows one to combine the two kinetic energies $p_1^2/2m_1$ and $p_2^2/2m_2$ into an effective single-particle kinetic

energy, $p^2/2\mu_{nr}$. This is extended to the relativistic case by two new tricks. The 16-component, free-particle version of (2.5) for $\mathbf{p}_2 = -\mathbf{p}$ is

$$[E - m_1\gamma_1^0 - m_2\gamma_2^0 - \mathbf{p}(\gamma_1^5\boldsymbol{\sigma}_1 - \gamma_2^5\boldsymbol{\sigma}_2)]\psi_0^{(16)} = 0. \quad (3.1)$$

The operator $\gamma_2^5\mathbf{p}\boldsymbol{\sigma}_2$ is completely eliminated by these tricks. In the first place, $\psi_0^{(16)}$ is divided into two chiral octets,

$$\psi_0 = \begin{pmatrix} \psi_{rr} \\ \psi_{ll} \end{pmatrix}, \quad \chi_0 = \begin{pmatrix} \psi_{rl} \\ \psi_{lr} \end{pmatrix}, \quad (3.2)$$

in which the first chirality index (r =righthanded, l =lefthanded) refers to particle 1, the other to particle 2. We call $\gamma_1^5 \equiv \gamma_5$ and find in the chiral basis (2.4)

$$\gamma_2^5\psi_0 = \gamma_5\psi_0, \quad \gamma_2^5\chi_0 = -\gamma_5\chi_0. \quad (3.3)$$

The matrix $\gamma_1^0\gamma_2^0$ which exchanges r_1 with l_1 and r_2 with l_2 will be called β :

$$\beta = \gamma_1^0\gamma_2^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.4)$$

When acting on the upper and lower chirality components of ψ_0 and χ_0 , it is only a 2×2 matrix, just as γ_5 . The matrix γ_2^0 exchanges only the second index, it thus exchanges ψ_0 with χ_0 . Finally, γ_1^0 is evaluated as $\beta\gamma_2^0$. We also introduce abbreviations for the sum and difference of the two Pauli matrices:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2, \quad \Delta\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2. \quad (3.5)$$

With all these abbreviations, the decomposition of (3.1) into two separate equations for ψ_0 and χ_0 remains reasonably compact:

$$(E - \gamma_5\mathbf{p}\Delta\boldsymbol{\sigma})\psi_0 = (m_2 + \beta m_1)\chi_0, \quad (E - \gamma_5\mathbf{p}\boldsymbol{\sigma})\chi_0 = (m_2 + \beta m_1)\psi_0. \quad (3.6)$$

Elimination of χ_0 gives the equation for ψ_0 :

$$(E - \gamma_5\mathbf{p}\boldsymbol{\sigma})(m_2 + \beta m_1)^{-1}(E - \gamma_5\mathbf{p}\Delta\boldsymbol{\sigma})\psi_0 = (m_2 + \beta m_1)\psi_0. \quad (3.7)$$

Its second-order derivatives vanish, $(\mathbf{p}\boldsymbol{\sigma})(\mathbf{p}\Delta\boldsymbol{\sigma}) = (\mathbf{p}\boldsymbol{\sigma}_1)^2 - (\mathbf{p}\boldsymbol{\sigma}_2)^2 = 0$. The remainder gives

$$\begin{aligned} &[(m_2 + \beta m_1)^{-1}(E^2 - E\gamma_5\mathbf{p}\Delta\boldsymbol{\sigma}) \\ &- E\gamma_5\mathbf{p}\boldsymbol{\sigma}(m_2 + \beta m_1)^{-1} - (m_2 + \beta m_1)]\psi_0 = 0. \end{aligned} \quad (3.8)$$

The 16 coupled first-order differential equations are now reduced to 8 coupled first-order differential equations.

The second trick was originally introduced to remove the denominators $(m_2 + \beta m_1)$ from (3.8). Multiplication by $(m_2 + \beta m_1)$ from the left alone does not remove the denominator of the term in the middle: The anticommutator

$$\beta\gamma_5 = -\gamma_5\beta \quad (3.9)$$

transforms this term into

$$-E\gamma_5 \mathbf{p}\boldsymbol{\sigma} \frac{m_2 - \beta m_1}{m_2 + \beta m_1}. \quad (3.10)$$

The quotient is removed by a somewhat intricate transformation of ψ_0 ,

$$\psi_0 = c\psi, \quad c = (m_2^2 - m_1^2)^{-1/2} [m_2 + \beta m_1 \frac{1}{2}(1 + \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2)]. \quad (3.11)$$

In addition, (3.8) is multiplied by $1/c$ from the left. One has $c^{-1}\gamma_5 = \gamma_5 c$ and

$$c\Delta\boldsymbol{\sigma}c = \Delta\boldsymbol{\sigma}, \quad c\boldsymbol{\sigma}c = \boldsymbol{\sigma} \frac{m_2 + \beta m_1}{m_2 - \beta m_1}, \quad (3.12)$$

which cancels the quotient of (3.10). To understand c , one must know that $\frac{1}{2}(1 + \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2)$ has the eigenvalue $+1$ for triplet spin states and -1 for the singlet spin state. All three matrices $\boldsymbol{\sigma}$ vanish when applied to the singlet state, while the $\Delta\boldsymbol{\sigma}$ transform the singlet state into triplet states.

After the transformation (3.11), $\mathbf{p}\boldsymbol{\sigma}$ and $\mathbf{p}\Delta\boldsymbol{\sigma}$ appear in the combination $\mathbf{p}\boldsymbol{\sigma} + \mathbf{p}\Delta\boldsymbol{\sigma} = 2\mathbf{p}\boldsymbol{\sigma}_1$. Noting moreover

$$(m_2 + \beta m_1)^2 = m_1^2 + m_2^2 + 2m_1 m_2 \beta, \quad (3.13)$$

the final version of (3.8) is

$$(E^2 - 2\gamma_5 E \mathbf{p}\boldsymbol{\sigma}_1 - m_1^2 - m_2^2 - 2m_1 m_2 \beta)\psi = 0. \quad (3.14)$$

By a minor miracle, $\mathbf{p}\boldsymbol{\sigma}_2$ has been removed by c that was constructed to remove the brackets of (3.10).

The remainder has the form of an effective Dirac equation. To see this more clearly, (3.14) may be divided by $2E$. With the definitions of a reduced mass μ and a reduced energy ε ,

$$\mu = \frac{m_1 m_2}{E}, \quad \varepsilon = \frac{E^2 - m_1^2 - m_2^2}{2E}, \quad (3.15)$$

one gets

$$(\varepsilon - \gamma_5 \mathbf{p}\boldsymbol{\sigma}_1 - \mu\beta)\psi = 0. \quad (3.16)$$

This looks familiar but complicates the E -dependence. It is better to divide the equation by $2m_1m_2$, which gives the Dirac equation in units of μ :

$$\left(\frac{\varepsilon}{\mu} - \frac{\gamma_5 \mathbf{p} \boldsymbol{\sigma}_1}{\mu} - \beta\right) \psi = 0, \quad \frac{\varepsilon}{\mu} = \frac{E^2 - m_1^2 - m_2^2}{2m_1m_2}. \quad (3.17)$$

The third trick is a simple rescaling of $\mathbf{r}$,

$$\boldsymbol{\rho} = \mu \mathbf{r}, \quad \mathbf{p}_\rho = \frac{\mathbf{p}}{\mu}. \quad (3.18)$$

Then E is eliminated in favour of E^2 everywhere. The equation is now invariant under the transformation $E \rightarrow -E$, and we shall see below that the invariance is not violated by the addition of the QED interaction operator. The last new trick concerns the reduction of the 16×16 T-matrix of QED to an 8×8 form M . In terms of the free Dirac spinors u_1, u_2 of the two incoming fermions and u'_1, u'_2 for the outgoing ones, one has

$$T_{if} = \bar{u}'_1 \bar{u}'_2 \hat{T} u_1 u_2. \quad (3.19)$$

The first Born approximation is the familiar $\hat{T} = \alpha \gamma_1^\mu \gamma_{2,\mu} / q^2$. The eight-component free-particle spinors of ψ_0 and χ_0 analogous to u_i are called v and w , respectively. By using the free-particle equations (3.6), T_{if} may be reduced to one of the two asymmetric forms

$$T_{if} = w'^+ M v = v'^+ M_\chi w. \quad (3.20)$$

The first form gives the interaction in the ψ -equation, the other one the interaction in an equivalent equation for χ . The interaction potential itself is essentially the Fourier transform of M . The complete ψ -equation with the potential from the first Born approximation $T_{if}^{(1)}$ is

$$\left[\frac{\varepsilon}{\mu} - V(\rho) - \beta - \gamma_5 \boldsymbol{\sigma}_1 \mathbf{p}_\rho + i \gamma_5 (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \frac{V(\rho) \mathbf{p}_\rho \mu}{E} \right] \psi = 0. \quad (3.21)$$

The last operator is a recoil-corrected hyperfine operator; in the original variable $\mathbf{r}$, one has $V(\rho) \mathbf{p}_\rho \mu / E = V(r) \mathbf{p} / \mu E$. With $\mu E = m_1 m_2$ according to (3.15), one recovers the mass dependence of the nonrelativistic limit H_F (2.8) (the static atomic hyperfine operator contains the nuclear Bohr magneton, which would be $e/2m_2$ in our notation. H_F is already recoil corrected).

4. Discussion and outlook

By the rescaling trick (3.18), the new equation (3.21) has become invariant under the replacement $E \rightarrow -E$. This symmetry is shared by all relativistic two-body equations including Dirac–Breit; it is merely violated by later approximations. Its physical basis is charge conjugation invariance. It would even survive the inclusion of C and P violating atomic interactions; the physical basis would then be CP or even CPT . If the eigenvalue $+E$ refers to an atom such as muonium ($e^-\mu^+$), the eigenvalue $-E$ refers to the corresponding antiatom (antimuonium $e^+\mu^-$). In particle theory, a field has the operator structure $ae^{-iEt} + a_-^\dagger e^{iEt}$, where a destroys a particle of energy E , and $a_-^\dagger$ creates an antiparticle of the same energy. In principle then, one may introduce fields for whole atoms. This is unlikely to be of any use, but in the analogous case of QCD, it should be possible to combine the equations for the inner structure of mesons with a meson field formalism.

From the point of view of Feynman rules, antiparticles correspond to particles moving backwards in time, with all additive quantum numbers reversed. One could add that antiatoms also have negative distances between their constituents: To find the spectrum of a differential equation, one must specify the range of its variables. For a spherically symmetric $V(r)$, one has $0 \leq r < \infty$. The relativistic formulation has $0 \leq \rho < \infty$. For $E < 0$, this corresponds to the interval $-\infty < r \leq 0$. With $r = (x^2 + y^2 + z^2)^{1/2}$ this is equivalent to a continuation on the other branch of the square root.

A point of more practical importance is the absence of cancellations. When the interaction is introduced at the 16-component level, E and the dominant Coulomb potential $V(r)$ occur linearly, in the combination $E - V(r)$. A reduction to 8 components produces the square of this operator, $E^2 - 2EV(r) + V^2(r)$. The second Born approximation provides additional operators, the dominant one being $-V(r)^2$. The net effect is thus $E^2 - 2EV(r) = E^2 - 2m_1m_2V(\rho)$. This agrees with the first-order interaction introduced at the 8-component level. Consequently, this interaction is not only simpler than the 16-component version, but it also includes the dominant operator of the second-order potential of that version.

The cancellations occur also in NRQED.

How much of this is applicable to QCD? At the level of NRQCD, the situation is good, as mentioned in the introduction. Only the nonrelativistic reductions of Breit and hyperfine operators are needed here. But is there any chance of describing the lighter pseudoscalar mesons and their vector partners by a single differential equation? In the extreme case of the π and ρ mesons, the hyperfine operator would have to explain $m_\rho^2/m_\pi^2 \approx 30$. The

factor $\mu/E = m_1 m_2/E^2$ in (3.21) of the hyperfine operator does increase for decreasing E^2 , but its singularity for $r \rightarrow 0$ excludes a nonperturbative treatment. In NRQED, a three-photon exchange correction to the hyperfine operator has been found by Pachucki [19], which shows a possibility for a nonperturbative treatment: It may be included by substituting in the hyperfine operator

$$\frac{\mu}{E} \rightarrow m_1 m_2 [E^2 - 2m_1 m_2 \alpha^2 \ln(\gamma\rho)]^{-1}, \quad (4.1)$$

with $\gamma = e^C$ and $C = 0.5772 = \text{Euler's constant}$.

A still larger problem is posed by the formal use of S-matrix theory. In QCD, this theory requires the existence of free in- and outgoing quark states. The requirement is also present in the Bethe–Salpeter equation, where it leads to the free-quark-propagators (2.11). Is it possible to avoid the concept of free quarks altogether? Again, the answer could be yes. The derivation of the potential $V(r)$ ($r = |\mathbf{r}_1 - \mathbf{r}_2|$) in the old Dirac–Breit Hamiltonian (2.5) did not require the existence of free leptons. The V arises from the operator form of A^0 in the Coulomb gauge,

$$A^0(\mathbf{r}_2) = -e \int d^3 r' \frac{\psi_D^\dagger(\mathbf{r}') \psi_D(\mathbf{r}')}{|\mathbf{r}_2 - \mathbf{r}'|} \quad (4.2)$$

and from the equal-time commutation relation of the field operator $\psi_D^\dagger(\mathbf{r}')$ with $\psi_D(\mathbf{r}_1)$ [18]. The Breit operator proper (2.7) is less general; it assumes the existence of a free photon field, which would have to be replaced by a free gluon field in QCD. In any case, this shows that the standard S-matrix theory for free fermions is avoidable.

From this point of view, it is relevant to know that the 8-component equation has an approximate derivation from the 16-component Dirac–Breit equation [18]. The derivation requires a transformation of the radial variable, which we now call r_{12} , contrary to (1.3):

$$r = \frac{r_{12}^2}{r_{12} + \alpha/2E}. \quad (4.3)$$

For $V = -\alpha/r_{12}$, this removes the operator V^2 from the 8-component reduction of the Dirac–Breit equation:

$$(E + V(r_{12}))^2 = E^2 + 2EV(r). \quad (4.4)$$

The square of the Breit operator is omitted by hand, which dispenses with the otherwise necessary projectors. Finally, ∇_{12} combines with the terms

linear in the Breit operators into the operators of (3.21). A more direct derivation of an eight-component Dirac–Breit equation does not yet exist. In such a derivation, the transformation (4.3) would be absent. The QCD extension to a potential which is essentially non-Coulombic could then also be possible.

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