EXOTIC SMOOTHNESS ON SPACETIME NEW DEVELOPMENTS*

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I discuss recent development in investigation of physical consequences of exotic differential structures on manifolds. I show, following T. Asselmayer, that corrections to the curvature after the change of differential structure produce a source like term in the Einstein equations. Then I give examples of topologically trivial spaces on which exotic differential structures act as a source of gravitational force even in the absence of matter.

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1. Introduction

The choice of mathematical model for spacetime has important physical significance. Riemann has already suggested that the geometry of space may be more than just a mathematical tool defining a stage for physical phenomena, and may in fact have profound physical meaning in its own right [1]. With the advent of general relativity physicists began to think of the spacetime as a differential manifold. Since then various assumptions about the spacetime topology and geometry have been put forward [2]. But until recently, the choice of differential structure of the spacetime manifold has been assumed to be trivial because most topological spaces used for modelling spacetime have natural differential structures and these structures where thought to be unique. Therefore the counterintuitive discovery of exotic four dimensional Euclidean spaces following from the work of Freedman [3] and Donaldson [4] raised various discussions about the possible physical consequences of this discovery. Exotic $\boldsymbol{R}^4_\theta$'s are smooth (C^∞)

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four-manifolds which are homeomorphic to the Euclidean four-space $\boldsymbol{R}^4$ but not diffeomorphic to it. Exotic $\boldsymbol{R}^4_\theta$'s are unique to dimension four, see [5–11] for details. Since then mathematician have shown that exotic (nonunique) smooth structures are abundant in dimension four. For example it is sufficient to remove one point from a given four-manifold to obtain a manifold with exotic differential structures [11] and every manifold of the form $M \times \boldsymbol{R}$, M being compact 3-manifold, has infinitely many inequivalent differential structures. Such manifolds play important rôle in theoretical physics and astrophysics and it became necessary to investigate the physical meaning of exotic smoothness. Unfortunately, this is not an easy task: we only know few complicated coordinate descriptions [12] and most mathematicians believe that there is no finite atlas on an exotic $\boldsymbol{R}^4$ and other exotic four-manifolds. To the best of my knowledge, only few possible physical manifestations have been discussed in the literature [2, 6, 7, 13, 14]. In this paper I would like to discuss some peculiarities that may happen while studying the theory of gravity on some exotic $\boldsymbol{R}^4$'s. First of all I will discuss the Asselmeyer's formula describing the corrections to the curvature after the change of differential structure produce a source like term in the Einstein equations. Then I will show that on some topologically trivial spaces there exist only "complicated" solutions of the Einstein equations. By this I mean that there may be no stationary cosmological model solutions and/or that empty space can gravitate. Such solutions are counterintuitive but I am aware of no physical principle that would require rejection of such spacetimes.

2. Corrections to Einstein equations induced by a change of differential structure

Exotic $\boldsymbol{R}^4$'s are defined as four-manifolds that are homeomorphic to the fourdimensional Euclidean space $\boldsymbol{R}^4$ but not diffeomorphic to it. There are infinitely many of such manifolds (at least a two parameter family of them) [5]. Note that exotic differential structures do not change the definition of the derivative. The essential difference is that the algebras of real differentiable functions are different on nondiffeomorphic manifolds. In the case of exotic $\boldsymbol{R}^4$'s this means that there are some continuous functions $\boldsymbol{R}^4 \mapsto \boldsymbol{R}$ that are smooth on one exotic $\boldsymbol{R}^4$ and only continuous on another and vice versa [9]. Brans conjectured that exotic smoothness can be a source of nonstandard solutions of Einstein equations [6–8]. But it is not easy to guess which physical observable will be modified by a change of differential structure. Asselmeyer gave a partial answer to this problem [13]. He considered two manifolds M and M' with different differential structures and found the change in covariant derrivative induced by exoticness. Then he was able to calculate the corresponding changes in the curvature tensor and Einstein equations. To this end he considered a 1-1 map $\alpha : M \to M'$ that is not a diffeomorphism. It must not be smooth at some point $p_0 \in M$ because M and M' are not equivalent. If one considers the splitting of the map $d\alpha : TM \to TM'$ in some neighborhood $U(p_0)$ of the point p_0:

$$d\alpha \mid_{U(p_0)} = (b_1, b_2) \tag{1}$$

then the change in the covariant derivative is given by [13]

$$\nabla' = \nabla + \left(b_1^{-1} db_1\right) \oplus \left(b_2^{-1} db_2\right) \tag{2}$$

The additional term disappears if the manifolds in questions have the same differential structure (are equivalent). The physical content of this formula can be found if one recall the formula expressing the curvature tensor in terms of the covariant derivative [15]:

$$R\,(X,Y)\,Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z, \tag{3}$$

where X, Y, Z are vector fields. Then the Einstein vacuum field equations take the form:

$$Ric\,(X,Y) - \tfrac{1}{2} g\,(X,Y) = 0, \tag{4}$$

where Ric denotes the Ricci tensor. So the exoticness correction to the covariant derivative leads to [13]:

$$Ric_{ik} - \tfrac{1}{2} g_{ik} R = 2\pi w \delta_i^j \left(\delta \left(b_i^j\right)_{jk} + \tfrac{1}{2} g_{ik} \left(g^{lm} \delta \left(b_m^j\right)_{jl} \right) \right), \tag{5}$$

where $\left(b_i^j\right)_{kl}$ are the coordinate representations of the functions b_i and w some constant describing the winding number of the b function, see [13] for details. This means that

$$Ric\,(X,Y) - \tfrac{1}{2} g\,(X,Y)\,R \neq 0 \tag{6}$$

in M'. Asselmeyer suggests a string-like interpretation of this source term. I would like to add the following. Suppose we have discovered some strange [19–24] astrophysical source of gravitation that do not fit to any acceptable solution of the Einstein equations. This may simply mean that we are using wrong differential structure on the spacetime manifold and this strange source is sort of an artefact of this mistake. If we change the differential structure then everything would be OK *e.g.* we would get an empty space solution.

3. General relativity on exotic $\boldsymbol{R}^4$'s with few symmetries

To proceed I will recall several definitions. A diffeomorphism $\phi : M \mapsto M$, where M is a (pseudo-)Riemannian manifold with (pseudo-)metric tensor g, is called an isometry if and only if it preserve g, $\phi^* g = g$ [15]. Such mappings form a group called the isometry group. We say that a smooth manifold has few symmetries provided that for every choice of differentiable metric tensor, the isometry group is finite. Recently, Taylor managed to construct examples of exotic $\boldsymbol{R}^4$'s with few symmetries [16]. Among these there are examples with nontrivial isometry groups. Taylor's result, although concerning Riemannian structures, has profound consequences for the analysis of the possible rôle of differential structures in physics where Lorentz manifolds are used. To show this let me define a (non-)proper actions of a group on manifolds as follows. Let G be a locally compact topological group acting on a metric space X. We say that G acts properly on X if and only if for all compact subsets $Y \subset X$, the set $\{g \in G : gY \cap Y \neq \emptyset\}$ is also compact. Restating this we say that G acts nonproperly on X if and only if there exist sequences $x_n \to x$ in X and $g_n \to \infty$ in G, such that $g_n x_n$ converges in X. Here $g_n \to \infty$ means that the sequence g_n has no convergent subsequence in the compact open topology on the set of all isometries [15]. My discussion would be based on the theorems proved by Kowalsky [17]. First of all let me quote [17]:

Theorem 1 *Let G be Lie transformation group of a differentiable manifold X. If G acts properly on X, then G preserves a Riemannian metric on X. The converse is true if G is closed in Diff(X).*

If we combine this theorem with the Taylor's construction of exotic $\boldsymbol{R}^4_\theta$ with few symmetries we immediately get:

Theorem 2 *Let G be a Lie transformation group acting properly on an exotic $\boldsymbol{R}^4$ with few symmetries and preserving a time-orientable Lorentz metric. Then G is finite.*

Further, due to Kowalsky, we also have [17]:

Theorem 3 *Let G be a connected noncompact simple Lie group with finite center. Assume that G is not locally isomorphic to SO(n, 1) or SO(n,2). If G acts nontrivially on a manifold X preserving a Lorentz metric, then G actually acts properly on X.*

and

Theorem 4 *If G acts nonproperly and nontrivially on X, then G must be locally isomorphic to SO(n,1) or SO(n,2) for some n.*

Now, suppose we are given an exotic $\boldsymbol{R}^4_\theta$ with few symmetries. We can try to solve the Einstein equations on this $\boldsymbol{R}^4_\theta$. Suppose we have found such a solution. Whatever the boundary conditions be we would face one of the two following situations [25].

- The isometry group G of the solution acts properly on $\boldsymbol{R}^4_\theta$. Then according to Theorem 3 G is finite. There is no nontrivial Killing vector field and the solution cannot be stationary [19]. The gravitation is quite "complicated" and even empty spaces do evolve.

- The isometry group G of the solution acts nonproperly on $\boldsymbol{R}^4_\theta$. Then G is locally isomorphic to SO(n,1) or SO(n,2) (Theorem 4). But the non-proper action of G on $\boldsymbol{R}^4_\theta$ means that there are points infinitely close together in $\boldsymbol{R}^4_\theta$ ($x_n \to x$) such that arbitrary large different isometries ($g_n \to \infty$) in G maps them into infinitely close points in $\boldsymbol{R}^4_\theta$ ($g_n x_n \to y \in \boldsymbol{R}^4_\theta$). There must exists quite strong gravity centers to force such convergence (even in empty spacetimes).

We see that in both cases Einstein gravity is quite nontrivial even in the absence of matter. Let us recall that if a spacetime has a Killing vector field ζ^a, then every covering manifold admits appropriate Killing vector field ζ'^a such that it is projected onto ζ^a by the differential of the covering map. This means that discussed above properties are "projected" on any space that has exotic $\boldsymbol{R}^4$ with few symmetries as a covering manifold *e.g.* quotient manifolds obtained by a smooth action of some finite group. Note that in that way a weaker form of the Brans conjecture [7] can be proven: there are examples of four-manifolds (spacetimes) on which differential structures act as sources of gravitational forces just as ordinary matter does.

4. Conclusions

The existence of topologically trivial spacetimes that admit only "non-trivial" solutions to the Einstein equations is very surprising. Such phenomenon might be also possible for other four-manifolds admitting exotic differential structures enumerated in the Introduction. The first reaction is to reject them as being unphysical mathematical curiosities. But this conclusion might be erroneous [6–8, 13]. If Nature has not used exotic smoothness we physicists should find out why only one of the existing differential structures has been chosen. Does it mean that the differential calculus, although very powerful, is not necessary (or sufficient) for the description of the laws of physics? It might not be easy to find any answer to these questions.

Let me conclude by saying that if exotic smoothness has anything to do with the physical world it may be a source/ explanation of various astrophysical and cosmological phenomena. Dark matter and vacuum energy substitutes and attracting centers are the most obvious among them [20–22]. "Exoticness" of the spacetime might be responsible for the recently discovered anomalies in the large redshift supernovae properties. The process of "elimination" of exotic differential structures might also result in the emergence time [23, 24] or spacetime signature.

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