ON FRACTIONAL SPIN IN THE CP^1 MODEL COUPLED TO THE HOPF TERM

B. Chakraborty* and A.S. Majumdar**

S.N.Bose National Centre for Basic Sciences
Block JD, Sector III, Salt Lake, Calcutta 700091, India

(Received August 17, 1998; Revised version received November 6, 1998)

We carry out a gauge independent Hamiltonian analysis of the CP^1 model coupled to the Hopf term. We show that no fractional spin is revealed at the classical level — a result that is different from the corresponding case for the O(3) nonlinear sigma model. We next show that if the former model is altered through an identity involving the time derivative, an expression of fractional spin emerges at the classical level itself, which is given in terms of the soliton number of the model. This result matches several other existing results, both for the CP^1 as well as the sigma model versions, obtained through canonical or path integral quantization.

PACS numbers: 11.15.Tk, 11.10.Ef, 11.10.Lm

Systems residing in $2+1$ dimensional spacetime dimensions have attracted much attention in recent years. This has been in the hope of obtaining critical insight into a large variety of phenomenological problems in divergent areas ranging from condensed matter physics to quantum gravity [1]. A peculiar aspect which has emerged from these investigations is that there exist several field theoretical models in $2+1$ dimensions which admit solitonic configurations imparting fractional spin and statistics to the coupled matter systems. It was first shown by Wilczek and Zee [2] that by performing a slow adiabatic rotation of 2π the wave function acquires a nontrivial phase, thus signalling fractional spin. Later, using the canonical Hamiltonian formalism, Bowick *et al.* [3] showed the existence of fractional spin in the nonlinear sigma model coupled to the Hopf term. That a purely bosonic classical field theory admitting topological solitons may have

* e-mail: `biswajit@bose.ernet.in`
** e-mail: `archan@bose.ernet.in`

fermionic characteristics was noted long ago [4]. For the case of $3+1$ dimensions, it was realized that the doubly connected configuration space allows for only Bose–Einstein and Fermi–Dirac statistics. In contrast, in $2+1$ dimensions all sorts of statistics may be permitted because of the possibility of having an infinitely connected configuration space. For example, for the case of the nonlinear sigma model, the configuration space $C = \{f\}$ is the set of all maps $f : S^2$ (compactified space of R^2) $\to S^2$ (field manifold) having the fundamental group $\pi_1(C) = \pi_3(S^2) = Z$. The model admits solitons as $\pi_0(C) = \pi_2(S^2) = Z$. These possibilities can be realized by introducing a Wess–Zumino term [5] in $3+1$ dimensions whereby solitons get half-integral spin, whereas solitons in $2+1$ dimensions can be imparted fractional spin by introducing a Chern–Simons or a Hopf term in the action. These issues have been typically exemplified by the detailed study of various models coupled to the Chern–Simons (CS) and Hopf terms in $2+1$ dimensions [6].

At the formal field theoretical level, the features of fractional spin and statistics in various $2+1$ dimensional models have been revealed using both the path integral [2] as well as the canonical Hamiltonian formalism [3,6,7]. In the latter scheme the explicit construction of the relevant angular momentum operators has been carried out in several models involving the Chern–Simons term in a gauge independent manner [6]. In these models the existence of fractional spin is usually revealed by computing the difference between the expression for the gauge invariant (physical) definition of the angular momentum operator J^s following from the symmetric energy momentum (EM) tensor, and that of the Noether angular momentum J^n. The latter expression J^n corresponds to only the orbital part of angular momentum (for scalar fields as in [3]), and turns out, in general, to be gauge invariant on the constraint surface *only* under those gauge transformations that reduce to identity at infinity [6].

The explicit construction of the angular momentum in the O(3) nonlinear sigma model coupled to a Hopf term was carried out by Bowick *et al.* [3] using the canonical Hamiltonian formalism showing the existence of fractional spin. In this model a gauge fixing had to be done right at the beginning in order to uniquely define the fictitious gauge field A_μ in terms of the current j_μ thereby making the Hopf term nonlocal. It is well known that at the classical level the nonlinear sigma model is completely equivalent to the CP^1 model in $2+1$ dimensions [8]. It has been claimed recently, that this equivalence can be established at the quantum level too [9]. The CP^1 model extended by a Hopf term is described by a Lagrangian which is local in terms of the basic fields, and gauge fixing is not required at the onset unlike as in the case of the nonlinear sigma model [3]. The model is interesting as it describes antiferromagnets [1]. On the other hand, its nonrelativistic version, the CP^1 model is intimately related to the Landau–Lifshitz model of ferromagnetism [10]. The nonrelativistic CP^1 model coupled to the Hopf term has been analyzed in [11]. A comprehensive Hamiltonian analysis in a *gauge independent manner* à la Dirac [12] of the CP^1 model with the Hopf term (this being a constrained system) is therefore desirable, in order to compare with the quantization carried out in the reduced phase space scheme using some gauge fixing condition, in light of its (possible) phenomenological relevance as well as the above-mentioned intricacies involved in the definition of the angular momentum in similar models. It is well established in literature that the schemes of Dirac and reduced phase space quantization might lead to entirely different physical results [13].

To this end, in this paper we perform a classical Hamiltonian analysis of the local CP^1 version of the relativistic nonlinear sigma model including a Hopf term in a consistent gauge independent manner [6,12]. We carry out the explicit construction of the translation and rotation symmetry generators using both the Noether prescription, as well as the symmetric energy momentum tensor. To begin with, let us briefly recall the essential characteristics of the nonlinear sigma model described by the Lagrangian [3,9]

$$\mathcal{L} = \tfrac{1}{4}\partial_\mu M_a\left(x\right)\partial^\mu M_a\left(x\right) - \lambda\left(M_a M_a - 1\right) . \tag{1}$$

λ is a Lagrange multiplier enforcing the constraints $M_a M_a = 1$. The field manifold is S^2. For finite energy static solutions the fields are required to tend to constant configurations asymptotically, so that the space R^2 is essentially compactified to S^2. The configuration space splits into disjoint unions of path connected sectors C_N, with N specifying the winding number of the soliton given by

$$N = \int d^2x j^0\left(x\right) , \tag{2}$$

where j^μ is the identically conserved ($\partial_\mu j^\mu = 0$) topological current

$$j^\mu = \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\varepsilon_{abc}M_a\partial_\nu M_b\partial_\lambda M_c\,. \tag{3}$$

The current j^μ can be expressed as the curl of a vector potential A_λ as

$$j^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda\,, \tag{4}$$

where A_λ is obtained by pulling back onto the spacetime, the Dirac monopole connection on CP$^1 \sim S^2$ [10].

The CP^1 version of the model (1) can be written as

$$\mathcal{L}_0 = \left(D_\mu Z\right)^\dagger\left(D^\mu Z\right) , \tag{5}$$

where

$$D_\mu \equiv \partial_\mu - iA_\mu \,. \tag{6}$$

Note that there is no dynamical term for A_μ in the Lagrangian. $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is a SU(2) doublet of complex scalar fields z_α ($\alpha = 1, 2$). The M_a fields in (1) are related to the z_α fields via the Hopf map $M_a = Z^\dagger \sigma_a Z$, with σ_a being the Pauli matrices. A Hopf term of the form $j^\mu A_\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$ can be added to $\mathcal{L}_0$, which has the appearence of a Chern–Simons (CS) term. However, this resemblance with the CS term is superficial since in this case A_μ is not an independent degree of freedom, but rather is obtainable in terms of the Z fields by inverting relation (4). In fact, up to a gauge transformation, A_μ is related to the Z fields, directly from geometrical considerations [10], by

$$A_\mu = -iZ^\dagger \partial_\mu Z \,. \tag{7}$$

This way of writing A_μ (7) and hence j_μ (4) in terms of the matter fields through local expressions can be done only for the CP^1 model with a topological current. (We shall see below that the above relation is reproduced by the constrained Hamiltonian analysis.) The CP^1 Lagrangian extended by the Hopf term is given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\mathrm{H} - \lambda \left(Z^\dagger Z - 1 \right) , \tag{8}$$

where

$$\mathcal{L}_\mathrm{H} = \Theta \varepsilon^{\mu\nu\lambda} \left[Z^\dagger \partial_\mu Z \partial_\nu Z^\dagger \partial_\lambda Z + \partial_\mu Z^\dagger Z \partial_\lambda Z^\dagger \partial_\nu Z \right] \tag{9}$$

with Θ being the Hopf parameter. Unlike the case of the nonlinear sigma model, the Hopf term $\mathcal{L}_\mathrm{H}$ is local in terms of the Z fields here. The configuration space variables are $z_\alpha, z^*_\alpha, A_i, A_0$ and λ. The corresponding momentum variables are given by

$$\pi_\alpha = \frac{\delta \mathcal{L}}{\delta \dot{z}_\alpha} = (D_0 z)^*_\alpha + \Theta \varepsilon^{ij} \left[\partial_i Z^\dagger \partial_j Z z^*_\alpha + Z^\dagger \partial_i Z \partial_j z^*_\alpha - \partial_i Z^\dagger Z \partial_j z^*_\alpha \right] , \tag{10}$$

$$\pi^*_\alpha = \frac{\delta \mathcal{L}}{\delta \dot{z}^*_\alpha} = (D_0 z)_\alpha + \Theta \varepsilon^{ij} \left[-Z^\dagger \partial_i Z \partial_j z_\alpha + \partial_j Z^\dagger \partial_i Z z_\alpha + \partial_i Z^\dagger Z \partial_j z_\alpha \right] , \tag{11}$$

$$\pi_i = \frac{\delta \mathcal{L}}{\delta \dot{A}^i} = 0 \,, \tag{12}$$

$$\pi_0 = \frac{\delta \mathcal{L}}{\delta \dot{A}^0} = 0 \,, \tag{13}$$

$$\pi_\lambda = \frac{\delta \mathcal{L}}{\delta \dot{\lambda}} = 0 \,. \tag{14}$$

The Eqs. (12)–(14) represent the primary constraints of this model. The Hopf Lagrangian (9) contains terms of the type $\dot{z}_\alpha$ and $\dot{z}^*_\alpha$ which are first order in the time derivative. Hence, the canonical Hamiltonian in terms of the phase space variables can be readily obtained as

$$\begin{aligned}\mathcal{H}_c &= \pi^*_\alpha \pi_\alpha - iA_0\left(\pi^*_\alpha z^*_\alpha - \pi_\alpha z_\alpha + z_\alpha \frac{\delta \mathcal{L}_{\rm H}}{\delta \dot{z}_\alpha} - z^*_\alpha \frac{\delta \mathcal{L}_{\rm H}}{\delta \dot{z}^*_\alpha}\right) \\ &\quad -\left(\pi_\alpha \frac{\delta \mathcal{L}_{\rm H}}{\delta \dot{z}^*_\alpha} + \pi^*_\alpha \frac{\delta \mathcal{L}_{\rm H}}{\delta \dot{z}_\alpha}\right) + \frac{\delta \mathcal{L}_{\rm H}}{\delta \dot{z}_\alpha}\frac{\delta \mathcal{L}_{\rm H}}{\delta \dot{z}^*_\alpha} + |D_i Z|^2 + \lambda(Z^\dagger Z - 1)\,. \end{aligned} \tag{15}$$

Preservation of the primary constraints (12)–(14) in time yield the following set of secondary constraints

$$A_i + \frac{i}{2(Z^\dagger Z)} Z^\dagger \overset{\leftrightarrow}{\partial_i} Z \approx 0\,, \tag{16}$$

$$\pi^*_\alpha z^*_\alpha - \pi_\alpha z_\alpha + z_\alpha \frac{\delta \mathcal{L}_{\rm H}}{\delta \dot{z}_\alpha} - z^*_\alpha \frac{\delta \mathcal{L}_{\rm H}}{\delta \dot{z}^*_\alpha} \approx 0\,, \tag{17}$$

$$Z^\dagger Z - 1 \approx 0\,, \tag{18}$$

respectively. From the constraint (18), a new tertiary constraint

$$\pi^*_\alpha z^*_\alpha + \pi_\alpha z_\alpha \approx 0 \tag{19}$$

is obtained. The constraint (17) can be simplified further using (19) to yield

$$\pi^*_\alpha z^*_\alpha - \pi_\alpha z_\alpha + 2\Theta \varepsilon^{ij} \partial_i Z^\dagger \partial_j Z \approx 0\,. \tag{20}$$

Finally, by demanding the preservation of (20) in time, one more constraint

$$\pi^*_\alpha \pi_\alpha + (D_i D_i Z)^\dagger Z - \lambda + \Theta - \text{dependent terms} \approx 0 \tag{21}$$

is obtained, where the last Θ -dependent terms are independent of λ. It can be checked that there exist no further constraints.

At this stage it is necessarry to classify the total set of constraints (12)–(14), (16), (18)–(21) into first and second class ones [12]. It can be checked that the pairs (18), (19), (14), (21) and (12), (16) are the second class constraints. Only the constraint (20) is first class, leaving apart the trivial constraint (13). The above pair of the second class constraints can be 'strongly' implemented by the Dirac Brackets (DB)

$$\{\lambda(x), \pi_\lambda(y)\} = 0\,, \tag{22}$$

$$\{A_i(x), \pi^j(y)\} = 0\,, \tag{23}$$

$$\{z_\alpha(x)\,, z_\beta(y)\} = \{z^*_\alpha(x)\,, z^*_\beta(y)\} = \{z^*_\alpha(x)\,, z_\beta(y)\} = 0\,, \tag{24}$$

$$\begin{aligned}
\{z_\alpha(x), \pi_\beta(y)\} &= [\delta_{\alpha\beta} - \frac{1}{2} z_\alpha(x) z^*_\beta(x)]\delta(x-y)\,, && (25)\\
\{z_\alpha(x), \pi^*_\beta(y)\} &= -\frac{1}{2} z_\alpha z_\beta \delta(x-y)\,, && (26)\\
\{\pi_\alpha(x), \pi_\beta(y)\} &= \frac{1}{2}[z^*_\beta \pi_\alpha - \pi_\beta z^*_\alpha]\delta(x-y)\,, && (27)\\
\{\pi_\alpha(x), \pi^*_\beta(y)\} &= \frac{1}{2}[z_\beta \pi_\alpha - \pi^*_\beta z^*_\alpha]\delta(x-y)\,. && (28)
\end{aligned}$$

Further, it follows that the constraint (20), (using the DB's (22)–(28))

$$G(x) \equiv i\Big(\pi_\alpha(x) z_\alpha(x) - \pi^*_\alpha(x) z^*_\alpha(x) - 2\Theta \varepsilon^{ij} \partial_i Z^\dagger(x) \partial_j Z(x)\Big) \approx 0 \quad (29)$$

generates a U(1) gauge transformation

$$\delta z_\alpha(x) = \int d^2y f(y) \{z_\alpha(x), G(y)\} = i f(x) z_\alpha(x) \quad (30)$$

and therefore can be identified with the Gauss constraint. This is in conformity with the fact that constraint (29) is obtained by preserving (13) in time, just as in Maxwell electrodynamics. It is easy to verify that the Gauss constraint has vanishing DB's with the two second class constraints (18) and (19). It should be noted that from (10), (11) and (29) one can solve for A_0 to get $A_0 = -iZ^\dagger \partial_0 Z$. The spatial components A_i are also given (using (16), (18)) as $A_i = -iZ^\dagger \partial_i Z$. Again, this is in conformity with the result (7) obtained from geometrical considerations [10]. As expected, the Hopf term being a total derivative [2,8], does not enter explicitly in the expressions for the DB's. However, the Gauss constraint modified by the presence of a Θ-dependent piece in (29), thereby distinguishing the present model from the case of a pure CP^1 model without any Hopf term.

To construct the various spacetime symmetry generators, one can either follow the Noether's prescription, or from the symmetric energy-momentum (EM) tensor obtained by functional differentiation of the action with respect to the metric. Using the latter method first, we get

$$T^s_{\mu\nu} = (D_\mu Z)^\dagger (D_\nu Z) + (D_\nu Z)^\dagger (D_\mu Z) - g_{\mu\nu} (D_\rho Z)^\dagger (D^\rho Z) \quad (31)$$

from which it follows that the expression for linear momentum in terms of the phase space variables is given by

$$\begin{aligned}
P^s_j &\equiv \int d^2x T^s_{0j} = P^n_j \\
&+ 2i\Theta \varepsilon^{ik} \int d^2x (A_i \partial_j Z^\dagger \partial_k Z - A_j \partial_i Z^\dagger \partial_k Z - A_i \partial_k Z^\dagger \partial_j Z) - \int d^2x A_j(x) G(x)\,,
\end{aligned} \quad (32)$$

where

$$P_j^n \equiv \int d^2x p_j^n = \int d^2x \Big(\pi_\alpha \partial_j z_\alpha + \pi_\alpha^* \partial_j z_\alpha^* \Big) \tag{33}$$

is the corresponding expression obtained from Noether's prescription.

Now using the fact that in two spatial dimensions one can write $\partial_i A_j - \partial_j A_i = \varepsilon_{ij} B$ (B being the magnetic field), it can be shown that the integrand in the Θ-dependent term in (32) vanishes exactly. However, because of the presence of the last term involving the Gauss constraint $G(x)$ in (32), P_j^s fails to generate the appropriate translation because $\{z_\alpha(x), P_j^s\} = D_j z_\alpha$ in contrast to P_j^n which, by construction, generates the appropriate translation, *i.e.*, $\{z_\alpha, P_j^n\} = \partial_j z_\alpha$. However, on the Gauss constraint surface (29) P_j^s (32) gets simplified to $\tilde{P}_j^s \equiv \int d^2x p_j^s$ and generates appropriate translations like P_j^n. This is equivalent to modifying P_j^s by an appropriate linear combination of first class constraint(s) (here only G (29)) to get $\tilde{P}_j^s$. In fact, $\tilde{P}_j^s$ is identically the same as P_j^n (33), also the same is holding for their respective densities *i.e.*,

$$p_j^s = p_j^n \, . \tag{34}$$

The generator of rotational symmetry, namely the angular momentum operator is given by

$$\begin{aligned} J^s &= \int d^2x \varepsilon_{mj} x_m p_j^s \, , \\ J^n &= \int d^2x \varepsilon_{mj} x_m p_j^n \end{aligned} \tag{35}$$

obtained from the symmetric EM tensor (31), and from Noether's prescription, respectively. It can be checked that J^n and J^s generate the appropriate rotations $\{Z, J\} = \varepsilon^{ij} x_i \partial_j Z$. By the adjective "appropriate" we mean that the bracket $\{Z(x), J\}$ is precisely the Lie derivative $\mathcal{L}_{\partial_\phi} Z(x) = \partial_\phi Z(x)$, where ∂_ϕ is the vector field associated with J (ϕ being the angular variable in the polar coordinate system in the $2D$ plane), thus showing that no anomalous term is obtained in this bracket, as expected. From (34) and (35) we have

$$J^s = J^n = J = \int d^2x \varepsilon_{mj} x_m \left[\pi_\alpha \partial_j z_\alpha + \pi_\alpha^* \partial_j z_\alpha^* \right] . \tag{36}$$

By looking at the above expression it is clear that the angular momentum J (which is of course gauge invariant) does not contain any term other than the orbital part usually present in a model containing spin zero scalar fields. Although a quantum mechanical concept, fractional angular momentum may be revealed at the classical level itself through the difference $(J^s - J^n)$ computed after a proper Hamiltonian analysis [3,6]. However, as we have seen above (36), in this case no fractional spin is exhibited by inclusion of the Hopf term at the classical level. This result should not be surprising since the Hopf term [8] is a total divergence, and thus should not alter any observable expression like angular momentum at the classical level. Nevertheless, a complete quantum mechanical analysis of this model is required to settle this question fully. Such an analysis in the Dirac scheme is rather involved due to operator ordering ambiguities, and as we argue below, a quantization of the model in a straightforward manner cannot be performed. This is because all the DB's (22-28) *cannot* be elevated to their respective quantum commutators, which is required for the second class constraints (18) and (19) to become "strongly" valid operator equations.

Note that this problem does not arise for the brackets (24)–(26), as the Z fields can be taken as commuting variables:

$$[\hat{z}_\alpha(x), \hat{z}_\beta(y)] = [\hat{z}_\alpha^\dagger(x), \hat{z}_\beta^\dagger(y)] = [\hat{z}_\alpha^\dagger(x), \hat{z}_\beta(y)] = 0\,. \tag{37}$$

With this, there is no operator ordering problem for (25) and (26):

$$[\hat{z}_\alpha(x), \hat{\pi}_\beta(y)] \;=\; i\hbar\left(\delta_{\alpha\beta} - \frac{1}{2}\hat{z}_\alpha(x)\hat{z}_\beta^\dagger(x)\right)\delta(x-y)\,, \tag{38}$$

$$\left[\hat{z}_\alpha(x), \hat{\pi}_\beta^\dagger(y)\right] \;=\; -\frac{i\hbar}{2}\hat{z}_\alpha(x)\hat{z}_\beta(x)\delta(x-y)\,. \tag{39}$$

The same, however, is not true for (27) and (28). Different orderings of $\hat{z}_\alpha$ and $\hat{\pi}_\beta$ and their hermitian conjugates give different quantum theory. For the quantum theory to be consistent, the "strong" constraints (18) and (19) should hold as operator identities, *i.e.*, one must have

$$\left[\hat{z}_\alpha^\dagger(x)\hat{z}_\alpha(x), \hat{z}_\beta(y)\right] \;=\; 0\,, \tag{40}$$

$$\left[\hat{z}_\alpha^\dagger(x)\hat{z}_\alpha(x), \hat{\pi}_\beta(y)\right] \;=\; 0\,, \tag{41}$$

$$\left[\hat{z}_\alpha^\dagger(x)\hat{\pi}_\alpha^\dagger(x) + \hat{\pi}_\alpha(x)\hat{z}_\alpha(x), \hat{z}_\beta(y)\right] \;=\; 0\,, \tag{42}$$

$$\left[\hat{z}_\alpha^\dagger(x)\hat{\pi}_\alpha^\dagger(x) + \hat{\pi}_\alpha(x)\hat{z}_\alpha(x), \hat{\pi}_\beta(y)\right] \;=\; 0 \tag{43}$$

and their hermitian conjugates. Note that we have taken the hermitian form of the constraints (18) and (19) at the quantum level. It can be seen that equations (40), (41) and (42) can be easily satisfied for any operator ordering. However, (43) is not satisfied for any of the possible operator orderings. For example, with the following ordering for the operators in the quantum commutators corresponding to (27) and (28):

$$
\begin{aligned}
[\hat{\pi}_\alpha(x), \hat{\pi}_\beta(y)] &= \frac{i\hbar}{2}(\hat{z}^\dagger_\beta \hat{\pi}_\alpha - \hat{\pi}_\beta \hat{z}^\dagger_\alpha)\delta(x-y)\,, &(44)\\
\left[\hat{\pi}_\alpha(x), \hat{\pi}^\dagger_\beta(y)\right] &= \frac{i\hbar}{2}(\hat{z}_\beta \hat{\pi}_\alpha - \hat{\pi}^\dagger_\beta \hat{z}^\dagger_\alpha)\delta(x-y) &(45)
\end{aligned}
$$

one gets for the left hand side of (43)

$$\frac{\hbar}{2}[\hat{z}^\dagger_\beta(x), G(y)]\delta(x-y) \tag{46}$$

which is a $O(\hbar^2)$ term and is clearly nonvanishing. It can be checked that with other orderings of the operators, the situation becomes worse. These are typical problems one encounters while quantizing theories with second class constraints (see [14], for example).

This shows that the DB's cannot be elevated to quantum commutators in a straightforward manner. Consequently, quantization of the model is a nontrivial job. However, it is hoped that quantization might be feasible by extending the configuration space whereby the second class constraints become first class, or by using BRST techniques. These options are presently under investigation.

After having analyzed in detail the CP^1 model, let us consider once again the classically equivalent nonlinear sigma model (1) coupled to the Hopf term. Note that in [3] this model was altered through simplification of the Hopf term ($\sim A^\mu j_\mu$) to ($\sim A^i j_i$) by making use of the identity

$$\int d^2x A_0(x) j_0(x) = -\int d^2x A_i(x) j_i(x) \tag{47a}$$

valid in the radiation gauge. The same identity can also be used to alter the Hopf term (9) in the present case as well. We emphasize that the model is *altered* as (47a) when rewritten entirely in terms of the Z variables (the only independent configuration space variables) takes the form

$$\int d^2x Z^\dagger \dot{Z} \vec{\nabla} Z^\dagger \times \vec{\nabla} Z = \int d^2x Z^\dagger \vec{\nabla} Z \times \left[\vec{\nabla} Z^\dagger \dot{Z} - \dot{Z}^\dagger \vec{\nabla} Z\right]. \tag{47b}$$

Clearly, this is not a constraint equation as it involves time derivatives, and therefore changes the dynamical content of the model. This will be borne out by explicit computation now.

To begin with, note that once the identity (47) is used, the Hopf term (9) is changed to $(\frac{\Theta}{\pi}\varepsilon^{i\nu\lambda}\partial_\nu Z^\dagger\partial_\lambda Z Z^\dagger\partial_i Z)$. Correspondingly, the model (8) is changed to

$$\mathcal{L} = |D_\mu Z|^2 + \frac{\Theta}{\pi}\varepsilon^{i\nu\lambda}\partial_\nu Z^\dagger\partial_\lambda Z Z^\dagger\partial_i Z - \lambda(Z^\dagger Z - 1)\,. \tag{48}$$

This just the CP^1 version of the model $\mathcal{L} = \frac{1}{4}(\partial_\mu M_a)^2 + 2\Theta J_i A_i$ considered in [3]. The canonically conjugate momenta corresponding to z_α and z^*_α are given by

$$\begin{aligned}\pi_\alpha &= (D_0 z)^*_\alpha + \frac{\Theta}{\pi}\varepsilon^{ij} Z^\dagger\partial_i Z\partial_j z^*_\alpha\,,\\ \pi^*_\alpha &= (D_0 z)_\alpha - \frac{\Theta}{\pi}\varepsilon^{ij} Z^\dagger\partial_i Z\partial_j z_\alpha\,.\end{aligned} \tag{49}$$

A rerun of the constraint analysis shows that certain differences crop up in the constraint stucture from that of the model (8). For example the Gauss constraint, the counterpart of (29), becomes

$$\pi^*_\alpha z^*_\alpha - \pi_\alpha z_\alpha \approx 0 \tag{50}$$

just as in case of a pure CP^1 model. Furthermore, in this model (48) the expressions for the various symmetry generators obtained from the symmetric EM tensor $T^s_{\mu\nu}$ differ from the expressions obtained through the Noether prescription. In particular, the symmetric angular momentum J^s is given by

$$J^s = \int d^2x\varepsilon_{mj}x_m\left[\pi_\alpha\partial_j z_\alpha + \pi^*_\alpha\partial_j z^*_\alpha\right] - 2\Theta\int d^2x\varepsilon_{ij}x_i A_j j^0\,. \tag{51}$$

The first term is just J^n. The second Θ-dependent term in (51) can be simplified on lines of the procedure used in [3] to get ΘN^2 (where N is the soliton number given by (2)), and interpreted to signify fractional spin

$$J^f = J^s - J^n = \Theta N^2 \tag{52}$$

in this model. This analysis clearly brings out the point that the model (48) is basically inequivalent to the model (8).

To conclude, we make the following observations. First, the absence of fractional spin (although a quantum mechanical concept, we use the definition $J^f = J^s - J^n$ at the classical level) in (8) and its presence in (48) are in conformity with the fact that the Hopf term in (8) is a total divergence, whereas in (48) it is not. Secondly, we want to emphasize that use of just the radiation gauge without making use of (47) will not yield any anomalous term in the algebra $\{Z(x), J\}$, and the difference $(J^s - J^n)$ will persist to be zero. If the radiation gauge condition is imposed, the corresponding symplectic structure of the reduced phase space will undergo modification, but the bracket $\{Z(x), J\} = \varepsilon^{ij} x_i \partial_j Z$ will remain unchanged as $\{G(x), J\} = 0 = \{\partial_i(-iZ^\dagger \partial_i Z), J\}$ (note that J is gauge invariant by construction), thus generating no anomalous transformation. The model (8) has to be altered to (48) by using the identity (47) (which is valid in the radiation gauge) in order to reveal fractional spin. Thirdly, the fractional spin (52) can be obtained at the classical level itself, as we have derived it. Although in [3], a result of fractional spin valid at the quantum level was claimed (although none of the operator ordering problems mentioned earlier were discussed there), it survives the classical limit ($\hbar \to 0$) as Θ has the dimensions of $\hbar$ itself. Finally, if Dirac quantization of the model (8) is eventually carried out, fractional spin may or may not appear at the quantum level. In case it does, it must contain a factor of $\hbar$, so that in the classical limit, $J_f = (J^s - J^N) = 0$ (36) is reproduced. Hence, the expression of J^f will be different from (52). Finally, we end by noting that the issue of fractional spin in the nonlinear sigma model and the CP^1 model coupled to the Hopf term is not yet completely settled at the quantum level. For instance, it has been observed recently [15], using the method of adjoint orbit parametrization, that the standard formula for fractional spin holds only for certain restricted configurations.

A.S.M. would like to acknowledge the financial support provided as a part of a project funded by the Department of Science and Technology, Government of India.

REFERENCES

[1] J.F. Schonfeld, *Nucl. Phys.* **B185**, 157 (1981); S. Deser, R. Jackiw, S. Templeton, *Phys. Rev. Lett.* **48**, 975 (1982); *Ann. Phys. (N.Y.)* **140**, 372 (1982); S. Deser, R. Jackiw, *Phys. Lett.* **B139**, 371 (1984); D. Boyanovsky, R. Blankenbecker, R. Yabalin, *Nucl. Phys.* **B270**, 483 (1986); R. Jackiw, *Nucl. Phys. B Proc. Suppl.* **18**, 107 (1991); E. Fradkin, *Field Theories in Condensed Matter Systems*, Addison Wesley, 1991; S. Forte, *Rev. Mod. Phys.* **64**, 193 (1992); W. Apel, Yu A. Bychkov, *Phys. Rev. Lett.* **78**, 2188 (1997); G.E. Vloviv, V.M. Yakovenko, *Phys. Rev. Lett.* **79**, 3791 (1997).

[2] F. Wilczek, A. Zee, *Phys. Rev. Lett.* **51**, 2250 (1983).

[3] M. Bowick, D. Karabali, L.C.R. Wijewardhana, *Nucl. Phys.* **B271**, 417 (1986).

[4] D. Finkelstein, *J. Math. Phys.* **7**, 1218 (1966); D. Finkelstein, R. Rubinstein, *J. Math. Phys.* **9**, 1762 (1968).

[5] E. Witten, *Nucl. Phys.* **B223**, 433 (1983).

[6] P.K. Panigrahi, S. Roy, W. Scherer, *Phys. Rev.* **D38**, 3199 (1988); A. Kovner, *Phys. Lett.* **B224**, 299 (1989); R. Banerjee, *Phys. Rev. Lett.* **69**, 17 (1992); *Phys. Rev.* **D48**, 2905 (1993); R. Banerjee, B. Chakraborty, *Phys. Rev.* **D49**, 5431 (1994); B. Chakraborty, *Ann. Phys. (N.Y.)* **244**, 312 (1995); Y. Cho, K. Kimm, *Phys. Rev.* **D52**, 7325 (1995); R. Banerjee, B. Chakraborty, *Ann. Phys. (N.Y.)* **247**, 188 (1996); B. Chakraborty, A.S. Majumdar, *Ann. Phys. (N.Y.)* **250**, 112 (1996).

[7] C.R. Hagen, *Ann. Phys.* **157**, 342 (1984); *Phys. Rev.* **D31**, 331 (1985).

[8] Y.S. Wu, A. Zee, *Phys. Lett.* **B147** , 325 (1984).

[9] R. Banerjee, *Phys. Rev.* **D49**, 2133 (1994).

[10] R. Banerjee, B. Chakraborty, *Nucl. Phys.* **B449**, 317 (1995).

[11] B. Chakraborty, T.R. Govindarajan, *Mod. Phys. Lett.* **A12**, 619 (1997).

[12] P.A.M. Dirac, *Lectures on Quantum Mechanics*, Belfar Graduate School of Science, Yeshiva University, New York 1964; A. Hanson, T. Regge, C. Teitelboim, *Constrained Hamiltonian Analysis*, Academia Nazionale dei Lincei, Rome 1976.

[13] J. Romano, R. Tate, *Classical Quantum Gravity* **6**, 1487 (1989); K. Schleich, *Classical Quantum Gravity* **7**, 1529 (1990); R. Loll, *Phys. Rev.* **D41**, 3785 (1990); M.S. Plyushchay, A.V. Razumov, hep-th/9412137; *Int. J. Mod. Phys.* **A11**, 1427 (1996).

[14] M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, Princeton 1992.

[15] I. Tsutsui, S. Tanimura, H. Kobayashi, *Nucl. Phys.* **B514**, 667 (1998); I. Tsutsui, M. Kimura, H. Kobayashi, hep-th/9801096.