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TWO HYPOTHETIC STERILE NEUTRINOS WHICH WANT TO MIX WITH ν_e AND ν_μ

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It is argued that the observed deficit of solar and atmospheric neutrinos can be explained by neutrino oscillations $\nu_e \to \nu_s$ and $\nu_\mu \to \nu_s'$ involving two hypothetic sterile neutrinos ν_s and ν_s' (blind to all Standard–Model interactions). They are keen to mix nearly maximally with ν_e and ν_μ, respectively, to form neutrino mass states ν_1, ν_4 and ν_2, ν_5. Our argument is presented in the framework of a model of fermion "texture" formulated previously, which implies the existence of two sterile neutrinos beside the three conventional.

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1. Introduction

The recent findings [1] of Super-Kamiokande atmospheric-neutrino experiment brought to us the important message that the observed deficit of atmospheric ν_μ's seems to be really caused by neutrino oscillations, related to nearly maximal mixing of ν_μ with another neutrino. This may be ν_τ or, alternatively, a new sterile neutrino (blind to all Standard Model interactions). The ν_e neutrino is here excluded from being a mixing partner of ν_μ by the negative result of CHOOZ long-baseline reactor experiment [2] which found no evidence for the disappearance modes of $\bar{\nu}_e$, in particular $\bar{\nu}_e \to \bar{\nu}_\mu$, in a parameter region overlapping the range of $\sin^2 2\theta_{\rm atm}$ and $\Delta m^2_{\rm atm}$ observed in the Super-Kamiokande experiment.

The survival probability for ν_μ, when analized experimentally in two-flavor form

$$P\left(\nu_\mu \to \nu_\mu\right) = 1 - \sin^2 2\theta_{\rm atm} \sin^2\left(1.27 \Delta m^2_{\rm atm}\, L/E\right) \,, \tag{1}$$

leads to the parameters [1]

$$\sin^2 2\theta_{\rm atm} = O(1) \sim 0.82 \;\text{ to }\; 1 \tag{2}$$

and

$$\Delta m^2_{\rm atm} \sim (0.5 \;\; \text{to} \;\; 6) \times 10^{-3} \;\; \text{eV}^2 \tag{3}$$

at the 90% confidence level (note that the value $\Delta m^2_{\rm atm} \sim 5 \times 10^{-3}$ eV^2 corresponds to the lower limit of the previous Kamiokande estimate of $\Delta m^2_{\rm atm}$ [3]). If ν_τ is responsible for this nearly maximal mixing of ν_μ, then the disappearance probability for ν_μ in the mode $\nu_\mu \to \nu_\tau$ is

$$P\left(\nu_\mu \to \nu_\tau\right) = \sin^2 2\theta_{\rm atm} \sin^2\left(1.27 \Delta m^2_{\rm atm}\, L/E\right) \,. \tag{4}$$

In the present paper, we conjecture that it is rather a sterile neutrino (denoted here by ν'_s) which is responsible for such a nearly maximal mixing of ν_μ(whether it is not or is ν_τ constitutes a crucial point of our conjecture which, unfortunately, is not at the moment easy to decide experimentally [1]). We conjecture moreover that another sterile neutrino (denoted by ν_s) mixes nearly maximally with ν_e, causing the observed deficit of solar ν_e's. In such a way, we introduce a unified picture of neutrino oscillations as being related to nearly maximal mixing of two sterile neutrinos ν_s and ν'_s with ν_e and ν_μ, respectively. Of course, this mixing of ν_s and ν'_s is not forbidden by the weak isospin I_3 and weak hypercharge Y of ν_e and ν_μ , as the conservation of these weak charges is spontaneously broken, except for their combination $Q \equiv I_3 + Y/2$ (equal to zero for ν_e and ν_μ). We should like also to remark that the sterile neutrinos ν_s and ν'_s, interacting only gravitionally, would be responsible for the existence of a Standard Model–inactive fraction of the dark matter.

Note that the existence of just two sterile neutrinos (blind to all Standard Model interactions), beside three families of Standard Model–active leptons and quarks, turns out to be natural in the model of lepton and quark "texture" we develop since some time [4,5] (*cf.* Eqs. (A.15) in Appendix). In this model, all neutrinos are Dirac particles having both lefthanded and righthanded parts.

For the Standard Model–active neutrinos $\nu_e\,, \nu_\mu\,, \nu_\tau$, charged leptons $e^-\,, \mu^-\,, \tau^-$, up quarks $u\,, c\,, t$ and down quarks $d\,, s\,, b$ we came to a proposal [5] (*cf.* Eq. (A.10) in Appendix) of unified algebraic structure of their mass matrices $\left(M^{(f)}_{ij}\right)$ $(f = \nu\,, e\,, u\,, d)$ in the three-dimensional family space $(i\,, j = 1\,, 2\,, 3)$. In the case of leptons $(f = \nu\,, e)$, this proposal reads

$$\left(M^{(f)}_{ij}\right) = \frac{1}{29}\begin{pmatrix} \mu^{(f)}\varepsilon^{(f)\,2} & 2\alpha^{(f)}e^{i\varphi^{(f)}} & 0 \\ 2\alpha^{(f)}e^{-i\varphi^{(f)}} & 4\mu^{(f)}(80+\varepsilon^{(f)\,2})/9 & 8\sqrt{3}\alpha^{(f)}e^{i\varphi^{(f)}} \\ 0 & 8\sqrt{3}\alpha^{(f)}e^{-i\varphi^{(f)}} & 24\mu^{(f)}(624+\varepsilon^{(f)\,2})/25 \end{pmatrix} . \tag{5}$$

Here, $\mu^{(f)}$, $\varepsilon^{(f)\,2}$, $\alpha^{(f)}$ and $\varphi^{(f)}$ denote real constants to be determined from the present and future experimental data for lepton masses and mixing parameters ($\mu^{(f)}$ and $\alpha^{(f)}$ are mass-dimensional).

For charged leptons, when assuming that the off-diagonal elements of the mass matrix $\left(M_{ij}^{(e)}\right)$ given in Eq. (5) can be treated as a small perturbation of its diagonal terms, we calculate in the lowest (quadratic) perturbative order in $\alpha^{(e)}/\mu^{(e)}$ [5]:

$$
\begin{aligned}
m_\tau &= \frac{6}{125}\left(351 m_\mu - 136 m_e\right) \\
&\quad + \frac{216\mu^{(e)}}{3625}\left(\frac{111550}{31696 + 29\varepsilon^{(e)\,2}} - \frac{487}{320 - 5\varepsilon^{(e)\,2}}\right)\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2 , \\
\varepsilon^{(e)\,2} &= \frac{320 m_e}{9m_\mu - 4m_e} + O\left[\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\right] , \\
\mu^{(e)} &= \frac{29}{320}\left(9m_\mu - 4m_e\right) + O\left[\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\right]\mu^{(e)} .
\end{aligned}
\tag{6}
$$

When the experimental m_e and m_μ [6] are used as inputs, Eqs. (6) give [5]

$$
\begin{aligned}
m_\tau &= \left[1776.80 + 10.2112\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\right]\,\mathrm{MeV} , \\
\varepsilon^{(e)\,2} &= 0.172329 + O\left[\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\right] , \\
\mu^{(e)} &= 85.9924\,\mathrm{MeV} + O\left[\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\right]\mu^{(e)} .
\end{aligned}
\tag{7}
$$

We can see that the predicted value of m_τ agrees very well with its experimental figure $m_\tau^{\rm exp} = 1777.00^{+0.30}_{-0.27}$ MeV [6], even in the zero-order perturbative calculation. To estimate $\left(\alpha^{(e)}/\mu^{(e)}\right)^2$, we take this experimental figure as another input. Then,

$$
\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2 = 0.020^{+0.029}_{-0.020} , \tag{8}
$$

so it is not inconsistent with zero.

The unitary matrix $\left(U^{(e)}_{ij}\right)$, diagonalizing the mass matrix $\left(M^{(e)}_{ij}\right)$ according to the relation $U^{(e)\,\dagger}\, M^{(e)}\, U^{(e)} = \mathrm{diag}(m_e\,,\, m_\mu\,,\, m_\tau)$, assumes in the lowest (quadratic) perturbative order in $\alpha^{(e)}/\mu^{(e)}$ the form

$$\left(U^{(e)}_{ij}\right) = \begin{pmatrix} 1-\frac{2}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2 & \frac{2}{29}\frac{\alpha^{(e)}}{m_\mu}e^{i\varphi^{(e)}} & 0 \\ -\frac{2}{29}\frac{\alpha^{(e)}}{m_\mu}e^{-i\varphi^{(e)}} & 1-\frac{2}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2-\frac{96}{841}\left(\frac{\alpha^{(e)}}{m_\tau}\right)^2 & \frac{8\sqrt{3}}{29}\frac{\alpha^{(e)}}{m_\tau}e^{i\varphi^{(e)}} \\ 0 & -\frac{8\sqrt{3}}{29}\frac{\alpha^{(e)}}{m_\tau}e^{-i\varphi^{(e)}} & 1-\frac{96}{841}\left(\frac{\alpha^{(e)}}{m_\tau}\right)^2 \end{pmatrix}, \tag{9}$$

where the small $\varepsilon^{(e)\,2}$ is neglected. Of course, in the limit of $\alpha^{(e)} \to 0$, we obtain $\left(U^{(e)}_{ij}\right) \to (\delta_{ij})$.

For neutrinos, we will assume in this paper that $\varepsilon^{(\nu)\,2}$ is very small and

$$\alpha^{(\nu)} = 0\,, \tag{10}$$

in contrast to the possibility of $\alpha^{(e)} \neq 0$ for charged leptons [*cf.* Eq. (8)]. Then, for conventional neutrinos $\left(U^{(\nu)}_{ij}\right) = (\delta_{ij})$ and so, ν_e, ν_μ, ν_τ can mix only by means of the trivial lepton CKM matrix $(V_{ij}) \equiv \left(\sum_k U^{(\nu)\,*}_{ki} U^{(e)}_{kj}\right) = \left(U^{(e)}_{ij}\right)$, what is a minor effect, vanishing in the limit of $\alpha^{(e)} \to 0$. Instead, allowing in this paper for the existence of two sterile neutrinos ν_s and ν'_s, we will extend the 3×3 neutrino mass matrix $\left(M^{(\nu)}_{ij}\right)$ $(i,j = 1,2,3)$, given through Eqs. (5) and (10), to a 5×5 neutrino mass matrix $\left(M^{(\nu)}_{IJ}\right)$ $(I,J = 1,2,3,4,5)$ with $M^{(\nu)}_{IJ} = M^{(\nu)\,*}_{JI}$. Explicitly, we will assume that

$$\left(M^{(\nu)}_{IJ}\right) = \begin{pmatrix} M^{(\nu)}_{11} & 0 & 0 & M^{(\nu)}_{14} & 0 \\ 0 & M^{(\nu)}_{22} & 0 & 0 & M^{(\nu)}_{25} \\ 0 & 0 & M^{(\nu)}_{33} & 0 & 0 \\ M^{(\nu)}_{41} & 0 & 0 & M^{(\nu)}_{44} & 0 \\ 0 & M^{(\nu)}_{52} & 0 & 0 & M^{(\nu)}_{55} \end{pmatrix}, \tag{11}$$

where $M^{(\nu)}_{11} = \mu^{(\nu)}\varepsilon^{(\nu)\,2}/29$, $M^{(\nu)}_{22} \simeq 320\mu^{(\nu)}/261$, $M^{(\nu)}_{33} \simeq 14976\mu^{(\nu)}/725$ due to Eq. (5), and $M^{(\nu)}_{44} \sim \mu^{(\nu)}\varepsilon^{(\nu)\,2}/7$, $M^{(\nu)}_{55} \sim 48\mu^{(\nu)}/7$ in consequence of Eqs. (A.19) and (A.20) (*cf.* Appendix). It will turn out that the matrix elements $M^{(\nu)}_{14} = M^{(\nu)\,*}_{41}$ and $M^{(\nu)}_{25} = M^{(\nu)\,*}_{52}$ lead to the mixing of neutrino flavor states ν_e with ν_s and ν_μ with ν'_s within neutrino mass states ν_1, ν_4 and ν_2, ν_5, respectively.

2. Neutrino mass states

The eigenvalues of the extended mass matrix $\left(M^{(\nu)}_{IJ}\right)$ given in Eq. (11) are Dirac masses of five neutrino mass states ν_1 , ν_2 , ν_3 , ν_4 , ν_5. They are

$$
\begin{aligned}
m_{\nu_1,\nu_4} &= \frac{M^{(\nu)}_{11}+M^{(\nu)}_{44}}{2} \mp \sqrt{\left(\frac{M^{(\nu)}_{11}-M^{(\nu)}_{44}}{2}\right)^2 + |M^{(\nu)}_{14}|^2}\,, \\
m_{\nu_3} &= M^{(\nu)}_{33}\,, \\
m_{\nu_2,\nu_5} &= \frac{M^{(\nu)}_{22}+M^{(\nu)}_{55}}{2} \mp \sqrt{\left(\frac{M^{(\nu)}_{22}-M^{(\nu)}_{55}}{2}\right)^2 + |M^{(\nu)}_{25}|^2}\,. \qquad (12)
\end{aligned}
$$

In Section 4, the masses m_{ν_1} and m_{ν_2} will turn out to be negative, what is irrelevant in the case of Dirac particles for which only masses squared are measurable (so, $|m_{\nu_1}|$ and $|m_{\nu_2}|$ will be the phenomenological masses of ν_1 and ν_2).

The corresponding 5×5 unitary matrix $\left(U^{(\nu)}_{IJ}\right)$, diagonalizing the mass matrix $\left(M^{(\nu)}_{IJ}\right)$ according to the equality $U^{(\nu)\,\dagger}\,M^{(\nu)}\,U^{(\nu)} = \mathrm{diag}(m_{\nu_1},\, m_{\nu_2},\, m_{\nu_3},\, m_{\nu_4},\, m_{\nu_5})$, takes the form

$$
\left(U^{(\nu)}_{IJ}\right) = \begin{pmatrix}
\frac{1}{\sqrt{1+Y^2}} & 0 & 0 & \frac{Y}{\sqrt{1+Y^2}}e^{i\varphi^{(\nu)}} & 0 \\
0 & \frac{1}{\sqrt{1+X^2}} & 0 & 0 & \frac{X}{\sqrt{1+X^2}}e^{i\varphi^{(\nu)\,\prime}} \\
0 & 0 & 1 & 0 & 0 \\
-\frac{Y}{\sqrt{1+Y^2}}e^{-i\varphi^{(\nu)}} & 0 & 0 & \frac{1}{\sqrt{1+Y^2}} & 0 \\
0 & -\frac{X}{\sqrt{1+X^2}}e^{-i\varphi^{(\nu)\,\prime}} & 0 & 0 & \frac{1}{\sqrt{1+X^2}}
\end{pmatrix}, \qquad (13)
$$

where $M^{(\nu)}_{14} = |M^{(\nu)}_{14}|\exp i\varphi^{(\nu)}$, $M^{(\nu)}_{25} = |M^{(\nu)}_{25}|\exp i\varphi^{(\nu)\,\prime}$ and

$$
\begin{aligned}
Y &= \frac{M^{(\nu)}_{11}-M^{(\nu)}_{44}}{2|M^{(\nu)}_{14}|} + \sqrt{1+\left(\frac{M^{(\nu)}_{11}-M^{(\nu)}_{44}}{2|M^{(\nu)}_{14}|}\right)^2} \\
&= \frac{M^{(\nu)}_{11}-m_{\nu_1}}{|M^{(\nu)}_{14}|} = -\frac{M^{(\nu)}_{44}-m_{\nu_4}}{|M^{(\nu)}_{14}|}\,, \\
X &= \frac{M^{(\nu)}_{22}-M^{(\nu)}_{55}}{2|M^{(\nu)}_{25}|} + \sqrt{1+\left(\frac{M^{(\nu)}_{22}-M^{(\nu)}_{55}}{2|M^{(\nu)}_{25}|}\right)^2} \\
&= \frac{M^{(\nu)}_{22}-m_{\nu_2}}{|M^{(\nu)}_{25}|} = -\frac{M^{(\nu)}_{55}-m_{\nu_5}}{|M^{(\nu)}_{25}|}\,. \qquad (14)
\end{aligned}
$$

The neutrino flavor states $\nu_\alpha \equiv \nu_e\,, \nu_\mu\,, \nu_\tau\,, \nu_s\,, \nu'_s$ (of which $\nu_e\,, \nu_\mu\,, \nu_\tau$, or rather their lefthanded parts, stand for the observed weak-interaction neutrino states and ν_s, ν'_s denote their unobserved sterile partners) are related to the neutrino mass states $\nu_I \equiv \nu_1\,, \nu_2\,, \nu_3\,, \nu_4\,, \nu_5$ through a five-dimensional unitary transformation

$$\nu_\alpha = \sum_J V^*_{J\alpha}\, \nu_J \tag{15}$$

with $\left(V^*_{J\alpha}\right) = \left(V_{\alpha J}\right)^\dagger$. Here,

$$V_{\alpha J} \equiv \sum_K U^{(\nu)\,*}_{K\alpha} U^{(e)}_{KJ} = \sum_k U^{(\nu)\,*}_{k\,\alpha} U^{(e)}_{k\,J} + U^{(\nu)\,*}_{4\alpha}\delta_{4J} + U^{(\nu)\,*}_{5\alpha}\delta_{5J}\,, \tag{16}$$

where $\left(U^{(e)}_{ij}\right)$ is the charged-lepton diagonalizing matrix given in Eq. (9) and

$$U^{(e)}_{i4} = 0 = U^{(e)}_{i5}\;,\;\; U^{(e)}_{4j} = 0 = U^{(e)}_{5j}\;,\;\; U^{(e)}_{44} = 1 = U^{(e)}_{55}\,. \tag{17}$$

The last equations follow from the fact that charged leptons get no sterile partners. Thus, from Eq. (16)

$$V_{\alpha\,j} = \sum_k U^{(\nu)\,*}_{k\,\alpha} U^{(e)}_{k\,j}\,,\; V_{\alpha 4} = U^{(\nu)\,*}_{4\alpha}\,,\; V_{\alpha 5} = U^{(\nu)\,*}_{5\alpha}\,. \tag{18}$$

Of course, the 5×5 unitary matrix $(V_{\alpha J})$ is a five-dimensional lepton counterpart of the familiar CKM matrix for quarks. The charged leptons $e^-\,, \mu^-\,, \tau^-$ are here counterparts of the up quarks $u\,, c\,, t$ (both with diagonalized mass matrix).

From Eqs. (18), with the use of Eqs. (13) and (9), we can calculate the matrix elements $V_{\alpha J}$ in the lowest (quadratic) perturbative order in $\alpha^{(e)}/\mu^{(e)}$. Writing for convenience $\alpha = I = 1, 2, 3, 4, 5$, we get

$$\begin{aligned}
V_{11} &= \left[1 - \frac{2}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2\right]\frac{1}{\sqrt{1+Y^2}}\,,\\
V_{22} &= \left[1 - \frac{2}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2 - \frac{96}{841}\left(\frac{\alpha^{(e)}}{m_\tau}\right)^2\right]\frac{1}{\sqrt{1+X^2}}\,,\\
V_{33} &= 1 - \frac{96}{841}\left(\frac{\alpha^{(e)}}{m_\tau}\right)^2\,,\\
V_{12} &= \frac{2}{29}\frac{\alpha^{(e)}}{m_\mu}\frac{1}{\sqrt{1+Y^2}}\,e^{i\varphi^{(e)}}\,,\quad V_{21} = -\frac{2}{29}\frac{\alpha^{(e)}}{m_\mu}\frac{1}{\sqrt{1+X^2}}\,e^{-i\varphi^{(e)}}\,,
\end{aligned}$$

$$\begin{aligned} V_{23} &= \frac{8\sqrt{3}}{29}\frac{\alpha^{(e)}}{m_\tau}\frac{1}{\sqrt{1+X^2}}\, e^{i\varphi^{(e)}}\,, V_{32} = -\frac{8\sqrt{3}}{29}\frac{\alpha^{(e)}}{m_\tau}\frac{1}{\sqrt{1+X^2}}\, e^{-i\varphi^{(e)}}\,, \\ V_{13} &= 0\,, \qquad\qquad\qquad V_{31} = 0 \end{aligned} \tag{19}$$

and

$$\begin{aligned} V_{14} &= -\frac{Y}{\sqrt{1+Y^2}}\, e^{i\varphi^{(\nu)}}\,, V_{41} = \left[1 - \frac{2}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2\right]\frac{Y}{\sqrt{1+Y^2}}\, e^{-i\varphi^{(\nu)}}\,, \\ V_{24} &= 0\,, \qquad\qquad V_{42} = \frac{2}{29}\frac{\alpha^{(e)}}{m_\mu}\,\frac{Y}{\sqrt{1+Y^2}}\, e^{-i(\varphi^{(\nu)} - \varphi^{(e)})}\,, \\ V_{34} &= 0\,, \qquad\qquad V_{43} = 0\,, \qquad V_{44} = \frac{1}{\sqrt{1+Y^2}}\,, \\ V_{15} &= 0\,, \qquad\qquad V_{51} = -\frac{2}{29}\frac{\alpha^{(e)}}{m_\mu}\,\frac{X}{\sqrt{1+X^2}}\, e^{-i(\varphi^{(\nu)\,\prime} + \varphi^{(e)})}\,, \\ V_{25} &= -\frac{X}{\sqrt{1+X^2}}\, e^{i\varphi^{(\nu)\,\prime}}\,, V_{52} = \left[1 - \frac{2}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2\right. \\ &\qquad \left. -\frac{96}{841}\left(\frac{\alpha^{(e)}}{m_\tau}\right)^2\right]\frac{X}{\sqrt{1+X^2}}\, e^{-i\varphi^{(\nu)\,\prime}}\,, \\ V_{35} &= 0\,, \qquad\qquad V_{53} = \frac{8\sqrt{3}}{29}\frac{\alpha^{(e)}}{m_\tau}\frac{X}{\sqrt{1+X^2}} e^{-i(\varphi^{(\nu)\,\prime} - \varphi^{(e)})}\,, \\ V_{45} &= 0\,, \qquad\qquad V_{54} = 0\,, V_{55} = \frac{1}{\sqrt{1+X^2}}\,. \end{aligned} \tag{20}$$

In the limit of $\alpha^{(e)} \to 0$, the only nonzero matrix elements $V_{\alpha J}$ are

$$V_{11} \to \frac{1}{\sqrt{1+Y^2}}\,, \quad V_{22} \to \frac{1}{\sqrt{1+X^2}}\,, V_{33} \to 1 \tag{21}$$

and

$$\begin{aligned} V_{14} = -\frac{Y}{\sqrt{1+Y^2}} e^{i\varphi^{(\nu)}} \quad , \quad V_{41} \to -V^*_{14}\,, \quad V_{44} = \frac{1}{\sqrt{1+Y^2}}\,, \\ V_{25} = -\frac{X}{\sqrt{1+X^2}} e^{i\varphi^{(\nu)\,\prime}} \quad , \quad V_{52} \to -V^*_{25}\,, \quad V_{55} = \frac{1}{\sqrt{1+X^2}}\,. \end{aligned} \tag{22}$$

3. Neutrino oscillations

Having once found the elements (19) and (20) of the extended lepton CKM matrix, we are able to calculate the probabilities of neutrino oscillations $\nu_\alpha \to \nu_\beta$ (in the vacuum), using the familiar formula:

$$P(\nu_\alpha \to \nu_\beta) = |\langle \nu_\beta | \nu_\alpha(t)\rangle|^2 = \sum_{K\,L} V_{L\,\beta} V^*_{L\,\alpha} V^*_{K\,\beta} V_{K\,\alpha} \exp\left(i \frac{m^2_{\nu_L} - m^2_{\nu_K}}{2|\vec{p}|}\, t \right) , \tag{23}$$

where $\nu_\alpha(0) = \nu_\alpha$, $\langle \nu_\beta| = \langle 0|\nu_\beta$ and $\langle \nu_\beta|\nu_\alpha\rangle = \delta_{\beta\,\alpha}$. Here, as usual, $t/|\vec{p}| = L/E$ $(c = 1 = \hbar)$, what is equal to $4 \times 1.2663 L/E$ if $m^2_{\nu_L} - m^2_{\nu_K}$, L and E are measured in eV^2, m and MeV, respectively. Of course, L is the source-detector distance (the baseline). In the following, it will be convenient to denote

$$x_{LK} = 1.2663 \frac{(m^2_{\nu_L} - m^2_{\nu_K})L}{E} \tag{24}$$

and use the identity $\cos 2x_{LK} = 1 - 2\sin^2 x_{LK}$.

From Eqs. (23), (19) and (20) we derive by explicit calculations the following neutrino-oscillation formulae valid in the lowest (quadratic) perturbative order in $\alpha^{(e)}/\mu^{(e)}$:

$$\begin{aligned}
P\,(\nu_e \to \nu_\mu) &= \frac{16}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2 \\
&\times \left[\frac{1}{(1+X^2)(1+Y^2)}\left(\sin^2 x_{21} + X^2 \sin^2 x_{51} + Y^2 \sin^2 x_{42} + X^2 Y^2 \sin^2 x_{54}\right) \right. \\
&\left. - \frac{X^2}{(1+X^2)^2}\sin^2 x_{52} - \frac{Y^2}{(1+Y^2)^2}\sin^2 x_{41} \right] , \\
P\,(\nu_e \to \nu_\tau) &= 0 , \\
P\,(\nu_\mu \to \nu_\tau) &= \frac{768}{841}\left(\frac{\alpha^{(e)}}{m_\tau}\right)^2 \left[\frac{1}{1+X^2}\left(\sin^2 x_{32} + X^2 \sin^2 x_{53}\right) - \frac{X^2}{(1+X^2)^2}\sin^2 x_{52} \right] , \\
P\,(\nu_e \to \nu_s) &= 4\left[1 - \frac{4}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2 \right] \frac{Y^2}{(1+Y^2)^2}\sin^2 x_{41} , \\
P\,(\nu_\mu \to \nu_s) &= \frac{16}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2 \frac{Y^2}{(1+Y^2)^2}\sin^2 x_{41} , \\
P\,(\nu_e \to \nu'_s) &= \frac{16}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2 \frac{X^2}{(1+X^2)^2}\sin^2 x_{52} , \\
P\,(\nu_\mu \to \nu'_s) &= 4\left[1 - \frac{4}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2 - \frac{192}{841}\left(\frac{\alpha^{(e)}}{m_\tau}\right)^2 \right] \frac{X^2}{(1+X^2)^2}\sin^2 x_{52} .
\end{aligned} \tag{25}$$

In the limit of $\alpha^{(e)} \to 0$, the only nonzero neutrino-oscillation probabilities are

$$\begin{aligned} P\left(\nu_e \to \nu_s\right) &\to 4\frac{Y^2}{(1+Y^2)^2}\sin^2 x_{41}\ , \\ P\left(\nu_\mu \to \nu'_s\right) &\to 4\frac{X^2}{(1+X^2)^2}\sin^2 x_{52}\ . \end{aligned} \tag{26}$$

The formulae (25) for the disappearance modes of ν_e and ν_μ imply the following survival probabilities for ν_e and ν_μ:

$$\begin{aligned} P\left(\nu_e \to \nu_e\right) &= 1 - P\left(\nu_e \to \nu_\mu\right) - P\left(\nu_e \to \nu_\tau\right) - P\left(\nu_e \to \nu_s\right) - P\left(\nu_e \to \nu'_s\right) \\ &= 1 - 4\left[1 - \frac{8}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2\right]\frac{Y^2}{(1+Y^2)^2}\sin^2 x_{41} \\ &\quad -\frac{16}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2\frac{1}{(1+X^2)(1+Y^2)} \\ &\quad \times\left(\sin^2 x_{21} + X^2\sin^2 x_{51} + Y^2\sin^2 x_{42} + X^2Y^2\sin^2 x_{54}\right) \end{aligned} \tag{27}$$

and

$$\begin{aligned} P\left(\nu_\mu \to \nu_\mu\right) &= 1 - P\left(\nu_\mu \to \nu_e\right) - P\left(\nu_\mu \to \nu_\tau\right) - P\left(\nu_\mu \to \nu_s\right) - P\left(\nu_\mu \to \nu'_s\right) \\ &= 1 - 4\left[1 - \frac{8}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2 - \frac{384}{841}\left(\frac{\alpha^{(e)}}{m_\tau}\right)^2\right]\frac{X^2}{(1+X^2)^2}\sin^2 x_{52} \\ &\quad -\frac{16}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2\frac{1}{(1+X^2)(1+Y^2)} \\ &\quad \times\left(\sin^2 x_{21} + X^2\sin^2 x_{51} + Y^2\sin^2 x_{42} + X^2Y^2\sin^2 x_{54}\right)\ . \end{aligned} \tag{28}$$

In the limit of $\alpha^{(e)} \to 0$, we obtain

$$P\left(\nu_e \to \nu_e\right) \to 1 - 4\frac{Y^2}{(1+Y^2)^2}\sin^2 x_{41} \tag{29}$$

and

$$P\left(\nu_\mu \to \nu_\mu\right) \to 1 - 4\frac{X^2}{(1+X^2)^2}\sin^2 x_{52}\ . \tag{30}$$

The last two formulae are to be compared with solar-neutrino and atmospheric-neutrino experiments, respectively.

4. Atmospheric and solar neutrinos

In the case of atmospheric neutrinos, we compare our formula (30) with Eq. (1). Then, for instance,

$$\frac{4X^2}{(1+X^2)^2} \sim 0.9 \tag{31}$$

(more generally: ~ 0.82 to 1) and

$$m^2_{\nu_5} - m^2_{\nu_2} \sim 5 \times 10^{-3} \;\; \mathrm{eV}^2 \tag{32}$$

(more generally: $\sim (0.5 \;\text{ to }\; 6) \times 10^{-3} \;\; \mathrm{eV}^2$).

From the input (31) we get

$$X \sim 0.721 \tag{33}$$

and, through the second Eq. (14),

$$\frac{M^{(\nu)}_{55} - M^{(\nu)}_{22}}{2|M^{(\nu)}_{25}|} = \frac{1-X^2}{2X} \sim \frac{1}{3} \tag{34}$$

or

$$|M^{(\nu)}_{25}| = \frac{X}{1-X^2}\left(M^{(\nu)}_{55} - M^{(\nu)}_{22}\right) \sim \frac{3}{2}\left(M^{(\nu)}_{55} - M^{(\nu)}_{22}\right) . \tag{35}$$

On the other hand, the third mass formula (12) and the input (32) give

$$\left(M^{(\nu)}_{22} + M^{(\nu)}_{55}\right)\sqrt{\left(M^{(\nu)}_{22} - M^{(\nu)}_{55}\right)^2 + 4|M^{(\nu)}_{25}|^2} = m^2_{\nu_5} - m^2_{\nu_2} \sim 5\times 10^{-3}\;\mathrm{eV}^2 \tag{36}$$

or, with the use of Eqs. (34) and (33),

$$M^{(\nu)\,2}_{55} - M^{(\nu)\,2}_{22} = \frac{1-X^2}{1+X^2}\left(m^2_{\nu_5} - m^2_{\nu_2}\right) \sim 1.58 \times 10^{-3} \;\; \mathrm{eV}^2 \,. \tag{37}$$

With the formulae $M^{(\nu)}_{22} \simeq 320\mu^{(\nu)}/261$ and $M^{(\nu)}_{55} \sim 48\mu^{(\nu)}/7$ we have $M^{(\nu)\,2}_{55} - M^{(\nu)\,2}_{22} \sim 45.5\mu^{(\nu)\,2}$. Hence, Eq. (37) leads to

$$\mu^{(\nu)} \sim 5.90 \times 10^{-3} \;\; \mathrm{eV}. \tag{38}$$

Then,

$$M^{(\nu)}_{22} \sim 7.25 \times 10^{-3} \;\; \mathrm{eV} \;\;,\;\; M^{(\nu)}_{55} \sim 4.04 \times 10^{-2} \;\; \mathrm{eV} \tag{39}$$

and so, from Eq. (35)

$$|M_{25}^{(\nu)}| \sim 4.97 \times 10^{-2} \ \text{eV}. \tag{40}$$

Finally, with the values (39) and (40) the third mass formula (12) gives

$$m_{\nu_2,\nu_5} \sim \left\{ \begin{array}{r} -2.86 \times 10^{-2} \ \text{eV} \\ 7.62 \times 10^{-2} \ \text{eV} \end{array} \right. . \tag{41}$$

In this way, all parameters appearing in our model of neutrino "texture", needed to explain the observed deficit of atmospheric ν_μ's in terms of neutrino oscillations $\nu_\mu \to \nu'_s$, are determined.

In the case of solar neutrinos, we compare our formula (29) with the survival probability for ν_e, usually analized experimentally in two-flavor form

$$P\left(\nu_e \to \nu_e\right) = 1 - \sin^2 2\theta_{\text{sol}} \sin^2\left(1.27 \Delta m^2_{\text{sol}}\, L/E\right) . \tag{42}$$

Taking into account the so-called vacuum fit [7] (*i.e.*, one that is not enhanced by the resonant MSW mechanism [8] in the Sun matter), we have the parameters

$$\sin^2 2\theta_{\text{sol}} \sim 0.65 \ \text{ to } \ 1\,, \ \Delta m^2_{\text{sol}} \sim (5 \ \text{ to } \ 8) \times 10^{-11} \ \text{eV}^2 \ , \tag{43}$$

what shows a large mixing and a very small difference of masses squared. Then, for instance,

$$\frac{4Y^2}{(1+Y^2)^2} \sim 0.8 \tag{44}$$

(more generally: ~ 0.65 to 1) and

$$m^2_{\nu_4} - m^2_{\nu_1} \sim 7 \times 10^{-11} \ \text{eV}^2 \tag{45}$$

(more generally: $\sim$ (5 to 8) $\times 10^{-11}$ eV^2).

From the input (44) we obtain

$$Y \sim 0.618 \tag{46}$$

and, due to the first Eq. (14),

$$\frac{M_{44}^{(\nu)} - M_{11}^{(\nu)}}{2|M_{14}^{(\nu)}|} = \frac{1-Y^2}{2Y} \sim \frac{1}{2} \tag{47}$$

or

$$|M_{14}^{(\nu)}| = \frac{Y}{1-Y^2}\left(M_{44}^{(\nu)} - M_{11}^{(\nu)}\right) \sim M_{44}^{(\nu)} - M_{11}^{(\nu)} \ . \tag{48}$$

On the other hand, the first mass formula (12) and the input (45) lead to

$$\left(M_{11}^{(\nu)}+M_{44}^{(\nu)}\right)\sqrt{\left(M_{11}^{(\nu)}-M_{44}^{(\nu)}\right)^2+4|M_{14}^{(\nu)}|^2}=m_{\nu_4}^2-m_{\nu_1}^2\sim 7\times 10^{-11}\ \mathrm{eV}^2 \tag{49}$$

or, through Eqs. (47) and (46), to

$$M_{44}^{(\nu)\,2}-M_{11}^{(\nu)\,2}=\frac{1-Y^2}{1+Y^2}\left(m_{\nu_4}^2-m_{\nu_1}^2\right)\sim 3.13\times 10^{-11}\ \mathrm{eV}^2\ . \tag{50}$$

With the formulae $M_{11}^{(\nu)}=\mu^{(\nu)}\varepsilon^{(\nu)\,2}/29$ and $M_{44}^{(\nu)}\sim\mu^{(\nu)}\varepsilon^{(\nu)\,2}/7$ we get $M_{44}^{(\nu)\,2}-M_{11}^{(\nu)\,2}\sim 0.0192\mu^{(\nu)\,2}\varepsilon^{(\nu)\,4}$. Hence, Eqs. (50) and (38) give

$$\varepsilon^{(\nu)\,2}\sim 6.85\times 10^{-3}\ . \tag{51}$$

Then,

$$M_{11}^{(\nu)}\sim 1.39\times 10^{-6}\ \mathrm{eV}\quad,\quad M_{44}^{(\nu)}\sim 5.77\times 10^{-6}\ \mathrm{eV} \tag{52}$$

and thus, from Eq. (48)

$$|M_{14}^{(\nu)}|\sim 4.38\times 10^{-6}\ \mathrm{eV}\ . \tag{53}$$

Eventually, with the values (52) and (53) the first mass formula (12) implies

$$m_{\nu_1,\nu_4}\sim\left\{\begin{array}{r}-1.32\times 10^{-6}\ \ \mathrm{eV}\\ 8.48\times 10^{-6}\ \ \mathrm{eV}\end{array}\right.\ . \tag{54}$$

In such a way, all parameters contained in our model of neutrino "texture", needed to describe the observed deficit of solar ν_e's in terms of neutrino oscillations $\nu_e\to\nu_s$ in the vacuum, are determined.

Our last item is concerned with the LSND accelerator experiment that reported the detection of $\bar{\nu}_\mu\to\bar{\nu}_e$ and $\nu_\mu\to\nu_e$ oscillations by observing $\bar{\nu}_e$'s and ν_e's in a beam of $\bar{\nu}_\mu$'s and ν_μ's produced in π^- and π^+ decays, respectively [9]. The observed excess of $\bar{\nu}_e$'s and ν_e's, analized in terms of two-flavor neutrino-oscillation formula, implies a considerable amplitude $\sin^2 2\theta_{\mathrm{LSND}}$, too large to be explained by our formula (25) for $P(\nu_\mu\to\nu_e)=P(\nu_e\to\nu_\mu)$, where the leading amplitude at $\sin^2 x_{21}$,

$$\frac{16}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2\frac{1}{(1+X^2)(1+Y^2)}\ , \tag{55}$$

is small:

$$0\leq\frac{16}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2\leq 6.2\times 10^{-4}\ , \tag{56}$$

as it follows from Eq. (8). Here, the central value is

$$\frac{16}{841}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2 = 2.5\times 10^{-4}\,. \tag{57}$$

If signs $\mp$ in the mass formulae (12) are replaced by $\pm$, then in Eqs. (14) for Y and X we ought to interchange $m_{\nu_1} \leftrightarrow m_{\nu_4}$, $m_{\nu_2} \leftrightarrow m_{\nu_5}$ and $M^{(\nu)}_{11} \leftrightarrow M^{(\nu)}_{44}$, $M^{(\nu)}_{22} \leftrightarrow M^{(\nu)}_{55}$ to keep Eq. (13) for $\left(U^{(\nu)}_{IJ}\right)$ unchanged. In the new situation, we may try the assumption $\mu^{(s)} = 0$ [instead of $\mu^{(s)} = \mu^{(\nu)}$, Eqs. (20)], and then with the use of $\sin^2 2\theta_{\rm atm} \sim 0.9$ and $\Delta m^2_{\rm atm} \sim 5\times 10^{-3}$ eV^2 we obtain $m_{\nu_2} \sim 8.28\times 10^{-2}$ eV and $m_{\nu_5} \sim -4.30\times 10^{-2}$ eV (and $\mu^{(\nu)} \sim 3.24\times 10^{-2}$ eV). Similarly, with the use of $\sin^2 2\theta_{\rm sol} \sim 0.8$ and $\Delta m^2_{\rm sol} \sim 7\times 10^{-11}$ eV^2 we get $m_{\nu_1} \sim 9.05\times 10^{-6}$ eV and $m_{\nu_4} \sim -3.46\times 10^{-6}$ eV (and $\varepsilon^{(\nu)2} \sim 5.00\times 10^{-3}$).

I would like to thank Jan Królikowski for several helpful discussions.

Appendix

Unified "texture dynamics"

In this Appendix the idea of a model of fermion "texture" that we develop since some time [4,5] is outlined. In particular, the existence of two sterile neutrinos ν_s and ν'_s turns out to follow naturally.

Let us introduce the following 3×3 matrices in the space of three fermion families:

$$\widehat{a} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}\,, \quad \widehat{a}^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \tag{A.1}$$

With the matrix

$$\widehat{n} = \widehat{a}^\dagger \widehat{a} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{A.2}$$

they satisfy the commutation relations

$$[\widehat{a}\,, \widehat{n}] = \widehat{a}\;,\; [\widehat{a}^\dagger\,, \widehat{n}] = -\widehat{a}^\dagger \tag{A.3}$$

characteristic for annihilation and creation matrices, while $\widehat{n}$ plays the role of an occupation-number matrix. However, in addition, they obey the "truncation" identities

$$\widehat{a}^3 = 0\,, \widehat{a}^{\dagger\,3} = 0\,. \tag{A.4}$$

Note that due to Eqs. (A.4) the bosonic canonical commutation relation $[\widehat{a},\widehat{a}^\dagger]=\widehat{1}$ does not hold, being replaced by the relation $[\widehat{a},\widehat{a}^\dagger]=\mathrm{diag}\,(1,1,-2)$.

In consequence of Eqs. (A.1), (A.2) and (A.3), we get $\widehat{n}|n\rangle = n|n\rangle$ as well as $\widehat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\widehat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ $(n=0,1,2)$, however, $\widehat{a}^\dagger|2\rangle = 0$ (*i.e.*, $|3\rangle = 0$) in addition to $\widehat{a}^\dagger|0\rangle = 0$ (*i.e.*, $|-1\rangle = 0$). Evidently, $n = 0,1,2$ may play the role of a vector index in our three-dimensional matrix calculus.

It is natural to expect that the Gell-Mann matrices (generating the horizontal SU(3) algebra) can be built up from $\widehat{a}$ and $\widehat{a}^\dagger$. In fact,

$$
\begin{aligned}
\widehat{\lambda}_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\left(\widehat{a}^2\widehat{a}^\dagger + \widehat{a}\widehat{a}^{\dagger\,2}\right)\,, \\
\widehat{\lambda}_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2i}\left(\widehat{a}^2\widehat{a}^\dagger - \widehat{a}\widehat{a}^{\dagger\,2}\right)\,, \\
\widehat{\lambda}_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\left(\widehat{a}^2\widehat{a}^{\dagger\,2} - \widehat{a}\widehat{a}^{\dagger\,2}\widehat{a}\right)\,, \\
\widehat{\lambda}_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\widehat{a}^2 + \widehat{a}^{\dagger\,2}\right)\,, \\
\widehat{\lambda}_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \frac{1}{i\sqrt{2}}\left(\widehat{a}^2 - \widehat{a}^{\dagger\,2}\right)\,, \\
\widehat{\lambda}_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\widehat{a}^\dagger\widehat{a}^2 + \widehat{a}^{\dagger\,2}\widehat{a}\right)\,, \\
\widehat{\lambda}_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \frac{1}{i\sqrt{2}}\left(\widehat{a}^\dagger\widehat{a}^2 - \widehat{a}^{\dagger\,2}\widehat{a}\right)\,, \\
\widehat{\lambda}_8 &= \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \frac{1}{\sqrt{3}}\left(\widehat{a}\widehat{a}^\dagger - \widehat{a}^\dagger\widehat{a}\right)\,, \\
\widehat{1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\left(\widehat{a}^2\widehat{a}^{\dagger\,2} + \widehat{a}\widehat{a}^{\dagger\,2}\widehat{a} + \widehat{a}^{\dagger\,2}\widehat{a}^2\right)\,. \qquad \text{(A.5)}
\end{aligned}
$$

Inversely, $\widehat{a} = (\widehat{\lambda}_1 + i\widehat{\lambda}_2)/2 + \sqrt{2}(\widehat{\lambda}_6 + i\widehat{\lambda}_7)/2$ and $\widehat{a}^\dagger = (\widehat{\lambda}_1 - i\widehat{\lambda}_2)/2 + \sqrt{2}(\widehat{\lambda}_6 -$

$i\widehat{\lambda}_7)/2$. A message we get from these relationships is that a horizontal field formalism, always simple (linear) in terms of $\widehat{\lambda}_A$ $(A = 1, 2, \ldots, 8)$ and $\widehat{1}$, is generally not simple in terms of $\widehat{a}$ and $\widehat{a}^\dagger$. In particular, a nontrivial SU(3)-symmetric horizontal formalism is not simple in $\widehat{a}$ and $\widehat{a}^\dagger$. Inversely, a nontrivial horizontal field formalism, if simple (linear and/or quadratic and/or cubic) in terms of $\widehat{a}$ and $\widehat{a}^\dagger$, cannot be SU(3)-symmetric.

Now, let us consider the following ansatz [5]:

$$\widehat{M}^{(f)} = \widehat{\rho}^{1/2}\widehat{h}^{(f)}\widehat{\rho}^{1/2} \quad (f = \nu\,, e\,, u\,, d)\,, \tag{A.6}$$

where

$$\widehat{\rho}^{1/2} = \frac{1}{\sqrt{29}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{24} \end{pmatrix}\;, \quad \mathrm{Tr}\widehat{\rho} = 1 \tag{A.7}$$

and

$$\begin{aligned} \widehat{h}^{(f)} &= \mu^{(f)}\left[(1+2\widehat{n})^2 + \left(\varepsilon^{(f)\,2} - 1\right)(1+2\widehat{n})^{-2} + \widehat{C}^{(f)}\right] \\ &+ \left(\alpha^{(f)}\widehat{1} - \beta^{(f)}\widehat{n}\right)\widehat{a}e^{i\varphi^{(f)}} + \widehat{a}^\dagger\left(\alpha^{(f)}\widehat{1} - \beta^{(f)}\widehat{n}\right)e^{-i\varphi^{(f)}} \end{aligned} \tag{A.8}$$

with $\widehat{n} = \widehat{a}^\dagger\widehat{a}$ and

$$\widehat{1} + 2\widehat{n} = \widehat{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}\;, \quad \widehat{C}^{(f)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C^{(f)} \end{pmatrix}\,. \tag{A.9}$$

It is the matter of an easy calculation to show that the matrices (A.6) get explicitly the form [5]:

$$\widehat{M}^{(f)} = \frac{1}{29}\begin{pmatrix} \mu^{(f)}\varepsilon^{(f)\,2} & 2\alpha^{(f)}e^{i\varphi^{(f)}} & 0 \\ 2\alpha^{(f)}e^{-i\varphi^{(f)}} & 4\mu^{(f)}(80+\varepsilon^{(f)\,2})/9 & 8\sqrt{3}(\alpha^{(f)}-\beta^{(f)})e^{i\varphi^{(f)}} \\ 0 & 8\sqrt{3}(\alpha^{(f)}-\beta^{(f)})e^{-i\varphi^{(f)}} & 24\mu^{(f)}(624+25C^{(f)}+\varepsilon^{(f)\,2})/25 \end{pmatrix}. \tag{A.10}$$

In this paper we write also $\widehat{M}^{(f)} = \left(M_{ij}^{(f)}\right)$ $(i, j = 1, 2, 3)$.

In a more detailed construction following from our idea about the origin of three fermion families [4], each eigenvalue $N = 1\,, 3\,, 5$ of the matrix $\widehat{N}$ corresponds (for any $f = \nu\,, e\,, u\,, d$) to a wave function carrying $N = 1\,, 3\,, 5$ Dirac bispinor indices: $\alpha_1, \alpha_2, \ldots, \alpha_N$ of which one, say α_1, is coupled to the external Standard Model gauge fields, while the remaining $N - 1 = 0\,, 2\,, 4\,:\;\; \alpha_2\,, \ldots,\; \alpha_N$ (that are not coupled to these fields) are fully antisymmetric under permutations. So, the latter obey Fermi statistics along with the Pauli principle implying that really $N - 1 \le 4$, because each

$\alpha_i = 1, 2, 3, 4$. Then, the three wave functions corresponding to $N = 1\,,\,3\,,\,5$ can be reduced to three other wave functions carrying only one Dirac bispinor index α_1 (and so, spin 1/2),

$$\begin{aligned}
\psi^{(f)}_{1\,\alpha_1} &\equiv \psi^{(f)}_{\alpha_1}\,, \\
\psi^{(f)}_{3\,\alpha_1} &\equiv \frac{1}{4}\left(C^{-1}\gamma^5\right)_{\alpha_2\alpha_3}\psi^{(f)}_{\alpha_1\alpha_2\alpha_3} = \psi^{(f)}_{\alpha_1\,12} = \psi^{(f)}_{\alpha_1\,34}\,, \\
\psi^{(f)}_{5\,\alpha_1} &\equiv \frac{1}{24}\varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5}\psi^{(f)}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = \psi^{(f)}_{\alpha_1\,1234}\,,
\end{aligned} \tag{A.11}$$

and appearing (up to the sign) with the multiplicities 1, 4 and 24, respectively. In this argument, for $N = 3$ the requirement of relativistic covariance of the wave function (and the related probability current) is applied explicitly [4]. The weighting matrix $\widehat{\rho}^{1/2}$ as given in Eq. (A.7) gets as its elements the square roots of these multiplicities, normalized in such a way that $\mathrm{Tr}\,\widehat{\rho} = 1$.

In Eqs. (A.11), the indices α_i $(i = 1, 2, \ldots, N)$ are of Jacobi type: α_1 is a "centre-of-mass" Dirac bispinor index, while $\alpha_2\,, \ldots,\ \alpha_N$ are "relative" Dirac bispinor indices. In fact, α_i $(i = 1, 2, \ldots, N)$ are defined by chiral representations of Γ^μ_i matrices $(i = 1, 2, \ldots, N)$ being the (properly normalized) Jacobi combinations of some individual γ^μ_i matrices $(i = 1, 2, \ldots, N)$, where, in particular, $\Gamma^\mu_1 \equiv (1/\sqrt{N})\sum_{i=1}^N \gamma^\mu_i$ [4]. For them $\left\{\Gamma^\mu_i, \Gamma^\nu_j\right\} = 2\delta_{ij}g^{\mu\nu}$ $(i, j = 1, 2, \ldots, N)$, in consequence of the anticommutation relations $\left\{\gamma^\mu_i, \gamma^\nu_j\right\} = 2\delta_{ij}g^{\mu\nu}$ valid for any γ^μ_i and γ^ν_j. Then, the Dirac-type equations $\{\Gamma_1 \cdot [p - gA(x)] - M\}\,\psi(x) = 0$ $(N = 1, 2, 3, \ldots)$ [4], independent of $\Gamma^\mu_2\,, \ldots,\ \Gamma^\mu_N$, hold for the fundamental-particle wave functions $\psi(x) = (\psi_{\alpha_1\alpha_2\ldots\alpha_N}(x))$, where $N = 1, 3, 5$ in the case of fermion wave functions (A.11). Here, $g\Gamma_1 \cdot A(x)$ symbolizes the Standard Model coupling.

Note that all four matrices $\widehat{M}^{(f)}$ $(f = \nu\,, e\,, u\,, d)$ defined by Eqs. (A.6)–(A.9) and (A.1) have a common structure, differing from each other only by the values of their parameters $\mu^{(f)}$, $\varepsilon^{(f)\,2}$, $\alpha^{(f)}$, $\beta^{(f)}$, $C^{(f)}$ and $\varphi^{(f)}$. We proposed the fermion mass matrices to be of this unified form [5]. Then, Eqs. (A.6) and (A.8) define a quantum-mechanical model for the "texture" of fermion mass matrices $\widehat{M}^{(f)}$ $(f = \nu\,, e\,, u\,, d)$. Such an approach may be called "texture dynamics".

The fermion mass matrix $\widehat{M}^{(f)}$, containing the kernel $\widehat{h}^{(f)}$ given in Eq. (A.8), consists of a diagonal part proportional to $\mu^{(f)}$, and of an off-diagonal part involving linearly $\alpha^{(f)}$ and $\beta^{(f)}$. The off-diagonal part of $\widehat{h}^{(f)}$ describes the mixing of three eigenvalues

$$\mu^{(f)}\left[N^2 + \left(\varepsilon^{(f)\,2} - 1\right)N^{-2} + \delta_{N\,5}C^{(f)}\right] \quad (N = 1, 3, 5) \tag{A.12}$$

of its diagonal part. Beside the term $\mu^{(f)}C^{(f)}$ that appears only for $N=5$, each of these eigenvalues is the sum of two terms containing N^2. They are: (i) a term $\mu^{(f)}N^2$ that may be interpreted as an "interaction" of N elements ("intrinsic partons") treated on the same footing, and (ii) another term

$$\mu^{(f)}\left(\varepsilon^{(f)\,2}-1\right)P_N^2 \ \text{ with } \ P_N=\left[N!/(N-1)!\right]^{-1}=N^{-1} \qquad (A.13)$$

that may describe an additional "interaction" with itself of one element arbitrarily chosen among N elements of which the remaining $N-1$ are undistinguishable. Therefore, the total "interaction" with itself of this (arbitrarily) distinguished "parton" is $\mu^{(f)}[1+\ (\varepsilon^{(f)\,2}-1)N^{-2}]$, so it becomes $\mu^{(f)}\varepsilon^{(f)\,2}$ in the first fermion family.

The form (A.11) of three fermion wave functions shows that each "intrinsic parton" carries a Dirac bispinor index (of the Jacobi type). For the (arbitrarily) distinguished "parton", this index, considered in the framework of a fermion wave equation, is coupled to the external gauge fields of the Standard Model. Thus, this "parton" carries the total spin 1/2 of the fermion as well as a set of its Standard Model charges corresponding to $f=\nu\,,\,e\,,\,u\,,\,d$. For the $N-1$ undistinguishable "partons", obeying Fermi statistics along with the Pauli principle, their Dirac bispinor indices are mutually coupled, resulting into Lorentz scalars, while their number $N-1=0,2,4$ differentiates between three fermion families (for each $f=\nu\,,\,e\,,\,u\,,\,d$). These "partons" are free of Standard Model charges.

Evidently, the intriguing question arises, how to interpret two possible boson families corresponding to the number $N-1=1,3$ of undistinguishable "partons" [10]. In the present paper this problem is not discussed. Here, we would like only to point out that three fermion families $N=1,3,5$ differ from these two hypothetic boson families $N=2,4$ by the full pairing of their $N-1=0,2,4$ undistinguishable "partons". So, the boson families, containing an odd number $N-1=1,3$ of such "partons", might be considerably heavier. Note that the wave functions corresponding to $N=2,4$ can be reduced (under some relativistic requirements) to two other wave functions carrying only spin 0,

$$\begin{aligned}\phi_2^{(f)} &\equiv \frac{1}{2\sqrt{2}}\left(C^{-1}\gamma^5\right)_{\alpha_1\alpha_2}\psi^{(f)}_{\alpha_1\alpha_2}=\frac{1}{\sqrt{2}}\left(\psi^{(f)}_{12}-\psi^{(f)}_{21}\right)=\frac{1}{\sqrt{2}}\left(\psi^{(f)}_{34}-\psi^{(f)}_{43}\right)\ ,\\ \phi_4^{(f)} &\equiv \frac{1}{6\sqrt{4}}\varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\psi^{(f)}_{\alpha_1\alpha_2\alpha_3\alpha_4}=\frac{1}{\sqrt{4}}\left(\psi^{(f)}_{1234}-\psi^{(f)}_{2134}+\psi^{(f)}_{3412}-\psi^{(f)}_{4312}\right)\ ,\end{aligned} \qquad (A.14)$$

and appearing (up to the sign) with the multiplicities 2 and 6, respectively.

Another important question also appears, namely, what is the interpretation of two fermions corresponding to the number $N=1\,,\,3$ of undistinguishable "partons" only. Such fermions can carry exclusively spin 1/2 (for

$N = 3$: under some relativistic requirements). Of course, they are free of Standard Model charges and so, can be considered as two sterile neutrinos with the wave functions

$$\begin{aligned} \nu_{s\,\alpha_1} &\equiv \psi_{1\,\alpha_1} \equiv \psi_{\alpha_1} \;, \\ \nu'_{s\,\alpha_1} &\equiv \psi_{3\,\alpha_1} \equiv \frac{1}{6}\left(C^{-1}\gamma^5\right)_{\alpha_1\,\alpha_2} \varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5}\psi_{\alpha_3\alpha_4\alpha_5} = \left\{ \begin{array}{r} \psi_{134} \text{ for } \alpha_1 = 1 \\ -\psi_{234} \text{ for } \alpha_1 = 2 \\ \psi_{312} \text{ for } \alpha_1 = 3 \\ -\psi_{412} \text{ for } \alpha_1 = 4 \end{array} \right. \end{aligned} \tag{A.15}$$

appearing (up to the sign) with the multiplicities 1 and 6, respectively.

For these sterile neutrinos one may introduce the 2×2 mass matrix $\widehat{M}^{(s)} = \widehat{\rho}^{(s)\,1/2}\widehat{h}^{(s)}\widehat{\rho}^{(s)\,1/2}$, where

$$\widehat{\rho}^{(s)\,1/2} = \frac{1}{\sqrt{7}}\left(\begin{array}{cc} 1 & 0 \\ 0 & \sqrt{6} \end{array}\right) \;,\; \mathrm{Tr}\widehat{\rho}^{(s)} = 1 \;, \tag{A.16}$$

while the diagonal part of $\widehat{h}^{(s)}$ is conjectured to have the eigenvalues

$$\mu^{(s)}\left[N^2 + \left(\varepsilon^{(s)\,2} - 1\right)P_N^2\right] \quad \text{with } P_N = N!/N! = 1 \;\; (N = 1\,,\,3)\;. \tag{A.17}$$

Now, one "intrinsic parton" is arbitrarily chosen (to carry the total spin 1/2 of the fermion) among N "intrinsic partons" that all are undistinguishable [in contrast to Eqs. (A.12) and (A.13)]. This gives the diagonal part of $\widehat{M}^{(s)}$ equal to

$$\frac{1}{7}\left(\begin{array}{cc} \mu^{(s)}\varepsilon^{(s)\,2} & 0 \\ 0 & 6\mu^{(s)}(8+\varepsilon^{(s)\,2}) \end{array}\right)\;. \tag{A.18}$$

Thus, the diagonal matrix elements $M^{(\nu)}_{44}$ and $M^{(\nu)}_{55}$ of the 5×5 neutrino mass matrix $\left(M^{(\nu)}_{IJ}\right)$ $(I, J = 1,2,3,4,5)$ introduced in Eq. (11) get the forms

$$M^{(\nu)}_{44} = \frac{\mu^{(s)}}{7}\varepsilon^{(s)\,2} \simeq 0 \;\;,\;\; M^{(\nu)}_{55} = \frac{6\mu^{(s)}}{7}\left(8 + \varepsilon^{(s)\,2}\right) \simeq \frac{48\mu^{(s)}}{7} \tag{A.19}$$

with $\varepsilon^{(s)\,2}$ expected to be very small. In the present paper we will assume that

$$\mu^{(s)} \sim \mu^{(\nu)} \;\;,\;\; \varepsilon^{(s)\,2} \sim \varepsilon^{(\nu)\,2}\;. \tag{A.20}$$

in Eqs. (A.19).

The possibility of existence of two bosons corresponding to the number $N = 2, 4$ of undistinguishable "partons" only ought to be also considered.

Such bosons can carry exclusively spin 0 (for $N = 2$: under some relativistic requirements). Obviously, they are free of Standard Model charges and so, may be considered as two "sterile scalars" with the wave functions

$$\phi_2 \equiv \frac{1}{4}\left(C^{-1}\gamma^5\right)_{\alpha_1\alpha_2}\psi_{\alpha_1\alpha_2} = \psi_{12} = \psi_{34}\ ,\ \phi_4 \equiv \frac{1}{24}\varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\psi_{\alpha_1\alpha_2\alpha_3\alpha_4} = \psi_{1234} \tag{A.21}$$

appearing (up to the sign) with the multiplicities 4 and 24, respectively.

A priori, the "intrinsic partons" may be either strictly *algebraic* objects providing fundamental fermions (leptons and quarks) with new family degrees of freedom, or may give us a signal of a new *spatial* substructure of fundamental fermions (built up of spatial "intrinsic partons" = preons, related to the individual γ_i^μ as well as x_i^μ and p_i^μ $(i = 1, 2, \ldots, N)$; note that here γ_i^μ's anticommute for different i !). Our idea about the origin of three fermion families [4] chooses the first option. The difficult problem of new non-Standard Model forces, responsible for the binding of N preons within fundamental fermions, does not arise in this option.

However, if the second option is true, then this irksome (though certainly profound) problem does arise and must be solved.

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