THE WEYL–WIGNER–MOYAL FORMALISM. III. THE GENERALIZED MOYAL PRODUCT IN THE CURVED PHASE SPACE

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Construction of the symplectic connection on phase space is proposed. This connection is in a sense defined by the Riemannian connection on the configuration space. The generalized Moyal product leading to the quantum multiplication in the curved phase space is given.

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1. Introduction

This paper is the third one in the series devoted to the formulation of the nonrelativistic quantum mechanics on the phase space. In previous works we considered the problem of operator ordering [1] and the formulation of nonrelativistic quantum mechanics in the phase space R^{2n} [2]. Here we are going to generalize the Weyl–Wigner–Moyal formalism [1–8] on the systems with nontrivial configuration space. In this article we construct the dynamics of the quantum spinless particle.

Let us consider only observables [1] belonging to the set of formal series $\Pi(\hbar, C^\infty(R^{6n}))$. We assume that there is no difference between classical and quantum observables represented the same physical quantity. In the presented formalism the quantum mechanics resides in the nonabelian but associative multiplication "$*_{(g)}$". For two observables $F, G \in \Pi(\hbar, C^\infty(R^2))$ the quantity defined as their product equals $F \cdot G$ is the observable too. The same quantity in the quantum mechanics is calculated as

[1] The observable is every real continuous function on the phase space.

$$F *_{(g)} G = \alpha^{-1}\left(-\hbar\frac{\partial^2}{\partial p\partial x}\right)$$
$$\times\left\{\left[\alpha\left(-\hbar\frac{\partial^2}{\partial p\partial x}\right)F(p,x)\right]\exp\left(\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}}\right)\left[\alpha\left(-\hbar\frac{\partial^2}{\partial p\partial x}\right)G(p,x)\right]\right\} \quad (1)$$

and it is not usually the observable. The operator $\mathcal{P}$ acts as

$$\overleftrightarrow{\mathcal{P}} \stackrel{\mathrm{def}}{=} \frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}, \quad (2)$$

and

$$F_g \stackrel{\mathrm{def}}{=} \alpha(-\hbar\frac{\partial^2}{\partial p\partial x})F. \quad (3)$$

From the definition the formal series

$$\alpha = \alpha\left(-\hbar\frac{\partial^2}{\partial p\partial x}\right) = \sum_{k=0}^{\infty}\alpha_k\cdot\left(-\hbar\frac{\partial^2}{\partial p\partial x}\right)^k, \ \alpha_k \ \varepsilon \ R \quad (4)$$

and

$$\alpha_0 = 1. \quad (5)$$

The expected value of the quantity F

$$\langle F(p,x)\rangle = \frac{1}{2\pi\hbar}\int\limits_{R^2} F(p,x)\alpha^2\left(-\hbar\frac{\partial^2}{\partial p\partial x}\right)\varrho(p,x)dpdx, \quad (6)$$

where $\varrho(p,x)$ is the generalized function representing the state [2].

The time evolution of the quantum observable F is given by the formula

$$\frac{dF}{dt} = \{F,H\}_M^{(g)}. \quad (7)$$

The bracket

$$\{F,H\}_M^{(g)} \stackrel{\mathrm{def}}{=} \frac{1}{i\hbar}(F *_{(g)} H - H *_{(g)} F) \quad (8)$$

is called the generalized Moyal bracket and $H(p,x)$ is the Hamiltonian.

The construction of the Weyl–Wigner–Moyal formalism in the curved phase space can be divided in two steps. First we construct the symplectic connection on the phase space which includes the Riemannian connection

[2] For $\alpha = 1$ the function $\frac{1}{2\pi\hbar}\varrho(p,x)$ is called the Wigner function.

on the configuration space. We prove that this construction is possible but not unique. In the next section we define the differential forms with values in the generalized Weyl algebras. We construct the most general abelian connection defined by the symplectic connection on the phase space $T^*\mathcal{M}$. Finally we give a formula for the generalized Moyal product in the curved phase space.

We prepared this paper under the inspiration of the excellent works by Fedosov [9] and [10].

2. The symplectic connection

Let the configuration space of the system be a real C^∞ paracompact differential manifold $\mathcal{M}$ $(\dim\mathcal{M} = m)$. The Riemannian connection $\Gamma^\gamma_{\alpha\beta}$ on this manifold is given uniquely by the quadratic form of kinetic energy.

In the atlas $\{(U_\varrho, \tilde{\varphi}_\varrho)\}_{\varrho\in I}$ on the configuration space $\mathcal{M}$ the information about dynamics of the system is given by the vector from the cotangent space $T^*_{\mathtt{p}}(\mathcal{M})$:

$$[(\mathtt{p}, (U_\varrho, \tilde{\varphi}_\varrho), \theta], \tag{9}$$

where $\theta = (p_1, \ldots, p_m)$ are momentum coordinates at the point $\mathtt{p} \in \mathcal{M}$ in the chart $\{(U_\varrho, \tilde{\varphi}_\varrho)\}$.

Define

$$T^*\mathcal{M} \stackrel{\text{def}}{=} \bigcup_{\mathtt{p}\in\mathcal{M}} T^*_{\mathtt{p}}(\mathcal{M}). \tag{10}$$

$T^*\mathcal{M}$ has the natural cotangent bundle structure with the symplectic structure ω and it is called **the phase space** of the system. The dimension of the phase space is $2m$.

Definition 2.1 The symplectic connection *on $T^*\mathcal{M}$ is the torsion free connection satisfying the condition*

$$\omega_{ij;k} = 0, \tag{11}$$

where a semicolon " ; " stands for the covariant derivative.

In every chart in $T^*\mathcal{M}$ coefficients Γ^i_{jk} of symplectic connection fulfill the conditions

$$\omega_{ij;k} = \frac{\partial\omega_{ij}}{\partial q^k} - \Gamma^l_{ik}\omega_{lj} - \Gamma^l_{jk}\omega_{il} = 0, \tag{12}$$

$$\Gamma^k_{ij} - \Gamma^k_{ji} = 0 \ \text{(torsion free)}. \tag{13}$$

In the Darboux coordinates we have

$$\omega_{ij;k} = -\Gamma^l_{ik}\omega_{lj} - \Gamma^l_{jk}\omega_{il} = \Gamma_{jik} - \Gamma_{ijk} = 0, \tag{14}$$

where

$$\Gamma_{ijk} \stackrel{\text{def}}{=} \omega_{il}\Gamma^{l}_{jk}. \tag{15}$$

Henceforth we also use the term "symplectic connection" for the coefficients Γ_{ijk}.

It is easy to see that in the Darboux coordinates the coefficients Γ_{ijk} are symmetric with respect to indices (i, j, k).

Note that the symplectic connection is not unique. In Darboux coordinates every set of completely symmetric coefficients Γ_{ijk} defines a symplectic connection. The difference

$$\Delta_{ijk} \stackrel{\text{def}}{=} \Gamma_{ijk} - \bar{\Gamma}_{ijk} \tag{16}$$

between two symplectic connections is the tensor symmetric with respect to indices (i, j, k).

Definition 2.2 *Let $\{(W_\varrho, \varphi_\varrho)\}_{\varrho\in I}$ be an atlas on the symplectic manifold $T^*\mathcal{M}$ such that in every chart the coordinates q^α, $1 \leq \alpha \leq m$ determine points on the basic manifold $\mathcal{M}$ and $q^{\alpha+m} = p_\alpha$, $1 \leq \alpha \leq m$, denote momenta in natural coordinates. Any atlas of this form is called* **the proper Darboux atlas.** *Adequately every chart of proper Darboux atlas is* **the proper Darboux chart.** *The transition functions are now the point transformations*

$$Q^\alpha = Q^\alpha(q^\beta) \;\;,\;\; P_\alpha = \frac{\partial q^\beta}{\partial Q^\alpha} p_\beta. \tag{17}$$

The relations (17) define **the proper Darbox transformation**.

The proper Darboux charts preserve the obvious from physical point of view difference between spatial coordinates and momenta.

Note that every proper Darboux atlas $\{(W_\varrho, \varphi_\varrho)\}_{\varrho\in I}$ on $T^*\mathcal{M}$ is uniquely connected with some atlas $\{(U_\varrho, \tilde{\varphi}_\varrho)\}_{\varrho\in I}$ on $\mathcal{M}$. For every $U_\varrho \subset \mathcal{M}$ the set

$$T^*\mathcal{M} \supset W_\varrho \stackrel{\text{def}}{=} U_\varrho \times R^m \tag{18}$$

is assigned. The diffeomorphism φ_ϱ is defined so that for every point $\mathtt{p} \in W_\varrho$

$$\varphi_\varrho(\mathtt{p}) \stackrel{\text{def}}{=} (q^1, \ldots, q^m, p_1, \ldots, p_m). \tag{19}$$

From here the small Greek letters: $\alpha, \beta, \ldots$ denote spatial coordinates. The capital Latin letters $A, B, \ldots$ $(m+1 \leq A, B \leq 2m)$ correspond to momenta coordinates.

Our aim is to construct the symplectic connection Γ_{ijk} on $T^*\mathcal{M}$ which includes Riemannian connection defined on the configuration space $\mathcal{M}$. It means that in every proper Darboux chart the coefficients $\Gamma_{I\alpha\beta}$ are the coefficients of the Riemannian connection.

The following theorem holds

Theorem 2.1 *For every proper Darboux chart $(W_\varrho, \varphi_\varrho)$ in $T^*\mathcal{M}$ there exists the connection fulfilling the equations (14) and containing the Riemannian connection.*

For example
1. $\Gamma_{K\alpha\beta}$ is Riemannian connection;
2. $\Gamma_{\alpha\beta\gamma} = \Gamma_{\alpha KI} = \Gamma_{KIJ} = 0$.

This construction works only in one chart [3].

From the transformation rule for the Christoffel symbols (compare [12])

$$\Gamma'_{ijk}(Q^1, \ldots, Q^{2m}) = \frac{\partial q^l}{\partial Q^i}\frac{\partial q^r}{\partial Q^j}\frac{\partial q^s}{\partial Q^k}\Gamma_{lrs}(q^1, \ldots, q^{2m}) + \omega_{rd}\frac{\partial q^r}{\partial Q^i}\frac{\partial^2 q^d}{\partial Q^j \partial Q^k} \tag{20}$$

and from (17) we can see that in the proper Darboux transformations

1. coefficients Γ_{KIJ} transform like tensors. Indeed, all summands

$$\omega_{rd}\frac{\partial q^r}{\partial Q^K}\frac{\partial^2 q^d}{\partial Q^I \partial Q^J}$$

are 0 because the second derivative $\frac{\partial^2 q^d}{\partial Q^I \partial Q^J}$ vanishes for arbitrary d. Then coefficients

$$\frac{\partial q^l}{\partial Q^I}\frac{\partial q^r}{\partial Q^J}\frac{\partial q^s}{\partial Q^K}\Gamma_{lrs}$$

could be inequal 0 only if l, r, s are momenta coordinates. It means that Γ_{KIJ} for proper Darboux transformations transform like tensor coordinates. If we put $\Gamma_{KIJ} = 0$ for every K, I, J in one of proper Darboux atlases, then they vanish in all proper Darboux atlases.

2. In the same way we can show that the coefficients $\Gamma_{\alpha IJ}$ transform like tensors [4] if we put $\Gamma_{KIJ} = 0$ for every K, I, J. There is no problem to put $\Gamma_{\alpha IJ} = 0$ for every α, I, J.

3. For the Riemannian connection $\Gamma_{K\alpha\beta}$ nontensorial summands in (20) do not vanish. Fortunately these summands depend only on coordinates on the base manifold $\mathcal{M}$. If the $\Gamma_{KIJ} = 0$ and $\Gamma_{\alpha IJ} = 0$ for every K, I, J and α, then coefficients $\Gamma_{K\alpha\beta}$ of the symplectic connection transform like the Riemannian connection on $\mathcal{M}$.

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[3] Another symplectic connection has been built by I. Gelfand, V. Retakh and M. Shubin [11].

[4] Under proper Darboux transformations.

4. The most difficult is to define coefficients $\Gamma_{\alpha\beta\gamma}$. Let $\{U_\varrho\}_{\varrho\in I}$ be a locally finite open covering of $\mathcal{M}$ $\mathcal{M}$. From the theorem of the partition of unity there exists a partition of unity $\{f_\varrho\}_{\varrho\in I}$ corresponding to $\{U_\varrho\}_{\varrho\in I}$. Using this we define $\Gamma_{\alpha\beta\gamma}$ by

$$\Gamma_{\alpha\beta\gamma} \stackrel{\text{def}}{=} \sum_{\varrho\in I} f_\varrho (\Gamma_{\alpha\beta\gamma})_\varrho , \tag{21}$$

where $(\Gamma_{\alpha\beta\gamma})_\varrho$ are coefficients given in the chart $(W_\varrho, \varphi_\varrho)$.

Thus we construct the symplectic connection on the phase space $T^*\mathcal{M}$ of the system for a paracompact base manifold $\mathcal{M}$. Unfortunately this construction is not unique because it depends on covering $\{U_\varrho\}_{\varrho\in I}$ on $\mathcal{M}$. We are still working on this problem.

3. The generalized Weyl algebra $\mathcal{V}_g$

Let $T^*\mathcal{M}$ be $2m-$dimensional cotangent bundle over a smooth paracompact manifold $\mathcal{M}$. The space $T^*\mathcal{M}$ has the natural symplectic structure. In an arbitrary point $\mathtt{p} \in T^*\mathcal{M}$ we define the formal series with respect to $\hbar$ and $X^1_{\mathtt{p}}, \ldots, X^{2m}_{\mathtt{p}}$.

$$a \stackrel{\text{def}}{=} \sum_{l=0}^{\infty} \hbar^k a_{k,i_1\ldots i_l} X^{i_1}_{\mathtt{p}} \cdots X^{i_l}_{\mathtt{p}} \quad k \geq 0\,. \tag{22}$$

For $l = 0$ we put $a = \hbar^k a_k$.

We use the Einstein convention and we summarize with respect to repeating indices.

Symbols in the formula (22) mean:

$\hbar$ is a positive parameter;

$X^1_{\mathtt{p}}, \ldots, X^{2m}_{\mathtt{p}}$ are components of an arbitrary vector belonging of the tangent space $T_{\mathtt{p}}(T^*\mathcal{M})$ to the symplectic manifold $T^*\mathcal{M}$ at the point $\mathtt{p}$. The components $X^1_{\mathtt{p}}, \ldots, X^{2m}_{\mathtt{p}}$ has been written in natural basis $(\frac{\partial}{\partial q^i})_{\mathtt{p}}$ determined by the chart $(W_\varrho, \varphi_\varrho)$ so that $\mathtt{p} \in W_\varrho$;

$a_{k,i_1,\ldots,i_l}$ are the components of covariant tensor symmetric with respect to indices $(i_1, \ldots, i_l)$, taken in the basis $dq^{i_1} \otimes \cdots \otimes dq^{i_l}$.

Note that *series like (22) are scalars.*

Our definition is different from that given in [9], [13]. Here (like in [10]) the parameter $\hbar$ is used only in nonnegative powers.

Let $V(X_{\mathtt{p}})$ be a set of all elements a at the point $\mathtt{p} \in T^*\mathcal{M}$.

Theorem 3.2 *The tetrad* $(V(X_{\mathtt{p}}), \mathcal{C}, +, \cdot)$ *is the linear space.* ■

$\mathcal{C}$ denotes the set of complex numbers.

Define the mapping $g : V(X_{\mathtt{p}}) \longrightarrow V(X_{\mathtt{p}})$

$$g(a) \stackrel{\mathrm{def}}{=} \alpha \left(-\frac{1}{2}\hbar\vartheta^{ij} \frac{\partial^2}{\partial X^i_{\mathtt{p}} \partial X^j_{\mathtt{p}}} \right) a, \tag{23}$$

where α is the formal series characterizing the generalized Weyl ordering. The covariant tensor ϑ^{ij} in the natural basis $(\frac{\partial}{\partial q^i})_{\mathtt{p}} \otimes (\frac{\partial}{\partial q^j})_{\mathtt{p}}$, which is determined by the proper Darboux chart $(W_\varrho, \varphi_\varrho)$, has the form

$$\left[\begin{array}{cc} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \boldsymbol{p}\boldsymbol{\Gamma} \end{array} \right], \tag{24}$$

where $\mathbf{0}$ and $\mathbf{1}$ are $m \times m$ matrices: null and unity, respectively

$$(\boldsymbol{p}\boldsymbol{\Gamma})^{\alpha\beta} \stackrel{\mathrm{def}}{=} 2p_\gamma \Gamma^\gamma_{\alpha\beta}. \tag{25}$$

$\Gamma^\gamma_{\alpha\beta}$ are components of the Riemannian connection on the configuration space $\mathcal{M}$ and p_γ are the momentum components at $\mathtt{p} \in T^*\mathcal{M}$.

Theorem 3.3 *The tensor ϑ^{ij} is invariant under the proper Darboux transformations.*

Proof

Nontrivial is only the transformation for $(\boldsymbol{p}\boldsymbol{\Gamma})^{\alpha\beta}$.

The new coordinates we denote by capital letters. Components of tensors in the new chart are denoted by " $'$." Greek indices, as usually, run from 1 to m.

From the transformation rule

$$\vartheta'^{\alpha+m,\beta+m} = \frac{\partial P_\alpha}{\partial q^\gamma}\frac{\partial P_\beta}{\partial p_\varepsilon}\delta^\gamma_\varepsilon + \frac{\partial P_\alpha}{\partial p_\gamma}\frac{\partial P_\beta}{\partial q^\varepsilon}\delta^\varepsilon_\gamma + \frac{\partial P_\alpha}{\partial p_\gamma}\frac{\partial P_\beta}{\partial p_\varepsilon}\, 2p_\tau \Gamma^\tau_{\gamma\varepsilon}. \tag{26}$$

Using (17)

$$\frac{\partial P_\alpha}{\partial p_\gamma} = \frac{\partial q^\gamma}{\partial Q^\alpha}, \tag{27}$$

$$\frac{\partial q^\xi}{\partial Q^\omega}\frac{\partial Q^\tau}{\partial q^\xi} = \delta^\tau_\omega \tag{28}$$

and

$$\frac{\partial P_\beta}{\partial q^\varepsilon} = \frac{\partial Q^\tau}{\partial q^\varepsilon}\frac{\partial^2 q^\omega}{\partial Q^\tau \partial Q^\beta} p_\omega = \frac{\partial Q^\tau}{\partial q^\varepsilon}\frac{\partial^2 q^\omega}{\partial Q^\tau \partial Q^\beta}\frac{\partial Q^\xi}{\partial q^\omega} P_\xi. \tag{29}$$

From (27) and the transormation rules for the connection

$$\frac{\partial P_\alpha}{\partial p_\gamma}\frac{\partial P_\beta}{\partial p_\varepsilon}\,2p_\tau \Gamma^\tau_{\gamma\varepsilon} = 2\frac{\partial q^\gamma}{\partial Q^\alpha}\frac{\partial q^\varepsilon}{\partial Q^\beta}\frac{\partial Q^\xi}{\partial q^\tau}P_\xi$$

$$\times\left(\frac{\partial q^\tau}{\partial Q^\theta}\frac{\partial Q^\phi}{\partial q^\gamma}\frac{\partial Q^\varphi}{\partial q^\varepsilon}\Gamma'^\theta_{\phi\varphi} + \frac{\partial q^\tau}{\partial Q^\varrho}\frac{\partial^2 Q^\varrho}{\partial q^\gamma \partial q^\varepsilon}\right). \tag{30}$$

(28), (29) and (30) give us

$$\vartheta'^{\alpha+m,\beta+m} = 2\frac{\partial^2 q^\varepsilon}{\partial Q^\alpha \partial Q^\beta}\frac{\partial Q^\xi}{\partial q^\varepsilon}P_\xi + 2P_\theta \Gamma'^\theta_{\alpha\beta} + 2\frac{\partial q^\gamma}{\partial Q^\alpha}\frac{\partial q^\varepsilon}{\partial Q^\beta}\frac{\partial^2 Q^\xi}{\partial q^\gamma \partial q^\varepsilon}P_\xi. \tag{31}$$

Using

$$\frac{\partial}{\partial Q^\alpha}\left(\frac{\partial Q^\xi}{\partial q^\varepsilon}\frac{\partial q^\varepsilon}{\partial Q^\beta}\right) = \frac{\partial q^\gamma}{\partial Q^\alpha}\frac{\partial^2 Q^\xi}{\partial q^\gamma \partial q^\varepsilon}\frac{\partial q^\varepsilon}{\partial Q^\beta} + \frac{\partial Q^\xi}{\partial q^\varepsilon}\frac{\partial^2 q^\varepsilon}{\partial Q^\alpha \partial Q^\beta} = 0 \tag{32}$$

we get finally

$$\vartheta'^{\alpha+m,\beta+m} = 2P_\theta \Gamma'^\theta_{\alpha\beta}. \ \blacksquare \tag{33}$$

We will write a_g instead of $g(a)$. Notice that a_g are elements of $V(X_{\mathtt{p}})$ and they are scalars.

In the set $V(X_{\mathtt{p}})$ we define the new product "$\circ_{(g)}$":

$$a \circ_{(g)} b \stackrel{\mathrm{def}}{=} \alpha^{-1}\left(-\frac{1}{2}\hbar\vartheta^{rs}\frac{\partial^2}{\partial X^r_{\mathtt{p}} \partial X^s_{\mathtt{p}}}\right)$$

$$\times \sum_{t=0}^{\infty}\left(\frac{i\hbar}{2}\right)^t \frac{1}{t!}\omega^{i_1 j_1}\cdots\omega^{i_t j_t}\frac{\partial^t a_g}{\partial X^{i_1}_{\mathtt{p}}\cdots\partial X^{i_t}_{\mathtt{p}}}\frac{\partial^t b_g}{\partial X^{j_1}_{\mathtt{p}}\cdots X^{j_t}_{\mathtt{p}}}. \tag{34}$$

This product *does not depend on the proper Darboux chart.* [5].

The five $(V(X_{\mathtt{p}}), \mathcal{C}, +, \cdot, \circ_{(g)})$ is also the noncommutative algebra with the unity. We will denote it by $\mathcal{V}_g(X_{\mathtt{p}})$. The centre of this algebra are elements (22) not containing $X^i_{\mathtt{p}}$.

Definition 3.3 *The algebra $\mathcal{V}_g(X_{\mathtt{p}})$ is called* **the generalized Weyl algebra.**

The algebras $\mathcal{V}_g(X_{\mathtt{p}})$ differ between themselves by the multiplication but they are isomorphic . We consider them separately because they give different physical results.

[5] For $\alpha = 1$, the formula (34) has the same form in every natural basis.

Let us take the sum

$$\mathcal{V}_g \stackrel{\mathrm{def}}{=} \bigcup_{\mathtt{p}\in T^*\mathcal{M}} \mathcal{V}_g(X_{\mathtt{p}}). \tag{35}$$

The tetrad $(\mathcal{V}_g, \pi, T^*\mathcal{M})$ is **the algebra bundle.**

Definition 3.4 The n-differential form with value in bundle $\mathcal{V}_g$ *is the form*

$$a = \sum_{l=0}^{\infty} \hbar^k a_{k,i_1\ldots i_l,j_1\ldots j_n}(q^1,\ldots,q^{2m}) X^{i_1}\cdots X^{i_l} dq^{j_1}\wedge\cdots\wedge dq^{j_n}, \tag{36}$$

where $0 \leq n \leq 2m$.

$a_{k,i_1\ldots i_l,j_1\ldots j_n}(q^1,\ldots,q^{2m})$ are smooth tensor fields symmetric with respect to indices $(i_1,\ldots,i_l)$ and antisymmetric with respect to $(j_1,\ldots,j_n)$. For simplicity we will omit names of variables $(q^1,\ldots,q^{2m})$ in $a_{k,i_1\ldots i_l,j_1\ldots j_n}$ $(q^1,\ldots,q^{2m})$. Product of two forms (36) we will denoted by "$\circ_{(g)}$".

The multiplication of forms is now the product "$\circ_{(g)}$" of elements of algebra $\mathcal{V}_g$ and the external product of forms. Let Λ^n be a smooth field of $n-$ form on the symplectic manifold $T^*\mathcal{M}$. The forms (36) are elements of the direct sum

$$\mathcal{V}_g \otimes \Lambda \stackrel{\mathrm{def}}{=} \oplus_{n=0}^{2m}(\mathcal{V}_g \otimes \Lambda^n). \tag{37}$$

(We denote the bundle and the cross section of the bundle by the same symbol.)

Definition 3.5 The commutator *of the forms* $a \in \mathcal{V}_g \otimes \Lambda^{n_1}$ *and* $b \in \mathcal{V}_g \otimes \Lambda^{n_2}$ *is the form* $[a,b]^{(g)} \in \mathcal{V}_g \otimes \Lambda^{n_1+n_2}$ *defined by*

$$[a,b]^{(g)} \stackrel{\mathrm{def}}{=} a \circ_{(g)} b - (-1)^{n_1\cdot n_2} b \circ_{(g)} a. \tag{38}$$

Definition 3.6 *The form* $a \in \mathcal{V}_g \otimes \Lambda$ *is called* **central**, *if for every* $b \in \mathcal{V}_g \otimes \Lambda$ *the commutator* $[a,b]^{(g)}$ *vanishes.*

In this sense only forms that do not contain elements X^i are central in the generalized Weyl algebra.

Let us define two operators acting on the forms from the algebra $\mathcal{V}_g \otimes \Lambda$.

Definition 3.7 *The operator* $\delta : \mathcal{V}_g \otimes \Lambda^n \longrightarrow \mathcal{V}_g \otimes \Lambda^{n+1}$ *acts as*

$$\delta a = dq^k \wedge \frac{\partial a}{\partial X^k}. \tag{39}$$

Definition 3.8 *The operator* $\delta^* : \mathcal{V}_g \otimes \Lambda^n \longrightarrow \mathcal{V}_g \otimes \Lambda^{n-1}$ *is defined by the relation*

$$\delta^* a \stackrel{\text{def}}{=} X^k \left(\frac{\partial}{\partial q^k} \right) \rfloor a. \tag{40}$$

One can easily prove the theorem:

Theorem 3.4 *The operators* δ *and* δ^**:*

1. *are linear;*
2. *their definitions do not depend on the chart;*
3. $$\delta^2 = (\delta^*)^2 = 0; \tag{41}$$
4. *for monomials* $X^{i_1} \ldots X^{i_l} dq^{j_1} \wedge \ldots \wedge dq^{j_n}$ *for which* $l+n>0$ *we have*
$$\delta\delta^* + \delta^*\delta = (l+n)Id, \tag{42}$$
where "Id" is the identity mapping. ■

Let

$$\delta^{-1} \stackrel{\text{def}}{=} \frac{\delta^*}{l+n} \ \text{ for } \ l+n>0 \ \text{ and } \ \delta^{-1} \stackrel{\text{def}}{=} 0 \ \text{ for } \ l+n=0. \tag{43}$$

The following theorem holds

Theorem 3.5 *Every form* $a \in \mathcal{V}_g \otimes \Lambda$ *can be written as*

$$a = \delta\delta^{-1}a + \delta^{-1}\delta a + a_{00}, \tag{44}$$

where a_{00} *is a central* $0-$*form.*

The proof is a consequence of linearity of the operators δ and δ^{-1} and the fourth property from the theorem 3.4.

Theorem 3.6 *The operators δ and α commute. So for every form $a \in \mathcal{V} \otimes \Lambda^n$*

$$g(\delta a) = \delta a_g. \ \blacksquare \tag{45}$$

This theorem is false for the operator δ^{-1}.

Theorem 3.7 *For every two forms $a \in \mathcal{V} \otimes \Lambda^{n_1}$ and $b \in \mathcal{V} \otimes \Lambda^{n_2}$*

$$\delta(a \circ_{(g)} b) = (\delta a) \circ_{(g)} b + (-1)^{n_1} a \circ_{(g)} (\delta b). \tag{46}$$

Theorem 3.8 *For every form $a \in \mathcal{V}_g \otimes \Lambda^{n_1}$*

$$\delta a = -[\frac{1}{i\hbar}\omega_{ij} X^i dq^j, a]^{(g)}. \ \blacksquare \tag{47}$$

These theorems are the results of the definitions of the operator δ and the linearity of the mapping g and the product "$\circ_{(g)}$".

4. The abelian connection

Definition 4.9 The exterior covariant derivative ∂ *is the linear operator*

$$\partial : \mathcal{V} \otimes \Lambda^n \longrightarrow \mathcal{V} \otimes \Lambda^{n+1},$$

defined by the formula

$$\partial a \stackrel{\text{def}}{=} dq^r \wedge a_{;r}, \quad a \in \mathcal{V} \otimes \Lambda^n. \tag{48}$$

The covariant derivative $a_{;r}$

$$\begin{aligned} a_{;r} = \sum_{l=0}^{\infty} \hbar^k \Big\{ & \frac{\partial a_{k,i_1\ldots i_l,j_1\ldots j_n}}{\partial q^r} X^{i_1} \cdots X^{i_l} dq^{j_1} \wedge \cdots \wedge dq^{j_n} \\ & -\Gamma^d_{i_1 r} a_{k,d\, i_2\ldots i_l,j_1\ldots j_n} X^{i_1} \cdots X^{i_l} dq^{j_1} \wedge \cdots \wedge dq^{j_n} \\ & - \cdots - \Gamma^d_{i_l r} a_{k,i_1\ldots i_{l-1}\, d,j_1\ldots j_n} X^{i_1} \cdots X^{i_l} dq^{j_1} \wedge \cdots \wedge dq^{j_n} \Big\}, \end{aligned} \tag{49}$$

so

$$\begin{aligned} \partial a = \sum_{l=0}^{\infty} \hbar^k \Big\{ & \frac{\partial a_{k,i_1\ldots i_l,j_1\ldots j_n}}{\partial q^r} X^{i_1} \cdots X^{i_l} dq^r \wedge dq^{j_1} \wedge \cdots \wedge dq^{j_n} \\ & -\Gamma^d_{i_1 r} a_{k,d\, i_2\ldots i_l,j_1\ldots j_n} X^{i_1} \cdots X^{i_l} dq^r \wedge dq^{j_1} \wedge \cdots \wedge dq^{j_n} \\ & - \cdots - \Gamma^d_{i_l r} a_{k,i_1\ldots i_{l-1}\, d,j_1\ldots j_n} X^{i_1} \cdots X^{i_l} dq^r \wedge dq^{j_1} \wedge \cdots \wedge dq^{j_n} \Big\}. \end{aligned} \tag{50}$$

Theorem 4.9 *For every* $a \in \mathcal{V} \otimes \Lambda^{n_1}$, $b \in \mathcal{V} \otimes \Lambda$

$$\partial(a \circ b) = \partial a \circ b + (-1)^{n_1} a \circ \partial b; \tag{51}$$

$$\partial \delta a + \delta \partial a = 0. \ \blacksquare \tag{52}$$

The proof is a consequence of the linearity and the definitions of the operators δ and ∂.

Let d be the exterior derivative and let the 1− form Γ be defined by

$$\Gamma \stackrel{\text{def}}{=} \tfrac{1}{2} \Gamma_{ijk} X^i X^j dq^k. \tag{53}$$

Note that *the form Γ is defined in Darboux coordinates.*

It easy to see that in the Darboux atlas the exterior covariant derivative

$$\partial a = da + \frac{1}{i\hbar}[\Gamma, a]. \tag{54}$$

The connection Γ_{ijk} is not a tensor but in every Darboux chart we can multiply [6] 1− form Γ and the elements of the algebra $\mathcal{V} \otimes \Lambda$.

Similarly as in the theory of vector bundles (see [12]) we define the curvature of the connection Γ.

Definition 4.10 2− *form* R

$$R \stackrel{\text{def}}{=} d\Gamma + \frac{1}{2i\hbar}[\Gamma, \Gamma]. \tag{55}$$

is called **the curvature of the connection** Γ.

In Darboux coordinates

$$R = \tfrac{1}{4} \omega_{im} R^m_{jkl} X^i X^j dq^k \wedge dq^l, \tag{56}$$

where

$$R^m_{jkl} \stackrel{\text{def}}{=} \frac{\partial \Gamma^m_{jl}}{\partial q^k} - \frac{\partial \Gamma^m_{jk}}{\partial q^l} + \Gamma^u_{jl} \Gamma^m_{uk} - \Gamma^u_{jk} \Gamma^m_{ul}. \tag{57}$$

From (54) and the definition 4.10 it is easy to see that for every form $a \in \mathcal{V} \otimes \Lambda^n$ the second exterior covariant derivative

$$\partial(\partial a) = dq^r \wedge (dq^s \wedge a_{;s})_{;r} = dq^r \wedge dq^s \wedge a_{;sr} = \frac{1}{i\hbar}[R, a]. \tag{58}$$

In the previous paper ([2]) we have showed that the mapping g is an isomorphism between algebras $(\Pi(\hbar, C^\infty(R^{6n})), *_{(g)})$ and $(\Pi(\hbar, C^\infty(R^{6n})), *)$. Here this mapping is also the isomorphism between $\mathcal{V}_g(X_{\mathtt{p}})$ and $\mathcal{V}(X_{\mathtt{p}})$. It seems to be natural that the mapping g defines some relation between the absoult covariant derivative ∂ acting on forms from $\mathcal{V} \otimes \Lambda$ and the linear operator $\partial^{(g)}$ defined on the cross sections of the boundle $\mathcal{V}_g \otimes \Lambda$.

[6] In the sense of product "$\circ$".

Definition 4.11 The exterior covariant derivative $\partial^{(g)}$ *is a linear operator*

$$\partial^{(g)} : \mathcal{V}_g \otimes \Lambda^n \longrightarrow \mathcal{V}_g \otimes \Lambda^{n+1},$$

defined by

$$g(\partial^{(g)} a) \stackrel{\mathrm{def}}{=} \partial a_g = dq^r \wedge a_{g;r}. \tag{59}$$

The following theorem holds

Theorem 4.10 *For every* $a \in \mathcal{V}_g \otimes \Lambda^{n_1},\ b \in \mathcal{V}_g \otimes \Lambda$

$$\partial^{(g)}(a \circ_{(g)} b) = \partial^{(g)} a \circ_{(g)} b + (-1)^{n_1} a \circ_{(g)} \partial^{(g)} b; \tag{60}$$

$$\partial^{(g)} \delta a + \delta \partial^{(g)} a = 0. \ \blacksquare \tag{61}$$

This theorem is a consequence of the theorems 3.6, 4.9 and the definition of the operator $\partial^{(g)}$.

Remark that

$$\partial^{(g)} a = g^{-1}(\partial a_g) \stackrel{(54)}{=} g^{-1}\left(da_g + [\Gamma, a_g]\right).$$

The difference $(\Gamma_g - \Gamma)$ is a central $1-$ form so

$$\partial^{(g)} a = da + g^{-1}[\Gamma_g, a_g] = da + [\Gamma, a]^{(g)}. \tag{62}$$

It is easy to check that

$$\partial^{(g)}(\partial^{(g)} a) = \frac{1}{i\hbar}[R, a]^{(g)}, \tag{63}$$

where

$$R \stackrel{\mathrm{def}}{=} d\Gamma + \frac{1}{2i\hbar}[\Gamma, \Gamma]^{(g)}. \tag{64}$$

Definition 4.12 *The connection* $\tilde{\Gamma}$ *in the bundle* $\mathcal{V}_g$ *is called* **abelian**, *if the exterior covariant derivative*

$$D^{(g)} a = da + \frac{1}{i\hbar}[\tilde{\Gamma}, a]^{(g)}$$

fulfills the equation

$$D^{(g)}(D^{(g)} a) = \frac{1}{i\hbar}[\Omega, a]^{(g)} = 0 \tag{65}$$

for every $a \in \mathcal{V}_g \otimes \Lambda$ *where* Ω *is the curvature of the connection* $\tilde{\Gamma}$

$$\Omega \stackrel{\mathrm{def}}{=} d\tilde{\Gamma} + \frac{1}{2i\hbar}[\tilde{\Gamma}, \tilde{\Gamma}]^{(g)}. \tag{66}$$

Formally the abelian connection is similar to the flat connection because $a_{;sd} = a_{;ds}$ for $1 \leq s, d \leq 2m$. But its curvature can be central form different from 0.

Using the symplectic connection given by 1− form Γ (look (53)) we construct the abelian connection $\tilde{\Gamma}$ in the bundle $\mathcal{V}_g$.

We assume that the abelian connection $\tilde{\Gamma}$ is the sum $(\Gamma + \gamma)$, where $\gamma \in \mathcal{V}_g \otimes \Lambda^1$. Now

$$D^{(g)}a = da + \frac{1}{i\hbar}[\Gamma + \gamma, a]^{(g)} \tag{67}$$

for every $a \in \mathcal{V}_g \otimes \Lambda$.

The curvature of the abelian connection from (64) and (66)

$$\Omega = R + \partial^{(g)}\gamma + \frac{1}{i\hbar}\gamma^2, \tag{68}$$

where

$$\gamma^2 \stackrel{\text{def}}{=} \gamma \circ_{(g)} \gamma.$$

It is more convenient to write the equation (67) in the form

$$D^{(g)}a = -\delta a + \partial^{(g)}a + \frac{1}{i\hbar}[r, a]^{(g)}, \tag{69}$$

where $r \in \mathcal{V}_g \otimes \Lambda^1$.

From the theorem 3.8

$$D^{(g)}a = \partial^{(g)}a + \frac{1}{i\hbar}[\omega_{ij}X^i dq^j + r, a]^{(g)}. \tag{70}$$

Comparing (67) and (70) we see that

$$\gamma = \omega_{ij}X^i dq^j + r. \tag{71}$$

The curvature Ω defined by (68) takes now the form

$$\Omega = -\frac{1}{2}\omega_{j_1 j_2} dq^{j_1} \wedge dq^{j_2} + R - \delta r + \partial^{(g)}r + \frac{1}{i\hbar}r^2. \tag{72}$$

The connection $\tilde{\Gamma}$ is abelian iff Ω is the central 2− form. It means that (see (72))

$$\delta r = R + \partial^{(g)}r + \frac{1}{i\hbar}r^2 + \Delta, \tag{73}$$

where Δ is some central 2− form. From the Bianchi identity

$$D^{(g)}\Omega = 0 \tag{74}$$

and the assumption that the connection $\tilde{\Gamma}$ is abelian we get that

$$d\Omega = 0. \tag{75}$$

and, consequently,

$$d\Delta = 0. \tag{76}$$

The most general form of r satisfying (73) is

$$r = \delta^{-1}R + \delta^{-1}\Delta + \delta^{-1}\left(\partial^{(g)}r + \frac{1}{i\hbar}r^2\right) + \delta S, \tag{77}$$

where $S = \delta^{-1}r$ is an arbitrary 0−form. From the formula (77) we can see that $\delta^{-1}r = S - S_{00}$, where S_{00} does not contain X^i. There exists also a bijection from the set of $S's$ with $S_{00} = 0$ onto the set of 0− forms $\delta^{-1}r$, where r fulfills (77).

We must prove that every solution of (77) for an arbitrary S such that $S_{00} = 0$ fulfills the condition (73).

From (77)

$$r - \delta S = \delta^{-1}\delta r = \delta^{-1}\left(R + \Delta + \partial^{(g)}r + \frac{1}{i\hbar}r^2\right),$$

so

$$\delta r = R + \Delta + \partial^{(g)}r + \frac{1}{i\hbar}r^2 + F, \tag{78}$$

where F is an arbitrary 2− form such that

$$\delta^{-1}F = 0. \tag{79}$$

If we prove that $F = 0$, then we get the equation (73). It means that every r fulfilling (77) is the solution of (73).

$$\delta F \stackrel{(78)}{=} -\delta R - \delta\Delta - \delta(\partial^{(g)}r) - \delta\frac{1}{i\hbar}r^2. \tag{80}$$

Δ is the central form so

$$\delta\Delta = 0. \tag{81}$$

Consider the last expression in (80).

$$\delta(\frac{1}{i\hbar}r^2) \stackrel{(46)}{=} \frac{1}{i\hbar}(\delta r \circ_{(g)} r - r \circ_{(g)} \delta r).$$

From the definition 3.5 of the commutator we get

$$\delta(\frac{1}{i\hbar}r^2) = \frac{1}{i\hbar}[\delta r, r]^{(g)} = -\frac{1}{i\hbar}[r, \delta r]^{(g)}. \tag{82}$$

Now

$$\delta F = -\delta R - \delta(\partial^{(g)} r) + \frac{1}{i\hbar}[r, \delta r]^{(g)}. \tag{83}$$

From the theorem 4.10

$$\delta(\partial^{(g)} r) = -\partial^{(g)}(\delta r) \tag{84}$$

and

$$\begin{aligned}
\delta R &= \tfrac{1}{4} R_{i_1 i_2, j_1 j_2} X^{i_2} dq^{i_1} \wedge dq^{j_1} \wedge dq^{j_2} \\
&\quad + \tfrac{1}{4} R_{i_1 i_2, j_1 j_2} X^{i_1} dq^{i_2} \wedge dq^{j_1} \wedge dq^{j_2} \\
&= \tfrac{1}{2} R_{i_1 i_2, j_1 j_2} X^{i_1} dq^{i_2} \wedge dq^{j_1} \wedge dq^{j_2} \\
&= \tfrac{1}{6} \Big(R_{i_1 i_2 j_1 j_2} X^{i_1} dq^{i_2} \wedge dq^{j_1} \wedge dq^{j_2} \\
&\quad + R_{i_1 j_2 i_2 j_1} X^{i_1} dq^{j_2} \wedge dq^{i_2} \wedge dq^{j_1} \\
&\quad + R_{i_1 j_1 j_2 i_2} X^{i_1} dq^{j_1} \wedge dq^{j_2} \wedge dq^{i_2} \Big) .
\end{aligned} \tag{85}$$

Using the identity ([12])

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0 \tag{86}$$

we get

$$\delta R = 0. \tag{87}$$

From (80)–(87) we can see that

$$\delta F = \partial^{(g)}(\delta r) + \frac{1}{i\hbar}[r, \delta r]^{(g)}. \tag{88}$$

This form of the equation is inconvenient for analysis. Let us modify (88).

$$\begin{aligned}
\partial^{(g)} R &= \tfrac{1}{4} R_{i_1 i_2 j_1 j_2; l} X^{i_1} X^{i_2} dq^{j_1} \wedge dq^{j_2} \wedge dq^l \\
&= \tfrac{1}{12} \Big(R_{i_1 i_2 j_1 j_2; l} X^{i_1} X^{i_2} dq^{j_1} \wedge dq^{j_2} \wedge dq^l \\
&\quad + R_{i_1 i_2 l j_1; j_2} X^{i_1} X^{i_2} dq^l \wedge dq^{j_1} \wedge dq^{j_2} \\
&\quad + R_{i_1 i_2 j_2 l; j_1} X^{i_1} X^{i_2} dq^{j_2} \wedge dq^l \wedge dq^{j_1} \Big) .
\end{aligned}$$

From the Bianchi identity

$$R_{nikl;m} + R_{nimk;l} + R_{nilm;k} = 0 \tag{89}$$

we can see that

$$\partial^{(g)} R = 0. \tag{90}$$

The formula (90) is of course the Bianchi identity for the curvature R (compare [12]) and it can be proved from (62) and (64) directly.

Using (63) we get

$$\partial^{(g)}(\partial^{(g)} r) = \frac{1}{i\hbar}[R, r]^{(g)}. \tag{91}$$

From the theorem 4.9

$$\partial^{(g)}(\frac{1}{i\hbar} r^2) = \frac{1}{i\hbar}[\partial^{(g)} r, r]^{(g)}. \tag{92}$$

It is easy to see that

$$[r, r^2]^{(g)} = 0. \tag{93}$$

The $2-$ form Δ is closed so

$$\partial^{(g)} \Delta = 0. \tag{94}$$

From (90)–(94)

$$\begin{aligned} \delta F &= \partial^{(g)} \left(\delta r - R - \Delta - \partial^{(g)} r - \frac{1}{i\hbar} r^2 \right) \\ &\quad + \frac{1}{i\hbar} \left[r, \delta r - R - \Delta - \partial^{(g)} r - \frac{1}{i\hbar} r^2 \right]^{(g)} \\ &= \partial^{(g)} F + \frac{1}{i\hbar}[r, F]^{(g)}. \end{aligned} \tag{95}$$

The most general shape $2-$ form F which fulfills (95) is

$$F = \delta^{-1} \left(\partial^{(g)} F + \frac{1}{i\hbar}[r, F]^{(g)} \right). \tag{96}$$

There is no element $\delta\delta^{-1} F$ because as we noticed earlier (the formula (79)), $\delta^{-1} F = 0$.

The equation (77) determines the abelian connection iff $F = 0$ is the only one solution of (96).

Let $r(l, n)$ denotes the component of r which contain $\hbar^l$ and n elements of X^i.

The equation (96) has the only one iterative solution $F = 0$ if $r(0, 0) = r(0, 1) = 0$.

The $1-$form r which fulfills this condition and the equation (77) gives us the abelian connection.

Let us solve the equation (77):

$$r = \delta^{-1}R + \delta^{-1}\Delta + \delta^{-1}(\partial^{(g)}r + \frac{1}{i\hbar}r^2) + \delta S.$$

The first term is $r(0,2)$. Hence,

1. Δ is the central form $\Delta(l,0)$ where $l > 0$. We can see that the operator δ is the only one component of r which *does not contain* $\hbar$. The same result has been presented in the paper [13];

2. We must put $S(0,1) = S(0,2) = 0$. because of the conditions $r(0,0) = r(0,1) = 0$. We remind that S determines $\delta^{-1}r$ if $S_{00} = 0$.

Now *the formula (77) is iterative and the first nonvanishing element* $r(0,2)$ *is determined by* S.

For the Weyl ordering when $\alpha = 1, S = 0$ i $\Delta = 0$,

$$r = \frac{1}{8}R_{ijkl}X^iX^jX^kdq^l + \frac{1}{20}\partial_m R_{ijkl}X^iX^jX^kX^mdq^l + \cdots \tag{97}$$

Next components also contain $\hbar$.

5. The generalized Moyal product "$*_{(g)}$" on the phase space $T^*\mathcal{M}$

Using the abelian connection $\tilde{\Gamma}$ constructed in the previous section we can define the generalized Moyal product "$*_{(g)}$" on the curved phase space $T^*\mathcal{M}$.

Theorem 5.11 *The set of forms* $a \in \mathcal{V}_g \otimes \Lambda$ *such that* $D^{(g)}a = 0$ *is the subalgebra* $\mathcal{V}_g^D \otimes \Lambda$ *of the algebra* $\mathcal{V}_g \otimes \Lambda$. ∎

Let $\sigma(a)$ be a "projection" of a $0-$ form $a \in \mathcal{V}_g$ onto the phase space $T^*\mathcal{M}$ *i.e.*, mapping which assigns to a its component that does not contain $X^1, \ldots, X^{2m}$. This mapping is unique.

The following theorem holds

Theorem 5.12 *Every formal series* $A \in \Pi(\hbar, C^\infty(T^*\mathcal{M}))$ *on the symplectic manifold* $T^*\mathcal{M}$ *defines uniquely the element* $\sigma^{-1}(A)$ *of the boundle* $\mathcal{V}_g^D$ *such that* $\sigma(\sigma^{-1}(A)) = A$.

Proof

The condition $D^{(g)}a = 0$ means (compare (69)) that

$$D^{(g)}a = -\delta a + \partial^{(g)}a + \frac{1}{i\hbar}[r,a]^{(g)} = 0, \tag{98}$$

so

$$\delta a = \partial^{(g)} a + \frac{1}{i\hbar}[r,a]^{(g)}. \tag{99}$$

a is $0-$form so from the theorem 3.5 the most general form of the solution (99) is

$$a = A + \delta^{-1}\delta a = A + \delta^{-1}\left(\partial^{(g)} a + \frac{1}{i\hbar}[r,a]^{(g)}\right), \tag{100}$$

where $A \stackrel{\mathrm{def}}{=} \sigma(a)$.

This equation is similar to (77). Per analogy we can see that this is an iterative equation with the solution determined by A.

The only problem is to prove that every solution of (100) fulfills (99).

$$a - A = \delta^{-1}\delta a \stackrel{(100)}{=} \delta^{-1}\left(\partial^{(g)} a + \frac{1}{i\hbar}[r,a]^{(g)}\right)$$

so

$$\delta a = \partial^{(g)} a + \frac{1}{i\hbar}[r,a]^{(g)} + F, \tag{101}$$

where F is a $0-$ form such that

$$\delta^{-1} F = 0. \tag{102}$$

We will prove that $F = 0$.

From (101)

$$F = \delta a - \partial^{(g)} a - \frac{1}{i\hbar}[r,a]^{(g)} = -D^{(g)} a. \tag{103}$$

We do not know if $D^{(g)} a = 0$ because the $0-$ form a is only the solution of (100). But $\tilde{\Gamma}$ is the abelian connection. Therefore

$$D^{(g)} F = D^{(g)}(D^{(g)} a) = 0. \tag{104}$$

It means that

$$\delta F = \partial^{(g)} F + \frac{1}{i\hbar}[r,F]^{(g)}. \tag{105}$$

From (102) we deduce that any $F(k,0)$ ($k \geq 0$) vanishes. From the previous section we know that the first nonvanishing component of r is $r(0,2)$. Calculating

$$F = \delta^{-1}\left(\partial^{(g)} F + \frac{1}{i\hbar}[r,F]^{(g)}\right) \tag{106}$$

we find that $F=0$. $F=0$ is the solution of (105) too. It means that every a defined by (100) fulfills (99). Thus we have proved that for every formal series $A \in \Pi(\hbar, C^{\infty}(T^*\mathcal{M}))$ exists one and only one $a \in \mathcal{V}_g^D$. ■

For the Weyl algebra $\mathcal{V}$ ($\alpha = 1$), if r is defined by (97),

$$\begin{aligned} a \ = \ & A + \partial_i A X^i + \tfrac{1}{2}\partial_i\partial_j A X^i X^j + \tfrac{1}{6}\partial_i\partial_j\partial_k A X^i X^j X^k \\ & -\frac{1}{24} R_{ijkl}\omega^{lm}\partial_m A X^i X^j X^k + \cdots . \end{aligned} \tag{107}$$

Now we can define the generalized Moyal product on the curved phase space.

Definition 5.13 *Let $A, B \in \Pi(\hbar, C^{\infty}(T^*\mathcal{M}))$ be formal series defined on the phase space $T^*\mathcal{M}$.* **The generalized Moyal product** *"$*_{(g)}$" of the formal series A, B is defined by*

$$A *_{(g)} B \stackrel{\mathrm{def}}{=} \sigma\left(\sigma^{-1}(A) \circ_{(g)} \sigma^{-1}(B)\right). \tag{108}$$

The reasons which have decided that the product defined above is the generalization of the Moyal product given in the phase space R^{6n} are the following ones:

1. For the phase space R^{6n} the product defined by the formula (108) is the generalized Moyal product defined in our previous paper [2];

2. In the classical limit $\hbar \to 0$ the product $*_{(g)}$ is the usual multiplication of formal series;

3. The product $*_{(g)}$ is nonabelian but associative;

4. The definition of the multiplication (108) is invariant under the proper Darboux transformations.

The nontrivial is only the proof that for the phase space R^{6n} the product defined by (108) is equivalent to the product (1).

Let us cover the space space R^{6n} by the cartesian chart. In this chart $\Gamma_{ijk} = 0$ so $\Gamma = 0$. It means (see (48) and (59)) that

$$\partial^{(g)} a = \partial a = da. \tag{109}$$

If we put $S=0$ and $\Delta = 0$ if $R=0$, then we get $r=0$. From (100)

$$\sigma^{-1}(A) - A = \delta^{-1} d(\sigma^{-1}(A)). \tag{110}$$

The solution of (110) reads

$$\sigma^{-1}(A) = \sum_{i_1=0}^{\infty}\cdots\sum_{i_{6n}=0}^{\infty}\frac{1}{i_1!\cdots i_{6n}!}\frac{\partial^{i_1}}{\partial(q^1)^{i_1}}\cdots\frac{\partial^{i_{6n}}}{\partial(q^{6n})^{i_{6n}}}A(q^1,\ldots,q^{6n}) \times\underbrace{X^1\cdots X^1}_{i_1\ \text{times}}\cdots\underbrace{X^{6n}\cdots X^{6n}}_{i_{6n}\ \text{times}}, \tag{111}$$

where $(q^1,\ldots,q^{6n})$ are coordinates on the phase space R^{6n}. But this formula is the expansion of A in the Taylor series at the point $(q^1,\ldots,q^{6n})$.

So $\sigma^{-1}(A) = A(q^1+X^1,\ldots,q^{6n}+X^{6n})$ and the same for $\sigma^{-1}B$. For every $1\leq k\leq 6n$

$$\frac{\partial}{\partial q^k}A(q^1+X^1,\ldots,q^{6n}+X^{6n}) = \frac{\partial}{\partial X^k}A(q^1+X^1,\ldots,q^{6n}+X^{6n}). \tag{112}$$

Now

$$\sigma^{-1}(A)\circ_{(g)}\sigma^{-1}(B) = \alpha^{-1}\left(-\frac{1}{2}\hbar\vartheta^{rs}\frac{\partial^2}{\partial X^r\partial X^s}\right) \times\sum_{k=0}^{\infty}\left(\frac{i\hbar}{2}\right)^k\frac{1}{k!}\omega^{i_1j_1}\cdots\omega^{i_kj_k} \times\frac{\partial^k A_g(q^1+X^1,\ldots,q^{6n}+X^{6n})}{\partial X^{i_1}\cdots\partial X^{i_k}}\ \frac{\partial^k B_g(q^1+X^1,\ldots,q^{6n}+X^{6n})}{\partial X^{j_1}\cdots\partial X^{j_k}}. \tag{113}$$

We can change the derivatives with respect to X^i in the formula(112) by the derivatives with respect to q^i. We see that

$$\sigma^{-1}(A)\circ_{(g)}\sigma^{-1}(B) = \alpha^{-1}\left(-\frac{1}{2}\hbar\vartheta^{rs}\frac{\partial^2}{\partial q^r\partial q^s}\right) \times\sum_{k=0}^{\infty}\left(\frac{i\hbar}{2}\right)^k\frac{1}{k!}\omega^{i_1j_1}\cdots\omega^{i_kj_k} \times\frac{\partial^k A_g(q^1+X^1,\ldots,q^{6n}+X^{6n})}{\partial q^{i_1}\cdots\partial q^{i_k}}\ \frac{\partial^k B_g(q^1+X^1,\ldots,q^{6n}+X^{6n})}{\partial q^{j_1}\cdots\partial q^{j_k}}, \tag{114}$$

so

$$A *_{(g)} B = \sigma\left(\sigma^{-1}(A)\circ_{(g)}\sigma^{-1}(B)\right) = \alpha^{-1}\left(-\frac{1}{2}\hbar\vartheta^{rs}\frac{\partial^2}{\partial q^r\partial q^s}\right) \times\sum_{k=0}^{\infty}\left(\frac{i\hbar}{2}\right)^k\frac{1}{k!}\omega^{i_1j_1}\cdots\omega^{i_kj_k}\frac{\partial^k A_g(q^1,\ldots,q^{6n})}{\partial q^{i_1}\cdots\partial q^{i_k}}\frac{\partial^k B_g(q^1,\ldots,q^{6n})}{\partial q^{j_1}\cdots\partial q^{j_k}}. \tag{115}$$

Using the definition of the Poisson bracket we get finally

$$= \alpha^{-1}\left(-\frac{1}{2}\hbar\vartheta^{rs}\frac{\partial^2}{\partial q^r\partial q^s}\right)A_g \exp\left(\frac{i\hbar}{2}\overset{\leftrightarrow}{\mathcal{P}}\right)B_g, \tag{116}$$

Note that we assumed that S *and* Δ *vanish for* $R=0$. ■

6. Conclusions

The construction of the generalized Moyal product on the phase space $T^*\mathcal{M}$ shows that the symplectic connection on $T^*\mathcal{M}$ is necessary to define the product "$*_{(g)}$"

The advantage of this formalism is that one uses the classical phase space of the system. The quantum mechanics appears as a deformation of the classical system with respect to the Planck constant $\hbar$ so the Weyl- Wigner-Moyal formalism seems to be very helpful for approximate calculations.

From mathematical point of view all orderings (determined by the series α) are equivalent. But they all give different physical results *i.e.*, different eigenvalues and the equation of motion.

This paper closes the series of papers devoted to mathematical foundations of the quantum dynamics of the spinless nonrelativistic particle. The main problems still waiting for the solutions are:

1. What kind of generalized function does represent the information about the state of the system or what is the equivalent of the density matrix on the phase spaces ?

2. Is quantum mechanics the only one deformation of the classical mechanics or there are many possibilities ? If it is the only one which of the series α is the physical one.

3. What does the integral form of the equation (108) look like? The answer to this question is important when the generalized function of the state is not the smooth function. This problem is probably closely related to the fact that one cannot find the solution of the eigenvalue equation for a rotator within the Moyal formalism (see for example [7]).

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