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COLOR DIELECTRIC MODELS FROM THE LATTICE SU($\boldsymbol{N}$)$_c$ GAUGE THEORY *

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The idea of coarse-grained gluon field is discussed. We recall motivation for introducing such a field. Next, we outline the approach to small momenta limit of lattice coarse-grained gluon field presented in our paper hep-ph/9803392. This limit points to color dielectric type models with a number of scalar and tensor fields instead of single scalar dielectric field.

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1. Introduction

The most prominent features of quantum chromodynamics such as asymptotic freedom or quark confinement follow from the presence of gluon fields with their intricate self-interactions. Therefore, the gluon sector of QCD may be regarded as the most interesting one. Unfortunately, at low energies and large distances the QCD dynamics becomes highly nontrivial. Practical method to study a complicated dynamics consists in constructing and solving appropriate effective models. Such approach has turned out to be extremely fruitful in condensed matter physics, *e.g.*, Ginzburg–Landau effective models play crucial role in physics of superconductors, superfluid Helium, or liquid crystals. Also examples from particle physics, like Skyrme or Nambu–Jona-Lasinio models are well known. The color dielectric models, also frequently called Friedberg–Lee models, can be regarded as effective

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models for QCD. Their characteristic and attractive feature is that they take into account the gluon fields, hence with these models one could study the gluon component of hadrons.

The color dielectric models have rather long history. In seventies models were proposed in which single real scalar field was coupled in a non-minimal, dilaton-like manner to a gauge field, [1–3]. The models implied that the electric displacement field $\vec{D}$ between two opposite charges was squeezed into a narrow flux-tube, and consequently the charges were confined. These models were a little bit artificial, nevertheless they evolved and eventually provided quite accurate description of static properties of hadrons [4–6]. Recently they have been applied also to dynamical phenomena: formation and breaking of flux-tubes in high energy collisions [7, 8].

The successes in describing the physics of hadrons motivated searches for connections of the color dielectric models with QCD. It seems that such connections indeed do exist — the models emerge from QCD when we focus on dynamics of so called coarse-grained gluon field, which is defined as an average of the original "microscopic" gauge field over a finite volume in space-time. This is analogous to averaging of microscopic electric or magnetic fields when deriving equations of macroscopic electrodynamics of dielectric or magnetic media. The difference is that in the non-Abelian case the nonlinearity of microscopic Yang–Mills equations, as well as the fact that non-Abelian field strength tensor is not gauge invariant, exclude straightforward averaging — such an average has to be taken in a special way to guarantee that it is gauge covariant. We shall present a definition of the coarse-grained field shortly, but first we would like to explain on theoretical grounds why we think that such a field is interesting.

There exists an evidence that the microscopic gauge field considered in the quantum theory based on Yang–Mills action is subjected to large infrared quantum fluctuations. The quantum tunelling between vacua with different winding numbers, described by instantons [9], is the example of such fluctuations. The notions of condensates of monopoles [10] or vortices [11] can be regarded as a way to describe and to understand such fluctuations. The fluctuations are the main *raison d'être* for the stochastic vacuum model [12, 13], which exploits specific assumptions about the probabilistic characteristics of the fluctuations in Euclidean non-Abelian gauge theory. Due to the fluctuations, the original microscopic gauge field is not the best variable to discuss the small momenta physics. One may find an analogy when considering position operators for a Dirac particle. In the standard representation that dynamical variable exhibits *Zitterbewegung*, which can be regarded as certain specific fluctuations. On the other hand, there exists another position operator, defined in Foldy–Wouthuysen representation [14] for which such fluctuations are absent — this dynamical variable has smooth behaviour in accordance with expectations based on classical limit. It is well-known that transformation to the Foldy–Wouthuysen representation can be regarded as averaging over a small space-time cell [15]. The coarse-grained gauge field can be regarded as a dynamical variable analogous to the position operator in the Foldy–Wouthuysen representation.

In the present article we would like to recall a definition of the coarse-grained gluon field given in [16] in the case of continuum space-time, and in papers [17, 18] in Euclidean lattice formulation. Next, in Section 3 we describe our proposal [19] how to interpret the lattice coarse-grained gluon field in terms of a continuum field theory. This should be possible at least in the small momenta limit. Section 4 is devoted to remarks on the problem how to construct the corresponding effective models for the low energy QCD dynamics in the gluon sector.

2. The coarse-grained gluon field

Let us start from the definition proposed by Nielsen and Patkos [16]. It is formulated for the gauge fields in continuum space-time. Let us divide the space-time into regular boxes of size b, and take the points x and $x+\varepsilon$ from one such box. Consider all continuous paths connecting these two points, with the restriction that they also lie inside that single box. With every such path Γ we associate the parallel transporter

$$W(x+\varepsilon, x) = P\exp\left(i\int_{\Gamma} \hat{a}_\mu dx^\mu\right), \tag{1}$$

where $\hat{a}_\mu$ denotes the microscopic $\mathrm{SU}(N)_c$ gauge field. The coarse-grained description of the gauge field is formulated in terms of an average $\overline{W(x+\varepsilon,x)}$ over all such paths Γ. Precise form of a weighting function defining the average is not essential in our considerations. Next, we expand $\overline{W(x+\varepsilon,x)}$ with respect to ε,

$$\overline{W(x+\varepsilon,x)} = \hat{\Sigma}(x) + i\varepsilon^\mu \hat{A}_\mu(x) + \dots \, . \tag{2}$$

The field $\hat{A}_\mu(x)$ is by definition the coarse-grained gluon field.

Further step, in practice rather difficult to perform, would consist in inserting the functional identity

$$1 = \int [d\overline{W}]\delta\left[\overline{W} - \overline{P\exp\left(i\int_{\Gamma} \hat{a}_\mu dx^\mu\right)}\,\right]$$

in a path integral for a partition function of the microscopic quantum gauge theory, and in integrating out the microscopic gauge field $\hat{a}_\mu$. As the result we would obtain a new quantum theory, with $\overline{W}$, or $\hat{\Sigma}, \hat{A}_\mu$, etc., as the basic fields. It is expected that in this theory the infrared quantum fluctuations will be less prominent, and that the small momenta physics can essentially be described in terms of a mean field approximation. This hope is based on the following heuristic picture of fluctuations of the microscopic non-Abelian gauge field in the vacuum state in a confining phase. Due to the self-interactions of the gauge fields a dynamically generated length scale Λ^{-1} can appear. Such dimensional transmutation [20] is suggested, *e.g.*, by results obtained within the framework of the instanton picture of QCD vacuum [21], in particular by the observation that the physically decisive contribution comes from instantons of definite size of the order $\Lambda^{-1} \approx 0.4$ fm. Then, if the box size b is larger than Λ^{-1}, one may assume that the vacuum expectation value of $\overline{W}$ vanishes, because the sum over Γ will involve independently fluctuating parallel transporters. If there is an external factor which forces nonvanishing of $\overline{W}$, like an external source or special boundary conditions, then still one can assume that the infrared fluctuations of $\overline{W}$ are small. On the other hand, if $b < \Lambda^{-1}$ then the infrared fluctuations will influence the parallel transporters present in $\overline{W}$ in a uniform coherent manner, hence $\overline{W}$ will be fluctuating too. In the picture sketched above we have completely neglected ultraviolet fluctuations. They are expected to introduce a finite multiplicative renormalization.

The coarse-grained gluon field has the following transformation law under the local $\mathrm{SU}(N)_c$ gauge transformations $\omega(x)$

$$\hat{A}'_\mu(x) = \omega(x)\hat{A}_\mu(x)\omega^{-1}(x) - i\partial_\mu\omega(x)\hat{\Sigma}(x)\omega^{-1}(x). \tag{3}$$

The other field $\hat{\Sigma}$ belongs to the adjoint representation of the gauge group. The effective model should involve at least $\hat{\Sigma}$ and $\hat{A}_\mu$. Unfortunately, due to the presence of $\hat{\Sigma}$ in the second term on the r.h.s. of formula (3) it is not clear how to build from $\hat{\Sigma}$ and $\hat{A}_\mu$ gauge invariant expressions which could be used in the corresponding effective action. In paper [16] it was assumed that $\hat{\Sigma}$ is proportional to the unit matrix, $\hat{\Sigma} = \sigma I$. Then one can construct an effective action of the Friedberg–Lie type with σ being the scalar dielectric field.

In 1984 Mack proposed a lattice version of the coarse-grained gluon field [17]. In this case the paths Γ run along links of the initial "microscopic" hipercubic lattice, and ε is an integer multiple of the microscopic link vector ae_k (e_k has length equal to 1, a is the length of the microscopic link). The average $\overline{W(x+\varepsilon,x)}$ is regarded as a link variable on a new coarser lattice with links of length $l_0 = na$, where n is the integer. For clarity, $\overline{W(x+\varepsilon,x)}$

as the link variable will be denoted by $\Phi_k(x)$. Here the index $k = 1, 2, 3, 4$ enumerates links of positive directions which start at the site x of the coarse lattice. The end point of the long link is $x + l_0 e_k$. As before, $\Phi_k(x)$ is expected to be insensitive to the quantum infrared fluctuations if $l_0 > \Lambda^{-1}$.

The lattice color dielectric model can be described as a non-standard lattice gauge theory in which instead of the usual unimodular link variables $U_k(x) \in \mathrm{SU}(N)_c$ we have more general N by N matrices

$$U_k(x) \rightarrow \Phi_k(x).$$

Upon inversion of the direction of the link

$$\Phi_{-k}(x + l_0 e_k) = \Phi_k^\dagger(x). \tag{4}$$

The link $-k$ in formula (4) starts at the point $x + l_0 e_k$ and it ends at the point x. Under the $\mathrm{SU}(N)_c$ gauge transformations the macroscopic link variables $\Phi_k(x)$ transform in the usual manner,

$$\Phi_k'(x) = \omega(x + l_0 e_k)\Phi_k(x)\omega(x)^{-1}. \tag{5}$$

Such a non-standard lattice model has been investigated with the help of analytic as well as numerical methods [22–24].

In the small momenta limit the lattice model certainly can be approximated by a continuum field theory. Such continuum description would offer the usual advantages, like translational and rotational symmetries. It has turned out that in spite of the naturalness of such continuum approximation it is not easy to come by it. The problem is how to identify the continuum field counterpart of the non-unitary link variable $\Phi_k(x)$. In the literature the polar decomposition of Φ_k is used,

$$\Phi_k(x) = \exp(i\theta_k(x)) V_k(x) \hat{\chi}_k(x), \tag{6}$$

Here $V_k(x)$ is a matrix of the $\mathrm{SU}(N)$ type, $\hat{\chi}_k(x) = \left(\Phi_k^\dagger(x)\Phi_k(x)\right)^{1/2}$ is a N by N Hermitean matrix with nonnegative eigenvalues, and $\theta_k(x)$ is a real number. Then, the problem is shifted to the continuum interpretation of $\hat{\chi}_k(x)$.

The polar decomposition (6) is analogous to a transformation from Cartesian to spherical coordinates, with θ_k and $\hat{A}_k$ (introduced in formula (10) below) playing the role of angle variables, while $\hat{\chi}_k$ corresponds to the radius. Such transformation has the drawback: it is singular at $\hat{\chi}_k = 0$. It is clear from formula (6) that $\theta_k(x)$ and $V_k(x)$ can be taken arbitrary if $\hat{\chi}_k = 0$. The polar decomposition has also the advantage: it gives a smooth correspondence with the original microscopic $\mathrm{SU}(N)_c$ gauge field theory, which is obtained when $\hat{\chi}_k = I$ and $\exp(i\theta_k(x)) = 1$ — in this region the polar decomposition is nonsingular.

In order to satisfy the relation (4) we assume that

$$V_{-k}(x + l_0 e_k) \; = \; V_k^\dagger(x)\,, \tag{7}$$

$$\exp[i\theta_{-k}(x + l_0 e_k)] \; = \; \exp[-i\theta_k(x)]\,. \tag{8}$$

Then, formulas (4), (6) give

$$\hat{\chi}_{-k}(x + l_0 e_k) = V_k(x)\hat{\chi}_k(x)V_k^\dagger(x). \tag{9}$$

V_k can be related in the standard manner to a traceless Hermitean gauge field $\hat{A}_\mu$ on M:

$$V_k(x) = \exp[i\hat{A}_k(x)], \tag{10}$$

where $\hat{A}_k(x)$ is the lattice coarse-grained gluon field. The $\theta_k(x)$ field has been called in the literature the bleached gluon.

By assumption, V_k transforms under the $\mathrm{SU}(N)_c$ gauge transformations like any lattice $\mathrm{SU}(N)_c$ gauge field — a formula analogous to (5) — and θ_k is gauge invariant. Then, it follows from (6) that $\hat{\chi}_k$ belongs to the adjoint representation of $\mathrm{SU}(N)_c$

$$\hat{\chi}'_k(x) = \omega(x)\hat{\chi}_k(x)\omega(x)^{-1}. \tag{11}$$

Let us note that $\mathrm{SU}(2)_c$ case is special: the θ_k field is absent, and each matrix $\hat{\chi}_k(x)$ is replaced by a nonnegative number $\chi_k(x)$, invariant under the $\mathrm{SU}(2)_c$ gauge transformations, [17]. In the following we assume that $N > 2$.

3. The continuum counterpart of the lattice field $\hat{\chi}_k(x)$

The main problem with the continuum description of the small momenta sector of the lattice model is the lack of an obvious continuum counterpart for the fields $\hat{\chi}_k(x)$. The properties of these fields are rather puzzling. By construction $\hat{\chi}_k(x)$ are defined on the links, like the lattice gauge field. In spite of that, they cannot be related to a gauge field in continuum space-time because then the gauge transformation law would have to be of the form (5), and not (11) which has the form typical for a matter field defined on the sites of the lattice. In [19] we have noticed that there exists a field transformation which relates the $\hat{\chi}_k(x)$ fields with components of a Hermitean vector field $\hat{B}_k(x)$, $k = 1, 2, 3, 4$, located on the sites of the coarse lattice. Namely,

$$\hat{\chi}_k(x) = G\left(D_k\hat{B}_k(x)\right) \tag{12}$$

(no summation over k). Here $G(\cdot)$ denotes a matrix function described below, and

$$D_j \hat{B}_k(x) = V_j^\dagger(x) \hat{B}_k(x + l_0 e_j) V_j(x) - \hat{B}_k(x) \tag{13}$$

is the lattice version of gauge-covariant derivative. The $\hat{B}_k(x)$ field has the following $\mathrm{SU}(N)_c$ gauge transformation

$$\hat{B}'_k(x) = \omega(x) \hat{B}_k(x) \omega(x)^{-1}. \tag{14}$$

Thus, $\hat{B}_k(x)$ is a matter field. The function $G(\cdot)$ is supposed to transform Hermitean matrices into Hermitean ones, hence coefficients of its Taylor expansion should be real numbers. Moreover, its matrix values should be positive definite because $\hat{\chi}_k(x)$ is positive definite. Finally, classical vacuum value of the $\hat{\chi}_k(x)$ fields is expected to vanish in the color dielectric models. If we require that the corresponding vacuum value of $D_k \hat{B}_k(x)$ also vanishes, then we may take

$$G(DB) = \left(c_1 DB + c_2 (DB)^2 + ...\right)^2, \tag{15}$$

where DB is a shorthand for $D_k \hat{B}_k(x)$, and all c_i are real. For DB close to its vacuum value, that is when

$$D_k \hat{B}_k(x) \ll 1, \tag{16}$$

we may neglect the higher powers of DB, and then

$$\hat{\chi}_k(x) = \left(D_k \hat{B}_k(x)\right)^2. \tag{17}$$

The constant c_1 has been removed by rescaling $\hat{B}_k(x)$. Below we shall use the field transformation (17).

Now, let us turn to the small momenta limit. The lattice fields $\hat{B}_k(x)$ in the small momenta sector are regarded as projections (on the coarse lattice four-vectors $l_0 e_k$) of the vector field $\hat{B}_\nu(x)$, which is defined on the continuum space-time and almost constant on distances l_0,

$$\hat{B}_k(x) = l_0 e_k^\nu \hat{B}_\nu(x). \tag{18}$$

Similarly,

$$\hat{A}_k(x) = l_0 e_k^\nu \hat{A}_\nu(x), \quad \theta_k(x) = l_0 e_k^\nu \Theta_\nu(x). \tag{19}$$

The lattice covariant derivative D_k is interpreted as

$$D_k = l_0 e_k^\mu D_\mu,$$

where D_μ is the covariant derivative in the continuum space-time. Then,

$$D_k\hat{B}_k(x) = l_0^2 e_k^\mu e_k^\nu D_\mu \hat{B}_\nu(x) + l_0^3 e_k^\mu e_k^\sigma e_k^\nu D_\mu D_\sigma \hat{B}_\nu(x) + ..., \tag{20}$$

where

$$D_\mu \hat{B}_\nu = \partial_\mu \hat{B}_\nu - i[\hat{A}_\mu, \hat{B}_\nu]$$

is the continuum covariant derivative in the adjoint representation. Thus, in the small momenta limit

$$D_k\hat{B}_k \cong l_0^2 e_k^\mu e_k^\nu D_\mu \hat{B}_\nu$$

(no summation over k). This formula implies that $D_k\hat{B}_k$ depends only on the symmetric part of $D_\mu\hat{B}_\nu$,

$$D_k\hat{B}_k(x) \cong l_0^2 e_k^\mu e_k^\nu \hat{G}_{\mu\nu}(x), \tag{21}$$

where

$$\hat{G}_{\mu\nu}(x) = \frac{1}{2}\Big(D_\mu\hat{B}_\nu(x) + D_\nu\hat{B}_\mu(x)\Big) = \hat{G}_{\nu\mu}(x). \tag{22}$$

It is clear that under the $\mathrm{SU}(N)_c$ gauge transformations

$$\hat{G}'_{\mu\nu}(x) = \omega(x)\hat{G}_{\mu\nu}(x)\omega(x)^{-1}.$$

Finally, formulas (17), (20) and (21) give

$$\hat{\chi}_k(x) \cong l_0^4 e_k^\mu e_k^\nu e_k^\rho e_k^\lambda \hat{G}_{\mu\nu}(x)\hat{G}_{\rho\lambda}(x) \tag{23}$$

(no summation over k).

The field $\hat{B}_\mu$ appears in formula (23) only through the symmetrized covariant derivatives, that is through $\hat{G}_{\mu\nu}$. Therefore, in most calculations we may use just the $\hat{G}_{\mu\nu}$ field. Nevertheless, the basic dynamical field in the small momenta limit is the $\hat{B}_\mu$ field. This makes difference when, for example, deriving mean field equations because variational derivative of the action should be taken with respect to $\hat{B}_\mu$ and not $\hat{G}_{\mu\nu}$.

The tensor field $\hat{G}_{\mu\nu}$ can be decomposed into two parts which are irreducible with respect to SO(4) group, the continuous Euclidean space-time symmetry group, namely

$$\hat{G}_{\mu\nu}(x) = \hat{\sigma}(x)\delta_{\mu\nu} + \hat{g}_{\mu\nu}(x), \tag{24}$$

where $(\hat{g}_{\mu\nu})$ has vanishing trace, $\hat{g}_{\mu\mu} = 0$. Formulas (22) and (24) imply that

$$\hat{\sigma}(x) = \frac{1}{4} D_\mu \hat{B}_\mu(x).$$

Because $e_k^\mu e_k^\mu = 1$ for each $k = 1, 2, 3, 4$, formulas (23) and (24) give

$$\hat{\chi}_k(x) = l_0^4 \left[\hat{\sigma}^2(x) + e_k^\mu e_k^\nu \left(\hat{\sigma}(x)\hat{g}_{\mu\nu}(x) + \hat{g}_{\mu\nu}(x)\hat{\sigma}(x)\right) + e_k^\mu e_k^\nu e_k^\rho e_k^\lambda \hat{g}_{\mu\nu}(x)\hat{g}_{\rho\lambda}(x)\right]$$

(no summation over k). Furthermore, $\hat{\sigma}$ and $\hat{g}_{\mu\nu}$ can be split into the color singlet and adjoint representation parts,

$$\hat{\sigma}(x) = \sigma(x)I + \frac{1}{2}\lambda_a\sigma^a(x), \quad \hat{g}_{\mu\nu}(x) = g_{\mu\nu}(x)I + \frac{1}{2}\lambda_a g^a_{\mu\nu}(x),$$

where λ_a are Gell-Mann matrices in the case of $\mathrm{SU}(3)_c$ gauge group. The fields σ, σ^a, $g_{\mu\nu}$ and $g^a_{\mu\nu}$ are real due to hermicity of $\hat{G}_{\mu\nu}$.

In the $\mathrm{SU}(2)_c$ case only the color singlet parts are present, that is $G_{\mu\nu} = \sigma(x)\delta_{\mu\nu} + g_{\mu\nu}(x)$.

4. Towards the continuum color dielectric action

We have seen that in the small momenta limit the lattice color dielectric model can be formulated in terms of the fields $\hat{A}_\mu, \hat{B}_\mu, \Theta_\mu$, which are defined on the continuum space-time. Now we would like to introduce certain lattice color dielectric action, and to discuss its small momenta limit. Our ultimate goal, not reached as yet, is to find the corresponding continuum effective action S_c for the fields $\hat{A}_\mu, \hat{B}_\mu, \Theta_\mu$. The discussion presented below has rather preliminary character. We concentrate on theoretical aspects, in particular on the problem of restoration of the SO(4) symmetry broken in the lattice model.

The action S_c is assumed to be invariant with respect to the coarse-grained $\mathrm{SU}(N)_c$ gauge transformations. Moreover, the terms with higher powers of Φ_k in general are less important, because Φ_k is expected to be close to its vacuum value equal to zero. As the first possible contribution to the action S_c let us consider

$$S_1[\Phi] = \sum_{x,k,j} \mathrm{Tr}\left(\Psi^\dagger_{k,j}(x)\Psi_{k,j}(x)\right), \tag{25}$$

where

$$\Psi_{k,j}(x) = \Phi_k(x) + \alpha_1 \Phi^\dagger_j(x + l_0 e_k)\Phi_k(x + l_0 e_j)\Phi_j(x). \tag{26}$$

In this formula α_1 is a real constant. The indices take the following values: $k = 1, ..., 4$ and $j = \pm 1, ..., \pm 4$. (Link with negative j starting at the point x ends at the point $x - l_0 e_{|j|}$.) The second term on the r.h.s. of formula (26) corresponds to a path of length $3l_0$ connecting the points x and $x + l_0 e_k$. For $j \neq \pm k$ the path has a staple-like shape. It is clear that if the constant

α_1 is not too large, all $\Psi_{k,j}$ vanish only when all Φ_k vanish — this ensures that $S_1[\Phi]$ has the absolute, nondegenerate minimum for $\Phi_k = 0$.

Let us introduce the field strength $\hat{F}_{kj}$ of the lattice coarse-grained $\mathrm{SU}(N)_c$ gauge field V_k

$$\exp(i\hat{F}_{kj}) = V_k^\dagger(x)V_j^\dagger(x + l_0 e_k)V_k(x + l_0 e_j)V_j(x). \tag{27}$$

Similarly, we define f_{kj} — the Abelian field strength corresponding to the lattice θ_k field,

$$\exp(if_{kj}) = \exp[i(\theta_k(x + l_0 e_j) + \theta_j(x) - \theta_k(x) - \theta_j(x + l_0 e_k))]. \tag{28}$$

In the case of weak fields $\hat{F}_{kj}, f_{kj}, D_j\hat{\chi}_k$, the first nontrivial approximation to $\Psi_{k,j}(x)$ has the form

$$\begin{aligned}\Psi_{k,j} \cong\ & \exp(i\theta_k(x))V_k(x)\Big[\hat{\chi}_k(x) + \alpha_1\Big(\hat{\chi}_j(x)\hat{\chi}_k(x)\hat{\chi}_j(x) \\ & + D_k\hat{\chi}_j(x)\hat{\chi}_k(x)\hat{\chi}_j(x) + \hat{\chi}_j(x)D_j\hat{\chi}_k(x)\hat{\chi}_j(x) \\ & + i\hat{\chi}_j(x)\Big(\hat{F}_{kj} + f_{kj}I\Big)\hat{\chi}_k(x)\hat{\chi}_j(x)\Big)\Big].\end{aligned} \tag{29}$$

The lattice color dielectric effective action can not be just equal to $S_1[\Phi]$ because in that case it would have too large symmetry group — $S_1[\Phi]$ is invariant under $\mathrm{U}(N)_c$ gauge transformations. In order to break this symmetry down to the $\mathrm{SU}(N)_c$ gauge symmetry we use $\mathrm{Det}\Phi_k$ which is invariant under the $\mathrm{SU}(N)_c$ gauge transformations only. The polar decomposition (6) gives

$$\mathrm{Det}\Phi_k = \exp(Ni\theta_k(x))\mathrm{Det}\hat{\chi}_k,$$

where $\mathrm{Det}\hat{\chi}_k$ can be expressed by traces of powers of $\hat{\chi}_k$ with the help of Hamilton–Cayley identity. Because $\mathrm{Det}\Phi_k$ can be a complex number it cannot be directly included into the action. Instead, we may take

$$S_2[\Phi] = \gamma\sum_{x,k}\left(\mathrm{ImDet}\Phi_k(x)\right)^2 = \gamma\sum_{x,k}\left(\mathrm{Det}\hat{\chi}_k(x)\right)^2\sin^2(N\theta_k(x)), \tag{30}$$

where γ is a positive constant. For $\theta_k \ll 1$ we may approximate $\sin^2(N\theta_k) \cong N^2 l_0^2 e_k^\mu e_k^\nu \Theta_\mu(x)\Theta_\nu(x)$.

We may also include in S_c terms which contain powers of $\hat{\chi}_k$ only. For example, up to the fourth power in $\hat{\chi}_k$

$$\begin{aligned} S_p \;=\; & \lambda_2 \sum_{x,k} \mathrm{Tr}\left(\Phi_k^\dagger(x)\Phi_k(x)\right) \\ & +\lambda_3 \sum_{x,k,j} \mathrm{Tr}\left(\Phi_k^\dagger(x)\Phi_k(x)\Phi_j^\dagger(x)\Phi_j(x)\right) \\ & +\lambda_4 \sum_{x,k,j} \mathrm{Tr}\left(\Phi_k^\dagger(x)\Phi_k(x)\right) \mathrm{Tr}\left(\Phi_j^\dagger(x)\Phi_j(x)\right)\,, \end{aligned} \tag{31}$$

where λ_i are positive constants. It is easy to see that $\exp(i\theta_k)$ and V_k cancel out in each term on the r.h.s. of this formula — only $\hat{\chi}_k$'s are left.

In the small momenta limit $\hat{F}_{kj}$ and f_{kj} are related to the projections on the lattice four-vectors of the corresponding tensors in the continuum space-time,

$$\hat{F}_{kj} \cong l_0^2 e_k^\mu e_j^\nu \hat{F}_{\mu\nu}, \quad f_{kj} \cong l_0^2 e_k^\mu e_j^\nu f_{\mu\nu}, \tag{32}$$

where $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]$ and $f_{\mu\nu} = \partial_\mu \Theta_\nu - \partial_\nu \Theta_\mu$. For $\hat{\chi}_k$ we have formula (23), and

$$D_j \hat{\chi}_k \cong l_0^5 e_j^\mu e_k^\nu e_k^\lambda e_k^\sigma e_k^\rho \left(D_\mu \hat{G}_{\nu\lambda}(x)\hat{G}_{\sigma\rho}(x) + \hat{G}_{\nu\lambda}(x) D_\mu \hat{G}_{\sigma\rho}(x)\right), \tag{33}$$

as follows from (23) by Leibniz rule for the covariant derivative of a product.

Using formulas given above and replacing $\sum_x$ by $l_0^{-4}\int d^4x$, we obtain the action functional $S_c[\hat{A}_\mu, \Theta_\nu, \hat{G}_{\mu\nu}] = S_1 + S_2 + S_p$ of the effective model in the Euclidean continuum space-time. It is clear that all terms in S_c have the form of contractions of multiple factors $\Theta_\mu, \hat{G}_{\mu\nu}, D_\mu \hat{G}_{\nu\lambda}, \hat{F}_{\lambda\sigma}, f_{\rho\omega}$ with the products $e_k^\mu e_k^\nu e_k^\rho e_k^\lambda e_j^\alpha e_j^\beta ...$ of components of the lattice unit four-vectors e_k. Such products are lattice artifacts, and they explicitly break the Euclidean SO(4) invariance. The contractions give SO(4) scalars, but their form in general is not invariant with respect to the SO(4) transformations. Thus, the action $S_c[\hat{A}_\mu, \Theta_\nu, \hat{G}_{\mu\nu}]$ is not SO(4) invariant, eventhough it is SO(4) scalar. To solve this problem, we apply the following trick. The four-vectors $e_k, k = 1, ..., 4$, form an orthonormal basis in the Euclidean space-time. This basis does not have any physical meaning and it can be arbitrary — the lattice can have any orientation with respect to a "laboratory" reference frame. Nevertheless, each concrete choice and the subsequent lattice formulation break the SO(4) invariance. We can restore this symmetry by integrating over all SO(4) orientations of the basis. Formally, this can be achieved by preceding each term in the effective action by the normalized to unity Haar integral $\int\limits_{\mathrm{SO}(4)} d\mathcal{O}$ over the SO(4) group, and by regarding the four four-vectors e_k as (orthonormal) columns of the matrix $\mathcal{O} \in \mathrm{SO}(4)$, that is

$$e_k^\mu = \mathcal{O}_{k\mu}.$$

Due to the SO(4) invariance of the Haar measure, the integrals are forminvariant with respect to simultaneous SO(4) transformations of all indices μ, ν, ρ, etc., which refer to the "laboratory" reference frame in the continuous Euclidean space-time. For example, the term $\mathrm{Tr}\left(\Phi_k^\dagger(x)\Phi_k(x)\right)$ acquires the form

$$\eta_{\mu_1\mu_2\ldots\mu_8} l_0^8 \ \mathrm{Tr}\left(\hat{G}_{\mu_1\mu_2}(x)\hat{G}_{\mu_3\mu_4}(x)\hat{G}_{\mu_5\mu_6}(x)\hat{G}_{\mu_7\mu_8}(x)\right),$$

where

$$\eta_{\mu_1\mu_2\ldots\mu_8} = \int\limits_{\mathrm{SO}(4)} d\mathcal{O}\mathcal{O}_{k\mu_1}\ldots\mathcal{O}_{k\mu_8}$$

(no summation over k). It is easy to see that $\eta_{\mu_1\mu_2\ldots\mu_8}$ does not depend on k, and that it is forminvariant under simultaneous SO(4) transformations of all its indices. Because $\eta_{\mu_1\mu_2\ldots\mu_8}$ is symmetric with respect to permutations of its indices, it is equal to a sum of products of Kronecker deltas $\delta_{\mu_i\mu_j}$, where $i, j = 1\ldots8$. In general case we encounter integrals of the type

$$\int\limits_{\mathrm{SO}(4)} d\mathcal{O} e_k^{\mu_1}\ldots e_k^{\mu_m} e_l^{\nu_1}\ldots e_l^{\nu_n},$$

where $e_k^\mu = \mathcal{O}_{k\mu}, \quad e_l^\nu = \mathcal{O}_{l\nu}$. They can be calculated with the help of generating functions [19].

5. Summary

1. Our main new result is the field transformation (12), which replaces the lattice color dielectric field $\hat{\chi}_k(x)$ by the square of the gauge-covariant derivative of the vector field $\hat{B}_k$. We have also shown how the integration over SO(4) group can be used to restore SO(4) invariance.

2. We have focused our attention on the theoretical problems which hampered the previous investigations of the small momenta limit of the gluonic part of the lattice $\mathrm{SU}(N)_c$ color dielectric model. Our results suggest that the Friedberg–Lee type phenomenological models in the $\mathrm{SU}(N)_c$ case can have natural extentions which would incorporate more color dielectric fields, in particular the scalar fields in the adjoint representation, as well as the tensor fields. We have not made any attempt to construct a phenomenologically viable model of that kind. Such a task would require many more steps, among them inclusion of quarks and a discussion of their coupling to the color dielectric fields. This is one direction in which one could continue our work.

We also see two very interesting topics which could be studied in the framework of the pure glue sector discussed in the present paper: QCD flux-tube and glueballs. Such investigations, while extremely interesting on their own rights, can also reveal the physical role played by the fields Θ_μ and $\hat{B}_\mu$. In the presented derivation of the continuum color dielectric model these fields appear in rather formal way as the mathematical consequence of $\Phi_k(x)$ being nonunitary matrix. Their physical role has not been elucidated, and it seems to us that at the present stage we do not have enough physical input for this.

3. For the physical applications one needs the Minkowski space-time version of the model. It can be obtained by the inverse Wick rotation, $x^4 \to +ix^0$, $\hat{B}_4 \to +i\hat{B}^0$, $\Theta_4 \to +i\Theta^0$, *etc.* The resulting Minkowski space-time metric has the signature (-,+,+,+).

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