STOCHASTIC FLOWS DRIVEN BY NON-MARKOVIAN DICHOTOMIC NOISE*

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(Received January 6, 1999)

Master equations governing the probability densities of the stochastic processes driven by explicitly non-Markovian dichotomic noise are derived and discussed. Such equations form the infinite hierarchy of equations for different probability densities or correlation functions defined at more and more time points. Approximations introducing decoupling of such hierarchies are constructed. Applications to special cases: random telegraph process and linear relaxation show that one class of approximations leads in these cases to correct (exact) results.

PACS numbers: 05.40.+j, 02.50.Ey

1. Introduction

In applications of stochastic theory to various physical, chemical, biological, *etc.* problems, the driving noises are assumed, almost without exception, to be Markovian stochastic processes. However, in real systems where the noise originates (at least partially) from the averaging out of very many fast variables [1], we may expect that system variables form a kind of hierarchies, in which the "higher-level" variables are driven by "lower-level" ones, the latter acting as driving stochastic processes. On the other hand, it is well-known that *(i)* a stochastic flow $\dot{X}(t)$ driven by Markovian white noise is a correlated process, which may act as a colored noise, and that *(ii)* almost any stochastic flow driven by a colored noise, even Markovian one, is a non-Markovian process by itself. Therefore in many cases the Markovianity of the driving process (*e.g.* of the internal fluctuations) is but an idealization. On the other hand, non-Markovian stochastic processes are more difficult to deal with than Markovian ones. This seems to be one of the reasons

* This work was partially supported by the Polish KBN grant No 2 P03B 209 08.

why in most of applications so far it is the Markovian processes which have been used as the driving noises. Only very recently a few papers have been published which deal with non-Markovian driving, either explicitly [2-5] or implicitly [6-8]. Implicitly non-Markovian seem to be also harmonic or quasi-monochromatic noises [9], interrupted Gaussian white noise [10], so-called real noise [11] and composite noises [12] (at least for some realizations of these processes). In the latter papers the non-Markovianity of the driving noise neither has been discussed, nor its implications on the behavior of driven processes worked out. Besides, specific properties of non-Markovian processes constructed as stochastic flows driven by Markovian noises [13], formal properties of stationary non-Markovian reversible measures [14], non-Markovian Brownian motion [15], and non-Markovian oscillatory system [16] were discussed. Again, these non-Markovian processes have not been used as driving noises for other stochastic flows.

In the preceding paper [17] the present author proposed a systematic theory of explicitly non-Markovian dichotomic noise (DN) with exponential damping of the memory, and of its "white" limits. In Ref. [17] the general properties of such noises have been found, together with preliminary discussion of the properties of processes driven by such non-Markovian noises. It was shown in the subsequent papers [18] that the behavior of the relaxation processes driven by this noise exhibits some unexpected features and is distinctly different from that of the process driven by Markovian DN. In this paper we are going to discuss in more detail the master equations governing the behavior of the probability densities describing the stochastic flows driven by non-Markovian DN. This investigation is motivated by the obvious observation that the non-Markovian (and in fact Markovian, too) stochastic process can be used as a working model of the noise driving some physical (chemical, biological, ...) process only when there is a workable — exact or approximate — scheme of calculation of quantities of interest, one of the latter being the probability functions describing the process under consideration.

The asymmetric dichotomic noise (DN) $\xi(t)$, called also the random telegraph signal, is the random two-state process with zero mean:

$$\xi(t) \in \{\Delta_1, -\Delta_2\}\,, \quad \xi^2(t) = \Delta^2 + \Delta_0\,\xi(t)\,, \quad \langle \xi(t) \rangle = 0\,, \tag{1.1}$$

where $\Delta^2 = \Delta_1\Delta_2$, $\Delta_0 = \Delta_1 - \Delta_2$.

The specific, non-Markovian dichotomic process considered in Ref. [17] is defined by the following non-Markovian master equation, fulfilled by every probability $\Pi(\xi,t) \equiv \Pi_{n+1}(\xi,t;\xi_1,t_1;\ldots;\xi_n,t_n)$, $n \geq 1$, $t \geq t_0 \equiv \max\{t_1,\ldots,t_n\}$:

$$\dot{\Pi}(\Delta_1,t) = -\dot{\Pi}(-\Delta_2,t) = -\int_{t_0}^{t} dt' K(t-t')[\lambda_1\Pi(\Delta_1,t') - \lambda_2\Pi(-\Delta_2,t')] \tag{1.2}$$

(overdot denotes d/dt). λ_j are parameters characterizing the process $\xi(t)$, which in the Markovian limit gain the interpretation of switching probabilities (per unit time) between states $\xi_1 = \Delta_1$ and $\xi_2 = -\Delta_2$.

The noise $\xi(t)$ will be fully defined when the initial conditions and the specific form of the kernel $K(\tau)$ are given. In preceding papers [12,17,18] and in the following the initial condition is the obvious relation (uniqueness condition):

$$\Pi_{n+1}(\xi,t=t_1;\xi_1,t_1;\ldots;\xi_n,t_n) = \delta_{\xi,\xi_1}\Pi_n(\xi_1,t_1;\ldots;\xi_n,t_n)\,, \tag{1.3}$$

and the kernel is assumed to contain both Markovian and non-Markovian contributions (the latter with exponentially damped memory):

$$K(t-t') = \gamma_0\delta(t-t') + \gamma_1\,\mathrm{e}^{-\nu(t-t')}\,. \tag{1.4}$$

However, part of the results obtained below remains valid for any form of the kernel $K(\tau)$.

In (1.4) the parameters γ_0 and γ_1 describe the relative contributions of Markovian and non-Markovian parts, and ν is the rate of damping of the non-Markovian memory. Two different kinds of transition from non-Markovian process $\xi(t)$ to Markovian one are possible, *viz.* *(i)* non-scaled transition of weights of both components:

$$\gamma_0 \to 1\,,\quad \gamma_1 \to 0\,,\quad \nu = \mathrm{const}\,, \tag{1.5}$$

and *(ii)* scaled transition of the memory time:

$$\gamma_1 = (1-\gamma_0)\nu\,,\quad \nu\to\infty\,,\quad \lim_{\nu\to\infty} K(t-t') = \delta(t-t')\,. \tag{1.6}$$

Both transitions are correct in the general case. However, when some approximations are being used, putting $\gamma_0 = 1$, $\gamma_1 = 0$ may lead to incorrect Markovian limit, whereas the procedure (1.6) will lead always to correct results.

Dichotomic noise described by Eqs. (1.2)–(1.4) is characterized by the following two-point correlation function:

$$\langle\xi(t_1)\xi(t_2)\rangle = \Delta^2\psi(|t_1 - t_2|)\,, \tag{1.7}$$

where

$$\psi(t) = \Gamma^{-1}[(\theta_1 - \nu)\,\mathrm{e}^{-\theta_1 t} - (\theta_2 - \nu)\,\mathrm{e}^{-\theta_2 t}]\,,$$

$$\theta_{1,2} = \tfrac{1}{2}(\nu + \gamma_0\Lambda \pm \Gamma)\,, \quad \Gamma = \sqrt{(\gamma_0\Lambda - \nu)^2 - 4\gamma_1\Lambda}\,, \quad \Lambda = \lambda_1 + \lambda_2\,. \tag{1.8}$$

Properties of the non-Markovian process $\xi(t)$ itself, and of related distributions, averages, *etc.* are given in detail in Refs. [12] and [17]. Here we shall consider the equations for probability densities describing the stochastic flows:

$$\dot{X} = f(X) + g(X)\xi(t)\,, \tag{1.9}$$

driven by the non-Markovian DN $\xi(t)$. It is to be noted that exact master equations describing the time dependence of probability densities of processes driven by dichotomic noises can be obtained only for the Markovian DN's [19,20]. In the non-Markovian case one must resort to approximations [12,17]. This paper is devoted to the systematic discussion of such approximations.

The rest of the paper is organized as follows: Section 2 contains general formulation of the hierarchy of master equations, whereas in Section 3 the approximations decoupling this hierarchy are considered. One family of these approximations is checked in Section 4 against the random telegraph process and against linear stochastic flows (relaxation processes), and it is shown — by comparison with exact solutions — that these approximations lead in these special cases to exact results either for the probability density $P(x,t)$, or at least for the first moments of $P(x,t)$. In Section 5 some final remarks are collected. The appendices list some properties of the functions of the dichotomic noise, definitions of several auxiliary functions (probability densities and correlation functions of higher order), the relations between these functions, and details of derivations of some formulas. Last Appendix is devoted to the detailed discussion of non-Markovianity of the dichotomic noise discussed in this paper.

2. Hierarchy of master equations

2.1. General formulation

We shall consider general one-dimensional stochastic flows (1.9). More general forms of the type of $\dot{X} = F(X,\xi(t))$, containing $\xi(t)$ in a nonlinear fashion, can be reduced to (1.9) by the use of the property (1.1).

We shall describe the flow (1.9) by the probability density $P(x,t)$ that at time interval $(t,t+dt)$ the value of the process $X(t)$ lies in the interval $(x,x+dx)$ and by the joint probability density $p(x,\xi_\alpha,t)$ that $X(t)\in(x,x+dx)$ and $\xi(t)=\xi_\alpha$, $\alpha=1,2$[1]:

$$P(x,t) \equiv \langle\delta(X(t,[\xi])-x)\rangle\,, \tag{2.1}$$
$$p_\alpha(t) \equiv p(x,\xi_\alpha,t)\equiv\langle\delta(X(t,[\xi])-x)\delta_{\xi(t),\xi_\alpha}\rangle\,. \tag{2.2}$$

The (Dirac) delta-function $\delta(X(t,[\xi])-x)$ is the probability density *for* k-th *realization*[2] *of the stochastic process* $\xi(t)$ that at time interval $(t,t+dt)$ the value of the process $X(t)$ lies in the interval $(x,x+dx)$, and the averaging is over all possible realizations of $\xi(t)$. Similarly, the (Kronecker) delta-function $\delta_{\xi(t),\xi_\alpha}$ is the probability for k-th realization of the stochastic process $\xi(t)$ that this process at time t is in the state ξ_α.

The standard method [19,20] leads to the following master equations for $p_\alpha(t)$ [17]:

$$\begin{aligned}\frac{\partial}{\partial t}p_\alpha(t) = &-\frac{\partial}{\partial x}[f(x)+\xi_\alpha g(x)]p_\alpha(t)-\varepsilon_\alpha\gamma_0[\lambda_1 p_1(t)-\lambda_2 p_2(t)]\\ &-\varepsilon_\alpha\gamma_1\int_{t_0}^{t}dt'\,\mathrm{e}^{-\nu(t-t')}[\lambda_1 h_1(t;t')-\lambda_2 h_2(t;t')]\,,\end{aligned} \tag{2.3}$$

where $\varepsilon_1=1$, $\varepsilon_2=-1$, and the auxiliary function $h_\beta(t,t')$, together with several other auxiliary probability densities and correlation functions which will be needed below, is defined in the Appendix A.

In the same way we find that the function $h_\alpha(t,t')$ fulfils the master equation (*cf.* (A5)):

$$\begin{aligned}\frac{\partial}{\partial t}h_\alpha(t,t') &= -\frac{\partial}{\partial x}f(x)h_\alpha(t,t')-\frac{\partial}{\partial x}g(x)\langle\delta(X(t,[\xi])-x)\xi(t)\delta_{\xi(t'),\xi_\alpha}\rangle\\ &= -\frac{\partial}{\partial x}[f(x)+\tfrac{1}{2}\Delta_0 g(x)]h_\alpha(t,t')\\ &\quad -D\frac{\partial}{\partial x}g(x)[{}^1k_{1\alpha}(t,t')-{}^1k_{2\alpha}(t,t')]\,,\end{aligned} \tag{2.4}$$

valid for $t>t'$ only. Here $2D=\Delta_1+\Delta_2$, and the function ${}^1k_{\beta\alpha}(t,t')$ is defined in the Appendix A. This function fulfils the master equation containing next higher-order auxiliary probability density, and so on. In general,

$$\begin{aligned}\frac{\partial}{\partial t}{}^m h_{\alpha\ldots}(t,t_m,\ldots)= &-\frac{\partial}{\partial x}[f(x)+\tfrac{1}{2}\Delta_0 g(x)]^m h_{\alpha\ldots}(t,t_m,\ldots)\\ &-D\frac{\partial}{\partial x}g(x)\left[{}^m k_{1\alpha\ldots}(t,t_m,\ldots)-{}^m k_{2\alpha\ldots}(t,t_m,\ldots)\right],\end{aligned} \tag{2.5}$$

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[1] To avoid confusion, we assume here the following convention: Greek subscripts will denote states of a given realization of two-state process, Roman subscripts — different realizations or different time moments of the same realization of the two-state process.

[2] *i.e.*, given definite series of switches between $+\Delta_1$ and $-\Delta_2$ at given specific times $0<t_1<t_2<\ldots<t_j<\ldots<t$.

$$\begin{aligned}\frac{\partial}{\partial t}{}^{m}k_{\beta\alpha\ldots}(t,t_m,\ldots) &= -\frac{\partial}{\partial x}[f(x)+\xi_\beta g(x)]^{m}k_{\beta\alpha\ldots}(t,t_m,\ldots)\\ &\quad -\varepsilon_\beta\int\limits_{t_m}^{t} dt_{m+1}\,K(t-t_{m+1})\Big[\lambda_1{}^{m+1}h_{1\alpha\ldots}(t,t_{m+1},\ldots)\\ &\quad -\lambda_2{}^{m+1}h_{2\alpha\ldots}(t,t_{m+1},\ldots)\Big]\,, \qquad (2.6)\end{aligned}$$

where the relations from Appendices A and B have been used, and where $t \geq t_m \geq \ldots$.

One may write down the above *hierarchy* of master equations in several different equivalent forms. Most convenient for our present purposes is the symmetric parametrization by functions R_m, Q_m defined in the Appendix A. This parametrization includes explicitly the master equation for the main function of interest, *i.e.* the probability density $P(x,t)$. By the use of the definitions and of the relations (A5), (A6) we get (*cf.* also Appendix B):

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}[f(x)P(x,t)+g(x)Q(x,t)]\,, \qquad (2.7)$$

$$\begin{aligned}&\frac{\partial}{\partial t}Q(x,t) - \frac{\partial}{\partial x}[f(x)+\Delta_0 g(x)]Q(x,t) - \Delta^2\frac{\partial}{\partial x}g(x)P(x,t)\\ &= \gamma_0\Lambda Q(x,t) - \gamma_1\Lambda\int\limits_{t_0}^{t} dt'\,\mathrm{e}^{-\nu(t-t')}\;R_1(x,t,t')\,, \qquad (2.8)\end{aligned}$$

and, in general,

$$\frac{\partial}{\partial t}R_m = -\frac{\partial}{\partial x}[f(x)R_m + g(x)Q_m]\,, \qquad (2.9)$$

$$\begin{aligned}\frac{\partial}{\partial t}Q_m &= -\frac{\partial}{\partial x}[f(x)+\Delta_0 g(x)]Q_m - \Delta^2\frac{\partial}{\partial x}g(x)R_m - \gamma_0\Lambda Q_m\\ &\quad -\gamma_1\Lambda\int\limits_{t_m}^{t} dt_{m+1}\;\mathrm{e}^{-\nu(t-t_{m+1})}\,R_{m+1}(t,t_{m+1},\ldots,t_1)\,. \qquad (2.10)\end{aligned}$$

The solutions of the above equations will provide the probability densities ${}^{m}h$, ${}^{m}k$, Q_m, R_m, as functions of x and of their first time argument, *i.e.*, of the actual time t only, and for $t \geq t_m$ only, whereas the integrals in Eqs. (2.3), (2.6), (2.8), and (2.10) (written for lower-order densities) require the knowledge of the densities ${}^{m}h$, R_m, as functions of their second time argument, t_m, for $t_{m-1} \leq t_m \leq t$. The master equations for t_m-dependence can be obtained in the same way as Eqs. (2.3)–(2.10). However, the non-Markovianity of the noise $\xi(t)$ implies that changes in the t_m-dependence influence t-dependence. This implies in turn that, when differentiating with respect to t_m, the argument t is to be written as $t = t_m + \tau$, with τ instead of t kept constant. Otherwise (*i.e.*, keeping t=const) incorrect results will be obtained (*cf.* Section 3.3).

Therefore we get:

$$\frac{\partial}{\partial t_m} R_m(t, t_m, t_{m-1}, \ldots) = -\frac{\partial}{\partial x}[f(x) R_m(t, t_m, t_{m-1}, \ldots) \\ + g(x) Q_m(t, t_m, t_{m-1}, \ldots)] - \gamma_o \Lambda R_m(t, t_m, t_{m-1}, \ldots) \\ -\gamma_1 \Lambda \int\limits_{t_{m-1}}^{t_m} dt' \; \mathrm{e}^{-(t-t')} R_m(t, t', t_{m-1}, \ldots) , \tag{2.11}$$

$$\frac{\partial}{\partial t_m} Q_m(t, t_m, t_{m-1}, \ldots) = -\frac{\partial}{\partial x}[f(x) + \Delta_0 g(x)] Q_m(t, t_m, t_{m-1}, \ldots) \\ -\Delta^2 \frac{\partial}{\partial x} g(x) R_m(t, t_m, t_{m-1}, \ldots) - 2\gamma_0 \Lambda Q_m(t, t_m, t_{m-1}, \ldots) \\ -\gamma_1 \Lambda \int\limits_{t_m}^{t} dt_{m+1} \; \mathrm{e}^{-\nu(t-t_{m+1})} R_{m+1}(t, t_{m+1}, \ldots) \\ -\gamma_1 \Lambda \int\limits_{t_{m-1}}^{t_m} dt' \; \mathrm{e}^{-\nu(t-t')} Q_m(t, t', t_{m-1}, \ldots) , \tag{2.12}$$

and analogous equations for ${}^m h$, ${}^m k$. In the same way the dependence of all these densities on $t_{m-1}, \ldots t_1$ can be obtained.

This means that for the non-Markovian case the standard procedure does not lead to a closed set of equations describing the probability densities of interest, but to an infinite hierarchy of equations (strictly speaking, to a branched set of such hierarchies). To obtain a workable scheme of calculation, this hierarchy must be decoupled by some approximation. In Ref. [17] we have proposed a simple approximation based on an *ansatz*. More systematic approximations will be discussed in the subsequent Section.

2.2. Differentiation theorem

The differentiation theorem for averages containing $\xi(t)$ can be obtained in the same way as Eqs. (2.8)–(2.10) (*cf.* also Appendix B). It reads:

$$\frac{\partial}{\partial t} \langle F(X(t;[\xi]), t) \xi(t) \rangle = \langle \xi(t) \frac{\partial}{\partial t} F(X(t;[\xi]), t) \rangle \\ -\Lambda \int\limits_{t_0}^{t} dt' K(t-t') \langle F(X(t;[\xi]), t) \xi(t') \rangle \langle \xi(t) \frac{\partial}{\partial t} F(X(t;[\xi]), t) \rangle . \tag{2.13}$$

Namely, the basic definition of averages gives:

$$\langle F(X(t;[\xi]),t)\xi(t)\rangle = \sum_{\alpha}\int\limits_{\mathcal{D}_x} dx\xi_\alpha F(x,t)\langle\delta(X(t;[\xi])-x)\delta_{\xi(t),\xi_\alpha}\rangle$$

$$= \int\limits_{\mathcal{D}_x} dxF(x,t)[\varDelta_1 p_1(x,t)-\varDelta_2 p_2(x,t)] = \int\limits_{\mathcal{D}_x} dxF(x,t)Q(x,t)\,, \quad (2.14)$$

and in the same way

$$\langle F(X(t;[\xi]),t)\xi(t')\rangle = \int\limits_{\mathcal{D}_x} dxF(x,t)R_1(t,t')\,. \quad (2.15)$$

In the above $\mathcal{D}_x$ denotes the domain of x, *i.e.* the domain of the stochastic process $X(t)$.

Eq. (2.13) is now obtained by differentiating (2.14), and taking into account (2.10), (2.15) and the following property:

$$\begin{aligned} &F(x,t)\tfrac{\partial}{\partial x}\{[f(x)+\varDelta_0 g(x)]Q(x,t)+\varDelta^2 g(x)P(x,t)\} \\ &= F(x,t)\tfrac{\partial}{\partial x}\langle\xi(t)\dot{X}(t)\delta(X(t,[\xi])-x)\rangle \\ &= -\langle\xi(t)\tfrac{\partial}{\partial t}\delta(X(t,[\xi])-x)F(X(t;[\xi]),t)\rangle + Q(x,t)\tfrac{\partial}{\partial t}F(x,t)\,. \quad (2.16)\end{aligned}$$

The theorem (2.13) is the generalization for the non-Markovian DN of the Shapiro–Loginov theorem [21] stating that for any (Markovian) exponentially correlated noise $\eta(t)$ of zero mean,

$$\tfrac{\partial}{\partial t}\langle F(X(t;[\eta]),t)\eta(t)\rangle = \langle\eta(t)\tfrac{\partial}{\partial t}\langle F(X(t;[\xi]),t)\rangle - \varLambda\langle F(X(t;[\eta]),t)\eta(t)\rangle\,. \quad (2.17)$$

The Shapiro–Loginov theorem proved to be very useful in treating linear stochastic problems [22-24], especially the systems of several linear kinetic equations [22], and the linear stochastic equations with time-dependent coefficients [24]. The application of differentiation theorem (2.13) will be demonstrated in the subsequent Section.

2.3. Hierarchy of telegrapher's equations

The hierarchy simplifies considerably for the random telegraph process:

$$\dot{X}(t) = \xi(t)\,, \quad (2.18)$$

i.e. when $f(x)=0$, $g(x)=1$. In this case the auxiliary functions Q_m can be easily eliminated and the m-th order master equations become the m-th order telegrapher's equation:

$$\left(\frac{\partial^2}{\partial t^2} + \gamma_0 \Lambda \frac{\partial}{\partial t} + \Delta_0 \frac{\partial^2}{\partial t \partial x} - \Delta^2 \frac{\partial^2}{\partial x^2}\right) R_m(x,t)$$

$$= \gamma_1 \Lambda \frac{\partial}{\partial x} \int\limits_{t_m}^{t} dt_{m+1}\ \mathrm{e}^{-\nu(t-t_{m+1})}\, R_{m+1}(x,t,t_{m+1},\ldots,t_1)\,, \qquad (2.19)$$

with $R_0(x,t) = P(x,t)$. In a similar way the hierarchy can be written for a linear process and for some other special cases.

3. Approximation schemes

To obtain a workable scheme of calculations, the hierarchy of master equations (2.5)–(2.6) or (2.9)–(2.10) must be decoupled in some way. There are two obvious systematic approximations which can be applied to achieve this goal: weak noise and short-memory expansions.

3.1. Weak-noise expansion

The condition for weak noise reads:

$$|\xi(t)| \ll 1\,, \quad i.e., \quad \Delta^2 \ll 1\,. \qquad (3.1)$$

Therefore, from Eqs. (A3), (A4):

$$Q_m \ll R_m\,, \quad R_{m+1} \ll R_m\,, \qquad (3.2)$$

which enables us to neglect the second term of the r.h.s. of Eq. (2.9) at the prescribed level of the hierarchy (which corresponds to the prescribed approximation order), so that we get in the m-th-order approximation:

$$R_m(t,t_m,\ldots) \approx \exp\left\{-\frac{\partial}{\partial x} f(x)(t-t_m)\right\} Q_{m-1}(t_m,t_{m-1},\ldots)\,. \qquad (3.3)$$

At the same level, we have, by virtue of Eqs. (A1), (A2), and (A5), equivalent approximations:

$${}^{m}k_\alpha \approx P_{st,\alpha}\, {}^{m}h_\alpha\,, \qquad (3.4)$$

which leads eventually to the m-th-order approximation:

$${}^{m}h_{\alpha\ldots}(t,t_m,\ldots) \approx \exp\{-\frac{\partial}{\partial x} f(x)(t-t_m)\}\ {}^{m-1}k_{\alpha\ldots}(t_m,t_{m-1},\ldots)\,. \qquad (3.5)$$

Zero-order approximation of this kind:

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}f(x)P(x,t)\,, \tag{3.6}$$

or equivalently

$$\frac{\partial}{\partial t}p_\alpha(t) = -\frac{\partial}{\partial x}f(x)p_\alpha(t)\,, \tag{3.7}$$

is equivalent to the deterministic description. First-order approximation gives:

$$\begin{aligned}\frac{\partial}{\partial t}p_\alpha(t) + \frac{\partial}{\partial x}[f(x)+\xi_\alpha g(x)]p_\alpha(t) &= -\varepsilon_\alpha\gamma_0[\lambda_1 p_1(t)-\lambda_2 p_2(t)]\\ -\varepsilon_\alpha\gamma_1\int\limits_{t_0}^{t} dt'\,\mathrm{e}^{-\nu(t-t')}\,\mathrm{e}^{-\frac{\partial}{\partial x}f(x)(t-t')}&[\lambda_1 p_1(t')-\lambda_2 p_2(t')]\,,\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\frac{\partial}{\partial t}Q(x,t) + \frac{\partial}{\partial x}[f(x)+\Delta_0 g(x)]Q(x,t) + \Delta^2\frac{\partial}{\partial x}g(x)P(x,t)\\ &= -\Lambda\gamma_0 Q(x,t) - \Lambda\gamma_1\int\limits_{t_0}^{t} dt' e^{-\nu(t-t')}\,\mathrm{e}^{-\frac{\partial}{\partial x}f(x)(t-t')}\,Q(x,t')\,,\end{aligned} \tag{3.9}$$

i.e., after removing the integral:

$$\begin{aligned}&\left[\nu+\frac{\partial}{\partial t}+\frac{\partial}{\partial x}f(x)\right]\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\Big(f(x)+\xi_\alpha g(x)\Big)\right]p_\alpha(t)\\ &= -\varepsilon_\alpha\left[\gamma_0\nu+\gamma_1+\gamma_0\frac{\partial}{\partial x}f(x)+\gamma_0\frac{\partial}{\partial t}\right][\lambda_1 p_1(t)-\lambda_2 p_2(t)]\,,\end{aligned} \tag{3.10}$$

$$\begin{aligned}&\left[\nu+\frac{\partial}{\partial t}+\frac{\partial}{\partial x}f(x)\right]\left\{\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\Big(f(x)+\Delta_0 g(x)\Big)+\gamma_0\Lambda\right]Q(x,t)\right.\\ &\left.+\Delta^2\frac{\partial}{\partial x}g(x)P(x,t)\right\} = -\gamma_1\Lambda Q(x,t)\,.\end{aligned} \tag{3.11}$$

The last of these equations is to be supplemented by Eq. (2.7). It will be shown below (Section 4) that this first-order approximation leads to correct (exact) results for linear flows. The same procedure can be applied at the m-th hierarchy level.

3.2. Short-memory expansion

Denote

$$W(x,t)\equiv\frac{\partial}{\partial t}Q(x,t)+\frac{\partial}{\partial x}[f(x)+\Delta_0 g(x)]Q(x,t)+\Delta^2\frac{\partial}{\partial x}g(x)P(x,t)+\gamma_0\Lambda Q(x,t). \tag{3.12}$$

Acting N times by the operator $(\nu + \partial/\partial t)$ on Eq. (2.8) we get:

$$\begin{aligned}&\left(\nu+\tfrac{\partial}{\partial t}\right)^N W(x,t)\\&= -\Lambda\gamma_1\left[\left(\nu+\tfrac{\partial}{\partial t}\right)^{N-1}+\left(\nu+\tfrac{\partial}{\partial t}\right)^{N-2}\tfrac{\partial}{\partial t}+\ldots+\tfrac{\partial^{N-1}}{\partial t^{N-1}}\right]Q(x,t)\\&-\gamma_1\Lambda\int_{t_0}^{t} dt'\, \mathrm{e}^{-\nu(t-t')}\ \tfrac{\partial^N}{\partial t^N}R_1(x,t,t')\,. \end{aligned}\tag{3.13}$$

In the short-memory limit: $\nu \equiv 1/\tau_m \to \infty$, and for $N \to \infty$, we get the short-memory expansion:

$$W(x,t) = -\Lambda\gamma_1 \sum_{n=0}^{\infty} \tau_m^{n+1}\, \tfrac{\partial^n}{\partial t^n} Q(x,t)\,, \tag{3.14}$$

or, with the scaling (1.6),

$$\begin{aligned}&\tfrac{\partial}{\partial t}Q(x,t) + \gamma_1\tau_m\Lambda\sum_{n=1}^{\infty}\tau_m^n\tfrac{\partial^n}{\partial t^n}Q(x,t)\\&= -\Lambda Q(x,t) - \tfrac{\partial}{\partial x}[f(x)+\Delta_0 g(x)]Q(x,t) - \Delta^2\tfrac{\partial}{\partial x}g(x)P(x,t)\,.\end{aligned}\tag{3.15}$$

However, the results obtained in Refs. [18] suggest that qualitatively the influence of non-Markovian DN with short memory is similar to that of Markovian DN. Therefore this expansion seems to be of little practical value.

3.3. Other approximations

Another type of seemingly obvious method of decoupling the hierarchy of master equations is to neglect the last (expressly non-Markovian) term in the hierarchy equation Eq. (2.10) (this is equivalent to putting $R_{m+1} = 0$ at the m-th order approximation). However, this leads to manifestly incorrect results even for the random telegraph process — we have checked this fact numerically up to 4-th order, obtaining both $P(x,t)$, and its second and fourth moments diverging strongly from exact results.

Another type of approximations, related to the *ansatz* approximation (4.17) of Ref. [17], can be obtained by neglecting the non-Markovian character of the process $\xi(t)$ in Eqs. (2.11)–(2.12). In this case we get:

$$\tfrac{\partial}{\partial t_m}S_m(t,t_m,\ldots) = -\Lambda\int_{t_{m-1}}^{t_m} dt'\, K(t-t')S_m(t,t',\ldots) \tag{3.16}$$

with solution:

$$
\begin{aligned}
&S_m(t, t_m, t_{m-1}, \ldots)\\
&= \psi(t_m - t_{m-1})[\Delta^2 S_{m-2}(t, t_{m-2}, \ldots) + \Delta_0 S_{m-1}(t, t_{m-1}, \ldots)]\,, \quad (3.17)
\end{aligned}
$$

where $S_m = R_m,\ Q_m$. Neither these approximations, nor more naive *ansatz* ones of the type of Eq. (4.17) of Ref. [17]:

$$
{}^{m+1}h_{\alpha\ldots}(t, t_{m+1}, t_m, \ldots) \approx \Delta^2 \psi(t - t_{m+1})\, {}^{m}h_{\alpha\ldots}(t_{m+1}, t_m, \ldots)\,, \quad (3.18)
$$

$$
R_{m+1}(t, t_{m+1}, t_m, \ldots) \approx \Delta^2 \psi(t - t_{m+1}) Q_m(t_{m+1}, t_m, \ldots)\,, \quad (3.19)
$$

give correct or near-to-correct results for simple flows considered in Section 4, although the second-order approximation (3.19) produces correct (up to numerical accuracy, at least) result for the second moment $\langle X^2(t)\rangle$, and reasonable results for $P(x,t)$, for the random telegraph process.

4. Special cases

In a few simplest cases the probability density $P(x,t)$, or at least its first moments, can be calculated directly and exactly. In this Section we shall discuss three such cases, the sole purpose being the demonstration that the approximation (3.11) leads in these cases to correct (*i.e.* identical with exact) results. For the sake of simplicity, only symmetric ($\Delta_0 = 0$) DN will be considered. Note that the solutions for the flows (4.3), (4.10), and (4.20) driven by Markovian DN, are well-known. The details of solutions for these flows driven by non-Markovian DN can be found in Refs. [17–18].

In the following we shall make use of the general expression for the characteristic function $T(k,t)$ of the stochastic process (1.9):

$$
T(k,t) = \int_{\mathcal{D}_x} dx\ \mathrm{e}^{ikx}\ P(x,t) = \left\langle\ \mathrm{e}^{ikX(t)}\ \right\rangle, \quad (4.1)
$$

where $\mathcal{D}_x$ denotes the domain of x, equal to the domain of the physical process $X(t)$. The above expression is related directly to the definition (2.1) of $P(x,t)$ by the well-known Fourier representation of the Dirac delta-function:

$$
\begin{aligned}
P(x,t) &= \frac{1}{2\Pi}\int_{-\infty}^{+\infty} dk\ \mathrm{e}^{-ikx}\,\langle \mathrm{e}^{ikX(t)}\rangle \quad \text{for } x \in \mathcal{D}_x\,,\\
&= 0 \qquad \text{otherwise}\,. \quad (4.2)
\end{aligned}
$$

4.1. Random telegraph process

Consider the symmetric ($\varDelta_0 = 0$) random telegraph process:

$$\dot{X}(t) = \xi(t)\,, \quad \mathcal{D}_x = (-\infty, +\infty)\,, \tag{4.3}$$

driven by non-Markovian DN $\xi(t)$. It was shown in Ref. [17] that the behavior of this process differs from that driven by Markovian DN: among others, the transients oscillate in wide range of noise parameters ν, $\varLambda$.

The average (4.1) for the process (4.3) can be calculated in a straightforward way [12] (*cf.* also Appendix C):

$$\langle \mathrm{e}^{ikX(t)} \rangle = \sum_{j=1}^{3} s_j \ \mathrm{e}^{z_j t}\,, \tag{4.4}$$

$$s_1 = (z_1 + \theta_1)(z_1 + \theta_2)/(z_1 - z_2)(z_1 - z_3)\,, \tag{4.5}$$

etc., and z_j are the solutions of Eq. (C6) (with $\alpha = ik$):

$$z^3 + (\theta_1 + \theta_2)z^2 + (\theta_1\theta_2 + k^2\varDelta^2)z + \nu k^2 \varDelta^2 = 0\,. \tag{4.6}$$

On the other hand, Eqs. (2.7) and (3.11) reduce in this case ($f = 0$, $g = 1$) to an equation which reads after Fourier transforming:

$$\Big(\frac{d^3}{dt^3} + A\frac{d^2}{dt^2} + B\frac{d}{dt} + C\Big)T(k,t) = 0\,, \tag{4.7}$$

$$A = \theta_1 + \theta_2\,, \quad B = \theta_1\theta_2 + k^2\varDelta^2\,, \quad C = \nu\varDelta^2 k^2\,, \tag{4.8}$$

the solution of which is just the exact result above for the following initial conditions:

$$T(k,0) = T_0(k)\,, \quad \dot{T}(k,0) = 0\,, \qquad \ddot{T}(k,0) = -k^2\varDelta^2 T_0(k)\,. \tag{4.9}$$

These conditions result from the obvious initial condition $Q(x,0) = 0$, from Eq. (2.8) differentiated with respect to t and from Eq. (2.9), and therefore contain no approximations.

4.2. Multiplicative linear relaxation

Consider now the stochastic flow

$$\dot{X}(t) = -aX(t) + \xi(t)X(t), \quad \mathcal{D}_x = [0, \infty)\,, \tag{4.10}$$

discussed in detail — in various contexts — in Refs. [18]. It was shown there that the behavior of the process (4.10) driven by non-Markovian DN differs significantly from that driven by Markovian DN: new transient most probable states may appear, average values may become non-monotonous functions of time, stochastic resonance may appear, *etc.*

In this case Eqs. (2.7) and (3.11) read:

$$\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}x\right)P(x,t) = -\frac{\partial}{\partial x}xQ(x,t)\,, \tag{4.11}$$

$$\left(\frac{\partial}{\partial t}+\nu-a\frac{\partial}{\partial x}x\right)\left[\left(\frac{\partial}{\partial t}+\gamma_0\Lambda-a\frac{\partial}{\partial x}x\right)Q(x,t)+\Delta^2\frac{\partial}{\partial x}xP(x,t)\right] = -\gamma_1\Lambda Q(x,t)\,. \tag{4.12}$$

Multiplying first of these equations by x, second by $x\frac{\partial}{\partial x}x$, and eliminating the function $Q(x,t)$, we get, after some rearrangements:

$$\begin{aligned}&\left\{\hat{D}^3 + (\theta_1+\theta_2)\hat{D}^2 + \left[\theta_1\theta_2 - \Delta^2\left(x\frac{\partial}{\partial x}\right)^2\right]\hat{D} - \nu\Delta^2\left(x\frac{\partial}{\partial x}\right)^2\right\}xP(x,t) = 0\,,\\ &\hat{D} = \frac{\partial}{\partial t} - ax\frac{\partial}{\partial x}\,.\end{aligned} \tag{4.13}$$

Assuming that $P(x,t)$ together with its first three x-derivatives goes to zero for $x\to\infty$ rapidly enough, *i.e.* that

$$\lim_{x\to\infty} x^m P(x,t) = 0 = \lim_{x\to\infty} x^m \frac{\partial^n}{\partial x^n}P(x,t)\,, \quad n=1,2,3, \quad m=0,1,2,\ldots \tag{4.14}$$

we get

$$\langle x^m(t)\rangle = \int\limits_0^\infty dx\, x^m\, P(x,t) = (-1)^n \int\limits_0^\infty dx\, x^{m-1}\left(x\frac{\partial}{\partial x}\right)^n x\, P(x,t)\,. \tag{4.15}$$

Therefore, multiplying Eq. (4.12) by x^{m-1} and integrating over x we get the kinetic equation for m-th moment of $P(x,t)$:

$$\begin{aligned}&\left[\left(\frac{d}{dt}+ma\right)^3 + (\theta_1+\theta_2)\left(\frac{d}{dt}+ma\right)^2\right.\\ &\left.+(\theta_1\theta_2 - m\Delta^2)\left(\frac{d}{dt}+ma\right) - m\nu\Delta^2\right]\langle X^m(t)\rangle = 0\,.\end{aligned} \tag{4.16}$$

On the other hand, by direct integration of the stochastic equation (4.10) we obtain:

$$\langle X^m(t)\rangle = \mathrm{e}^{-mat}\, x_0^m\left\langle \exp\left[m\int\limits_0^t dt'\xi(t')\right]\right\rangle = \mathrm{e}^{-mat}\, x_0^m \sum_{j=1}^{3} s_{mj}\,\mathrm{e}^{z_{mj}t}\,, \tag{4.17}$$

with s_{mj} given by Eq. (4.5), and z_{mj} given by Eq. (C6) with $\alpha = m$. It is easy to check that the above expression is the solution of Eq. (4.16) with initial conditions equivalent to conditions (4.9):

$$\langle X^m(0)\rangle = x_0^m\,, \quad \frac{d}{dt}\langle X^m(t)\rangle\Big|_{t=0} = -max_0^m\,,$$
$$\frac{d^2}{dt^2}\langle X^m(t)\rangle\Big|_{t=0} = -m(\varDelta^2 + a^2)x_0^m\,. \tag{4.18}$$

Therefore Eq. (4.13) correctly reproduces all moments of $P(x,t)$, *i.e.*, reproduces correctly $P(x,t)$ for linear process with multiplicative noise. $P(x,t)$ itself can be calculated directly by noting that Eq. (4.10) can be transformed into Eq. (4.3) by putting $y = \ln x + at$. Therefore $P(x,t)$ for multiplicative linear process, $x \in [0,\infty)$ is given by $P(y,t)$, $y \in (-\infty,+\infty)$, *i.e.*,

$$P(x,t) = \frac{1}{2\varPi}\int_{-\infty}^{+\infty} dk\ \mathrm{e}^{-ik(\ln x + at)}\,T(k,t)\,, \quad x \geq 0\,, \tag{4.19}$$

with $T(k,t)$ given by Eq. (4.1)–(4.4).

4.3. Additive linear relaxation

Consider the linear relaxation driven by additive symmetric non-Markovian DN:

$$\dot{X}(t) = -aX(t) + \xi(t)\,, \quad \mathcal{D}_x = (-\infty,\infty)\,, \tag{4.20}$$

discussed in Ref. [17]. Again, properties of this flow differ significantly from those of such process driven by Markovian DN.

In this case the probability density cannot be calculated directly by the procedures of Appendix C. It is possible, however, to calculate first few moments of the process (4.20). The first moment is just

$$x(t) = \langle X(t)\rangle = \mathrm{e}^{-at}\,x_0\,, \tag{4.21}$$

the second one:

$$\begin{aligned} x_2(t) &= \langle X^2(t)\rangle = x^2(t) + \int\limits_0^t dt_1 \int\limits_0^t dt_2\ \mathrm{e}^{-a(2t-t_1-t_2)}\langle \xi(t_1)\xi(t_2)\rangle \\ &= x^2(t) + 2\Delta^2 \int\limits_0^t dt_1 \int\limits_0^{t_1} dt_2\ \mathrm{e}^{-a(2t-t_1-t_2)}\,\psi(t_1-t_2) \\ &= x^2(t) + \frac{\Delta^2(a+\nu)}{a(a+\theta_1)(a+\theta_2)} + \frac{2\Delta^2(a-\nu)}{a(a-\theta_1)(a-\theta_2)}\,\mathrm{e}^{-2at} \\ &\quad + \frac{2\Delta^2}{\Gamma}\left[\frac{\nu-\theta_1}{a^2-\theta_1^2}\,\mathrm{e}^{-(a+\theta_1)t} - \frac{\nu-\theta_2}{a^2-\theta_2^2}\,\mathrm{e}^{-(a+\theta_2)t}\right], \end{aligned} \tag{4.22}$$

etc. On the other hand, such quantities can be also calculated as follows. Define:

$$x(t) = \langle X(t)\rangle\,, \quad y(t) = \langle X(t)\xi(t)\rangle\,, \quad z(t) = x_2(t) = \langle X^2(t)\rangle\,. \tag{4.23}$$

This leads to the set of kinetic equations (*cf.* Appendix B):

$$\dot{x} = -ax\,, \tag{4.24}$$

$$\dot{z} = -2az + 2y\,, \tag{4.25}$$

$$\dot{y} = \langle \dot{X}\xi + X\dot{\xi}\rangle = -(a+\gamma_0\Lambda)y + \Delta^2 - \gamma_1\Lambda \int\limits_0^t dt'\,\mathrm{e}^{-\nu(t-t')}\langle X(t)\xi(t')\rangle\,. \tag{4.26}$$

The approximation (3.2) gives:

$$\begin{aligned} \langle X(t)\xi(t')\rangle &= \int\limits_0^\infty dx\,x\,R_1(x,t,t') \\ &\approx \int\limits_0^\infty dx\,x\ \mathrm{e}^{a(t-t')\frac{\partial}{\partial x}x}\,Q(x,t')\,, \end{aligned} \tag{4.27}$$

$$= \mathrm{e}^{a(t-t')\frac{\partial}{\partial x}x}\langle X(t')\xi(t')\rangle\,. \tag{4.28}$$

Note that the interpretation of the form (4.28) is given by (4.27). Assuming that $Q(x,t)$ together with all its logarithmic derivatives vanishes sufficiently rapidly at $x\to\infty$:

$$\lim_{x\to\infty}\left(x\frac{\partial}{\partial x}\right)^m x\,Q(x,t) = 0, \quad m = 0,1,2,\dots\,, \tag{4.29}$$

we get ($\tau = t - t'$):

$$\begin{aligned}
\mathrm{e}^{a\tau \frac{\partial}{\partial x} x} \langle X(t')\xi(t')\rangle &= \int_0^\infty dx\, x\; \mathrm{e}^{a\tau \frac{\partial}{\partial x} x}\, Q(x,t') \\
&= \sum_{n=0}^{\infty} \frac{(a\tau)^n}{n!} \int_0^\infty dx \left(x \frac{\partial}{\partial x}\right)^n x Q(x,t') \\
&= \sum_{n=0}^{\infty} \frac{(a\tau)^n}{n!} (-1)^n \int_0^\infty dx\, x Q(x,t') \\
&= \mathrm{e}^{-a\tau} \langle X(t')\xi(t')\rangle \,.
\end{aligned} \tag{4.30}$$

Therefore,

$$\dot{y} = -(a + \gamma_0 \Lambda) y + \Delta^2 - \gamma_1 \Lambda \int_0^t dt'\, \mathrm{e}^{-(\nu+a)(t-t')}\, y(t')\,, \tag{4.31}$$

with the solution:

$$y(t) = C_0 + C_1\, \mathrm{e}^{-\mu_1 t} + C_2\, \mathrm{e}^{-\mu_2 t}\,, \tag{4.32}$$

$$\begin{aligned}
\mu_j &= a + \theta_j\,, \qquad\qquad C_0 = \frac{\Delta^2(\nu + a)}{(a+\theta_1)(a+\theta_2)}\,, \\
C_1 &= -\frac{\Delta^2}{\Gamma}\, \frac{\theta_1 - \nu}{a+\theta_1}\,, \quad C_2 = \frac{\Delta^2}{\Gamma}\, \frac{\theta_2 - \nu}{a+\theta_2}\,.
\end{aligned} \tag{4.33}$$

This result substituted into Eq. (4.25) for z leads to the correct expression for the second moment, Eq. (4.22), which proves that the approximation (3.2) leads to the correct results for the linear relaxation driven by non-Markovian additive DN, at least up to second moment of the probability density.

5. Final remarks

As we have mentioned in Section 1, part of the results obtained in this paper do not depend on the detailed form of the kernel $K(\tau)$ in the general non-Markovian master equations. These are, especially: the general form of the differentiation theorem, Eq. (2.13), and general forms of the hierarchy of master equations, Eqs. (2.5)–(2.6). Other forms — *viz.* Eqs. (2.3), (2.10), (2.12), (2.19), the approximations (3.12), and like, can be easily cast into the form containing general kernel $K(\tau)$. On the other hand, the specific results are true only for the kernel (1.4) — esp. Section 3.2, the equations with removed time integral, *e.g.* Eqs. (3.10)–(3.11), (4.7) (4.12), (4.13), (4.16), *etc.*, the averages (4.4), (4.17), (4.22), and like.

For the derivation of formulas (4.4)–(4.6) (and the formulas of Appendix C) crucial is the formula (C2) (Eq. (3.11) of Ref. [17]). Assumptions about initial conditions and about behavior of $P(x,t)$, $Q(x,t)$ for $x \to \infty$ seem to be justified by the fact that they lead to correct (*i.e.* identical with exact) results. The same can be said about the radius of convergence of the weak noise expansion, which is difficult to estimate otherwise.

Minimal requirements for the approximate equations seem to be: (*i*) they should be simple enough to enable practical applications (calculations); (*ii*) they should lead to correct or almost-correct results in simpler cases and/or in well-defined limits. These requirements seem to be satisfied by approximation (3.10)–(3.11), or its higher-order generalizations (3.3),(3.5). All $m = 1$ approximations of this type lead to correct (exact) results for the random telegraph process and for linear stochastic processes. As we have mentioned above, other types of approximations, although similar at first sight to those generated by (3.3)–(3.5), lead to manifestly incorrect results.

The approximations (3.10)–(3.11) have been checked against simplest stochastic flows only. For nonlinear kinetic equations these approximations may turn out to be not so satisfying. Nevertheless, the dichotomic noises are powerful tools mainly for linear systems [22–24]. Therefore the approximations proposed in this paper seem to be of practical significance.

Appendix A

Definitions of and relations between auxiliary functions

We introduce the following auxiliary and higher-order distributions (densities), which will be of use below, for hierarchy of master equations describing the non-Markovian case[3]:

$$
\begin{aligned}
{}^m h_{\beta\ldots\alpha}(t,t_m,\ldots t_1) &= \langle \delta(X(t,[\xi])-x)\delta_{\xi(t_m),\xi_\beta}\ldots\delta_{\xi(t_1),\xi_\alpha}\rangle\,, &\text{(A.1)}\\
{}^m k_{\gamma\beta\ldots\alpha}(t,t_m,\ldots t_1) &= \langle \delta(X(t,[\xi])-x)\delta_{\xi(t),\xi_\gamma}\delta_{\xi(t_m),\xi_\beta}\ldots\delta_{\xi(t_1),\xi_\alpha}\rangle, &\text{(A.2)}\\
R_m(t,t_m,\ldots t_1) &= \langle \delta(X(t,[\xi])-x)\xi(t_m)\ldots\xi(t_1)\rangle\,, &\text{(A.3)}\\
Q_m(t,t_m,\ldots t_1) &= \langle \delta(X(t,[\xi])-x)\xi(t)\xi(t_m)\ldots\xi(t_1)\rangle\,, &\text{(A.4)}
\end{aligned}
$$

$t \geq t_m \geq \cdots \geq t_1$, and $R_0 = P(t) = {}^0h$, $Q_0 = Q(t)$, ${}^1h_\alpha = h_\alpha$.

[3] To keep the notation short, the explicit indication of the dependence of these functions on x, $\xi_\alpha \ldots$, *etc.*, is omitted.

The definition (1.1) of DN implies that

$$\delta_{\xi(t),\xi_\alpha} = P_{st,\alpha} + \frac{\varepsilon_\alpha}{2D}\xi(t)\,, \quad P_{st,\alpha} = P_1(\xi_\alpha) = 1 - \frac{\Delta_\alpha}{2D}\,, \tag{A.5}$$

$$\begin{aligned}\xi(t) &= 2D\Big[\delta_{\xi(t),\Delta_1} - \frac{\Delta_2}{2D}\Big] = -2D\Big[\delta_{\xi(t),-\Delta_2} - \frac{\Delta_1}{2D}\Big] \\ &= \frac{1}{2}\Delta_0 + D\Big[\delta_{\xi(t),\Delta_1} - \delta_{\xi(t),-\Delta_2}\Big]\,,\end{aligned} \tag{A.6}$$

where $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, $D = (\Delta_1 + \Delta_2)/2$. This leads to the following relations (sum rules) between these various probability densities:

$$p_1(t) + p_2(t) = h_1(t,t') + h_2(t,t') = P(t)\,, \tag{A.7}$$

$${}^1k_{1,\beta}(t,t') + {}^1k_{2,\beta}(t,t') = h_\beta(t,t')\,, \quad {}^1k_{\beta,1}(t,t') + {}^1k_{\beta,2}(t,t') = p_\beta(t)\,, \tag{A.8}$$

$${}^m h_{1\beta\ldots\alpha}(t,t_m,\ldots t_1) + {}^m h_{2\beta\ldots\alpha}(t,t_m,\ldots t_1) = {}^{m-1}h_{\beta\ldots\alpha}(t,t_{m-1},\ldots t_1)\,, \tag{A.9}$$

$${}^m k_{1\beta\ldots\alpha}(t,t_m,\ldots t_1) + {}^m k_{2\beta\ldots\alpha}(t,t_m,\ldots t_1) = {}^m h_{\beta\ldots\alpha}(t,t_m,\ldots t_1)\,, \tag{A.10}$$

$${}^0k_\alpha = p_\alpha(t)\,, \quad {}^0h = P(t)\,, \tag{A.11}$$

$$\Delta_1 p_1(t) - \Delta_2 p_2(t) = Q(t)\,, \tag{A.12}$$

$$\Delta_1 h_1(t,t') - \Delta_2 h_2(t,t') = R_1(t,t')\,, \tag{A.13}$$

$$\Delta_1 {}^1k_{1\alpha}(t,t') - \Delta_2 {}^1k_{2\alpha}(t,t') = P_{st,\alpha}Q(t) + \frac{\varepsilon_\alpha}{2D}Q_1(t,t')\,, \tag{A.14}$$

$$\Delta_1 {}^1k_{\alpha 1}(t,t') - \Delta_2 {}^1k_{\alpha 2}(t,t') = P_{st,\alpha}R_1(t,t') + \frac{\varepsilon_\alpha}{2D}Q_1(t,t')\,, \tag{A.15}$$

$$p_\alpha(t) = P_{st,\alpha}P(t) + \frac{\varepsilon_\alpha}{2D}Q(t)\,, \tag{A.16}$$

$$h_\alpha(t,t') = P_{st,\alpha}P(t) + \frac{\varepsilon_\alpha}{2D}\,R_1(t,t')\,. \tag{A.17}$$

We have also the following *boundary conditions*:

$${}^1h_\alpha(t',t') = p_\alpha(t')\,, \quad {}^1k_{\beta\alpha}(t',t') = \delta_{\beta\alpha}p_\alpha(t')\,, \tag{A.18}$$

$$R_1(t',t') = Q(t')\,, \quad Q_1(t',t') = \Delta^2 P(t') + \Delta_0 Q(t')\,, \tag{A.19}$$

$$R_m(t,t,t_{m-1}\ldots t_1) = Q_{m-1}(t,t_{m-1}\ldots t_1)\,, \tag{A.20}$$

$$Q_m(t,t,t_{m-1}\ldots t_1) = \Delta^2 R_{m-1}(t,t_{m-1}\ldots t_1) + \Delta_0 Q_{m-1}(t,t_{m-1}\ldots t_1)\,. \tag{A.21}$$

These functions may serve for calculation of various averages — *cf.* (2.14), (2.15).

Appendix B

Derivation of master equations

Master equations of Section 2 can be also derived by the method due to Haken [25]:

$$\begin{aligned}\frac{\partial}{\partial t}P(x,t) &= \frac{\partial}{\partial t}\langle\delta(X(t;[\xi])-x)\rangle = \left\langle \frac{\partial}{\partial X(t)}\delta(X(t;[\xi])-x)\dot{X}(t)\right\rangle \\ &= -\frac{\partial}{\partial x}\left\langle\delta(X(t;[\xi])-x)\dot{X}(t)\right\rangle , \end{aligned} \qquad (B.1)$$

$$\begin{aligned}\frac{\partial}{\partial t}p_1(x,t) &= \left\langle \frac{\partial}{\partial X(t)}\delta(X(t;[\xi])-x)\dot{X}(t)\delta_{\xi(t),\Delta_1}\right\rangle + \left\langle\delta(X(t;[\xi])-x)\frac{d}{dt}\delta_{\xi(t),\Delta_1}\right\rangle \\ &= -\frac{\partial}{\partial x}\left\langle\delta(X(t;[\xi])-x)\dot{X}(t)\delta_{\xi(t),\Delta_1}\right\rangle \\ &\quad -\int_{t_0}^{t} dt' K(t-t')\left\langle\delta(X(t;[\xi])-x)\left[\lambda_1\delta_{\xi(t'),\Delta_1}-\lambda_2\delta_{\xi(t'),-\Delta_2}\right]\right\rangle , \end{aligned} \qquad (B.2)$$

which leads directly to Eqs. (2.3) and (2.7). Here use has been made of the well-known properties of the Dirac delta-functions. In particular, the distribution $\frac{\partial}{\partial x}\delta(X(t)-x)f(x)$ is equivalent to the distribution $f(X(t))\frac{\partial}{\partial x}$ $\delta(X(t)-x)$: multiply both distributions by a trial function $q(x)$ and integrate (by parts) over a small interval around $x = X(t)$; in both cases the result is $-f(X)[dq(X)/dX]$, which proves the equivalence.

Besides, in the same manner we have:

$$\frac{\partial}{\partial t}\langle\delta(X(t;[\xi])-x)\xi(t)\rangle = -\frac{\partial}{\partial x}\langle\delta(X(t;[\xi])-x)\dot{X}(t)\xi(t)\rangle + \langle\delta(X(t;[\xi])-x)\dot{\xi}(t)\rangle, \qquad (B.3)$$

which, compared with Eqs. (2.8), (2.3), (B10), (A4) and (A5) gives

$$\langle\delta(X(t;[\xi])-x)\dot{\xi}(t)\rangle = -\Lambda\int_{t_0}^{t} dt' K(t-t')\langle\delta(X(t;[\xi])-x)\xi(t')\rangle . \qquad (B.4)$$

The above relation can be generalized to averages containing arbitrary function of time (*cf.* also Eq. (2.13)):

$$\frac{\partial}{\partial t}\langle F(t)\xi(t)\rangle = \langle\dot{F}(t)\xi(t)\rangle - \Lambda\int_{t_0}^{t} dt' K(t-t')\langle F(t)\xi(t')\rangle . \qquad (B.5)$$

Appendix C

Derivation of the formula (4.4)

For the sake of completeness, we present here the derivation of the averages of the type of (4.4), given in Ref. [12].

For symmetric DN ($\Delta_0 = 0$):

$$\begin{aligned}
A(t,t_0;\alpha) &= \left\langle \exp\left[\alpha\int_{t_0}^{t} dt'\xi(t')\right]\right\rangle = \left\langle \exp\left[\alpha\int_{0}^{\tau} dt'\xi(t'+t_0)\right]\right\rangle \\
&= 1+\sum_{n=2}^{\infty}\frac{\alpha^n}{n!}\int_0^\tau dt_1\ldots\int_0^\tau dt_n\langle\xi(t_1+t_0)\ldots\xi(t_n+t_0)\rangle \\
&= 1+\sum_{n=2}^{\infty}\alpha^n\int_0^\tau dt_1\int_0^{t_1} dt_2\ldots\int_0^{t_{n-1}} dt_n\langle\xi(t_1+t_0)\xi(t_2+t_0)\ldots\xi(t_n+t_0)\rangle \\
&= 1+\sum_{m=1}^{\infty}\alpha^{2m}\Delta^{2m}\int_0^\tau ds_1\int_0^{s_1} dt_1\psi(s_1-t_1)\ldots\int_0^{t_{m-1}} ds_m\int_{t_0}^{s_m} dt_m\psi(s_m-t_m).
\end{aligned} \tag{C.1}$$

In the above, the following property (Eq. (3.11) of Ref. [17]) of averages of the products of symmetric DN's (both non-Markovian and Markovian[4] has been used:

$$\langle\xi(t_1)\ldots\xi(t_n)\rangle = \Delta^2\psi(t_1-t_2)\langle\xi(t_3)\ldots\xi(t_n)\rangle\,, \quad t_1\geq t_2\geq\ldots\geq t_n\,. \tag{C.2}$$

Using the Laplace transform and its well-known properties [26] we get, subsequently:

$$\begin{aligned}
\hat{\phi}_1(z) &= \int_0^\infty d\tau\, \mathrm{e}^{-z\tau}\int_0^\tau dt_1\int_0^{t_1} dt_2\,\psi(t_1-t_2) = \frac{1}{z^2}\hat{\psi}(z) \\
&= \frac{z+\nu}{z^2(z+\theta_1)(z+\theta_2)}\,,
\end{aligned} \tag{C.3}$$

[4] This property, and some other, resemble those of Markovian DN. This is the result of the initial conditions (1.3) used in Ref. [17]. Other properties — *e.g.* the form of two-point correlation function, Eq. (1.7) — are distinctly non-Markovian. Especially, the conditional probabilities do not satisfy the Smoluchowski–Chapman–Kolmogorov functional equation [20,27,28].

$$\int_0^\infty d\tau\, \mathrm{e}^{-z\tau} \int_0^\tau ds_1 \int_0^{s_1} dt_1\, \psi(s_1-t_1) \int_0^{t_1} ds_2 \int_0^{s_2} dt_2\, \psi(s_2-t_2)$$
$$= \int_0^\infty d\tau\, \mathrm{e}^{-z\tau}\, \frac{1}{z} \int_0^\tau dt_1\, \psi(\tau-t_1)\phi_1(t_1) = \frac{1}{z^3}[\hat{\psi}(z)]^2 \qquad \text{(C.4)}$$

etc., which leads eventually to:

$$\begin{aligned}\hat{A}(z) &= \frac{1}{z} + \frac{1}{z}\sum_{m=1}^{\infty}\left[\alpha^2\Delta^2\hat{\psi}(z)/z\right]^m = \frac{1}{z-\alpha^2\Delta^2\hat{\psi}(z)} \\ &= \frac{(z+\theta_1)(z+\theta_2)}{(z-z_1)(z-z_2)(z-z_3)}\,,\end{aligned} \qquad \text{(C.5)}$$

where z_j are the solutions of the cubic equation:

$$z^3 + (\theta_1+\theta_2)z^2 + (\theta_1\theta_2 - \alpha^2\Delta^2)z - \nu\alpha^2\Delta^2 = 0\,. \qquad \text{(C.6)}$$

The inverse Laplace transform of the above gives the formula (4.4)–(4.5).

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