DIFFERENTIAL EQUATIONS FOR MULTI-POINT FEYNMAN FUNCTIONS CALCULATION*

Roberto Bonciani

Dipartimento di Fisica dell'Università di Bologna
and INFN Sezione di Bologna
Via Irnerio, 46 I-40126 Bologna, Italy

(Received November 2, 1999)

A non traditional method to calculate multi-point Feynman functions is presented. In the approach, D-dimensional loop integrals defining a Feynman amplitude are not directly performed, but a system of linear differential equations for the Feynman amplitudes themselves is found. The solution of the differential equations provides then with the actual value of the amplitudes.

PACS numbers: 11.10.–z, 11.10.Kk, 11.15.Bt

1. Introduction

The importance of more and more accurate tests of Quantum Field Theory push physicists to the evaluation of quantum corrections at higher and higher level in perturbation theory, *i.e.* to the calculation of multi-loop Feynman diagrams.

A new method, proposed in [1] and developed in [2], consists in establishing a system of linear differential equations for the required Feynman amplitudes. This system of equations can always be solved numerically without a big effort, even if the analytic solution is not available.

In this paper the method is recalled and its application to the 1-loop box graph, relevant for the Bhabha scattering, is considered.

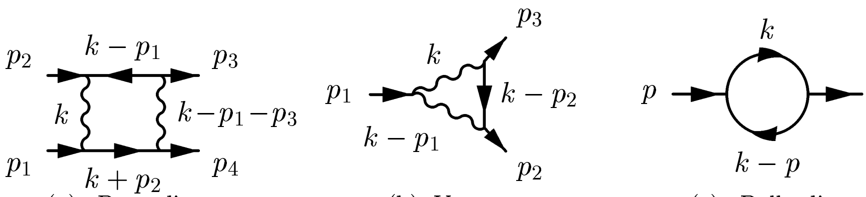
We will work in D-dimensional Euclidean space (the corresponding Minkowski integrals are recovered by the Wick rotation) in a scalar theory.

* Presented at the XXIII International School of Theoretical Physics "Recent Developments in Theory of Fundamental Interactions", Ustroń, Poland, September 15–22, 1999.

2. System of differential equations

Let us take the 1-loop scalar box diagram (see Fig.1(a)) with external momenta p_1 and p_2 (initial particles), p_3 and p_4 (final particles) and with two massless and two massive propagators (with mass squared $m^2 = a$). The corresponding amplitude is the following:

$$F = \int \frac{d^D k}{(2\pi)^{(D-2)}} \left\{ \frac{1}{k^2 \left[(k-p_1)^2 + a\right] \left[(k+p_2)^2 + a\right] (k-p_1+p_3)^2} \right\} . \quad (1)$$



(a) Box diagram. (b) Vertex diagram. (c) Ball diagram.

Fig. 1. 1-loop diagrams

We know, from kinematical considerations, that F is a function of six invariants M_i with $i = 1, ..., 6$. We can choose them to be the Mandelstam variables $M_1 = s = -(p_1 + p_2)^2$, $M_2 = t = -(p_1 - p_3)^2$ and the external square momenta $M_{i+2} = -p_i^2$, which we will put later on the mass-shell.

Let us construct the following object:

$$O_{ij} = p_i^{\mu} \frac{\partial F}{\partial p_j^{\mu}} . \quad (2)$$

By derivation of the function F with respect to M_i, we have:

$$O_{ij} = p_i^{\mu} \sum_{\xi} \left[\frac{\partial F}{\partial M_\xi} \frac{\partial M_\xi}{\partial p_j^{\mu}} \right] = \sum_{\xi} a_\xi (p_i \cdot p_j) \frac{\partial F}{\partial M_\xi} = \sum_{\xi} a_{\xi,ij}(M_l) \frac{\partial F}{\partial M_\xi} , \quad (3)$$

where $i, j = 1, ..., 3$ (from the conservation of 4-momentum, p_4 can be expressed in terms of p_1, p_2 and p_3) and where the functions $a_{\xi,ij}(M_l)$ are linear combinations of invariants.

On the other hand, if we perform the direct derivation of the integrand in Eq. (1), we find a combination of integrals: F itself, integrals with one denominator squared and integrals in which we have one denominator less. In this way a linear system is derived that we can solve with respect to the derivatives $\frac{\partial F}{\partial M_\xi}$ to obtain a system of partial differential equations of the first order:

$$\frac{\partial F}{\partial M_\xi} = a_\sigma(M_l)\, F_{\xi,\sigma(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} + b_\xi(M_l)\, F + \Omega'_\xi\,, \tag{4}$$

where $\sigma(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ is a permutation of $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, with only one α_i set equal to 2 and the others equal to 1 and where Ω'_ξ is a term containing all the functions with a smaller number of denominators.

3. Integration by parts identities

The system of Eq. (4) is not useful because it still involves the functions $F_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$. We want to find a system expressed only in terms of F and some other known terms.

We know that the following identities, called Integration by Parts Identities [3], hold:

$$\int \frac{d^D k}{(2\pi)^{\frac{D}{2}}} \frac{\partial}{\partial k_\mu} \left\{ \frac{v_\mu}{k^{2\alpha_1}[(k-p_1)^2+a]^{\alpha_2}[(k+p_2)^2+a]^{\alpha_3}(k-p_1+p_3)^{2\alpha_4}} \right\} = 0, \tag{5}$$

for every value of α_i and with $v_\mu = k_\mu, p_{1\,\mu}, p_{2\,\mu}, p_{3\,\mu}$.

Eqs. (5) form a linear system with $F_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ as unknown functions and using it we can reexpress $F_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$, appearing in Eq. (4), in terms of F and functions with a smaller number of denominators. This gives us the possibility to construct for F a system od partial differential equations of the first order, solved with respect to the derivatives.

If we put the external momenta on the mass-shell, we remain with two differential equations, with respect to $M_1 = s$ and $M_2 = t$:

$$\begin{cases} \dfrac{\partial F}{\partial s} = g(s,t,a)\, F + \Omega_1\,, \\ \dfrac{\partial F}{\partial t} = h(s,t,a)\, F + \Omega_2\,. \end{cases} \tag{6}$$

Let us comment the structure of the system, which is a general feature of the method. The equations involve the function under consideration, F, and a function Ω, which is considered known, containing multi-point Feynman functions with a smaller number of denominators. One can proceed in the following way: At first one finds the two-point two-denominators bubble function. It appears as inhomogeneous term in the system of differential equations for three-point three-denominators vertex function. One solves this system and the solution appears in the system of differental equations for the four-point four-denominators box function, and so on.

In special cases (like, *e.g.*, the box in figure 1(a)) some cancelations hold and Ω does not contain all the sub-diagrams of F, but only some of them. It is also possible to find some problems in which F is determined entirely in terms of Ω, without solving any differential equation, but solving only the Integration by Parts Identities.

4. 1-loop case

Let us come back to the problem of the determination of the function F (Eq. (1)).

The system in Eq. (6) can be written explicitely as follows:

$$
\begin{aligned}
\frac{\partial F}{\partial s} = & -\frac{1}{2}\left[\frac{1}{s} - \frac{(D-5)}{(s-4a)} + \frac{(D-4)}{(s+t-4a)}\right] F - (D-4)\left[\frac{1}{4as} - \frac{(t-4a)}{4at(s-4a)}\right. \\
& \left. - \frac{1}{t(s+t-4a)}\right] V(t) + \frac{2(D-3)}{t}\left[\frac{1}{(s-4a)^2} - \frac{1}{t(s-4a)}\right. \\
& \left. + \frac{1}{t(s+t-4a)}\right] B(a,a,s) + \frac{(D-3)}{2at}\left[\frac{1}{s} - \frac{1}{(s-4a)}\right] B(0,0,t) \\
& + \frac{(D-2)}{at}\left[\frac{1}{(s-4a)^2} - \frac{1}{t(s-4a)} + \frac{1}{t(s+t-4a)}\right] T \, , \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F}{\partial t} = & \frac{1}{2}\left[\frac{(D-6)}{t} - \frac{(D-4)}{(s+t-4a)}\right] F + \frac{(D-4)}{(s-4a)}\left[\frac{1}{t} - \frac{1}{(s+t-4a)}\right] V(t) \\
& - \frac{2(D-3)}{(s-4a)^2}\left[\frac{1}{t} - \frac{1}{(s+t-4a)}\right] B(a,a,s) \\
& - \frac{(D-2)}{a(s-4a)^2}\left[\frac{1}{t} - \frac{1}{(s+t-4a)}\right] T, \qquad (8)
\end{aligned}
$$

where $V(t)$ is the vertex function of figure 1(b), as a function of t, $B(a,a,s)$ is the bubble function of figure 1(c), with equal internal masses $m^2 = a$ and T is the tadpole. Let us note that in this case we can find the solution of the problem using only one of the equations of the system (7), (8). In fact, we are able to give the boundary conditions $F(s=0,t)$ and $F(s,t=0)$, respectively. Let us multiply, for example, Eq. (7) by s. As F is regular with all its derivatives in $s=0$, we can obtain:

$$
F(s=0,t) = \frac{(D-4)}{2a} V(t) - \frac{(D-3)}{at} B(0,0,t) \, . \qquad (9)
$$

Now we need the espressions for V and B.

For the vertex function V, we can construct the following differential equation in $s=-p_1^2$:

$$\frac{\partial V}{\partial s}=\frac{1}{2}\left[\frac{(D-5)}{s}-\frac{(D-3)}{(s-4a)}\right]V+\frac{(D-3)}{s(s-4a)}B(0,0,s)+\frac{(D-2)}{2as(s-4a)}T, \quad (10)$$

where $B(0,0,s)$ is the 1 loop bubble function with zero internal masses, [1]. as a function of s. The initial condition, derived in the same way as Eq. (9), reads:

$$V(s=0)=\frac{1}{4\,(4\pi)^{\frac{(D-4)}{2}}}\Gamma\left(3-\frac{D}{2}\right)\frac{a^{\frac{(D-6)}{2}}}{(D-4)(D-5)}\,. \quad (11)$$

Let us note that Eq. (10) contains two scales, s and a, and then we can reexpress it in terms of some dimensionless ratio. As a result we can write the solution in terms of hypergeometric functions [4]:

$$\begin{aligned}V(s)=\; & -\frac{(4\pi)^{-\frac{(D-4)}{2}}\Gamma\left(3-\frac{D}{2}\right)}{(D-4)(s-4a)}\left[(D-3)\frac{\Gamma^2\left(\frac{(D-4)}{2}+1\right)}{\Gamma(D-2)}(-s)^{\frac{(D-4)}{2}}\right.\\ & \times {}_2F_1\left(\frac{(5-D)}{2},1;\frac{3}{2};\frac{s}{s-4a}\right)+\frac{a^{\frac{(D-4)}{2}}}{(D-5)}\,{}_2F_1\\ & \left.\times\left(\frac{(5-D)}{2},1;\frac{(7-D)}{2};\frac{s}{s-4a}\right)\right]. \end{aligned} \quad (12)$$

For the bubble function $B(a,a,s)$, we can construct the following differential equation in $s=-p^2$:

$$\frac{\partial B}{\partial s}=-\frac{1}{2}\left(\frac{1}{s}-\frac{(D-3)}{s-4a}\right)B-\frac{(D-2)}{4a}\left(\frac{1}{s}-\frac{(D-3)}{s-4a}\right)T\,, \quad (13)$$

with the initial condition $B(s=0)=-\frac{(D-2)}{2a}T$, which gives the following solution:

$$B(a,a,s)=\frac{2\Gamma\left(3-\frac{D}{2}\right)}{(4\pi)^{\frac{(D-4)}{2}}}\frac{a^{\frac{(D-2)}{2}}}{(D-4)}\,{}_2F_1\left(\frac{(D-1)}{2},1;\frac{3}{2};\frac{s}{s-4a}\right)\,. \quad (14)$$

Knowing the inhomogeneous part of the equations of the system (7,8) we can solve it numerically or analitically in some limits; for example for $s,t\gg a$.

Special thanks to prof. Ettore Remiddi, with whom the presented calculation was done, and to the Institut für Theoretische Teilkenphysik, Universität Karlsruhe, for the hospitality.

REFERENCES

[1] E. Remiddi, *Nuovo Cimento* **110A**, 1435 (1997).
[2] M. Caffo, H. Czyż, S. Laporta, E. Remiddi, *Acta Phys. Pol.* **B29**, 2627 (1998); M. Caffo, H. Czyż, S. Laporta, E. Remiddi, *Nuovo Cimento* **A111**, 365 (1998).
[3] K.G. Chetyrkin, F.V. Tkachov, *Nucl. Phys.* **B192**, 159 (1981); F.V. Tkachov, *Phys. Lett.* **B100**, 65 (1981).
[4] See for example: J. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*, Ed. Alan Jeffrey, Academic Press; M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, Inc., New York.