SPECTRUM OF THE ODDERON CHARGE FOR ARBITRARY CONFORMAL WEIGHTS

Michał Praszałowicz† and Andrzej Rostworowski

Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Kraków, Poland

(Received November 11, 1998)

The odderon equation is studied in terms of the variable suggested by the modular invariance of the 3 Reggeon system. Odderon charge is identified with the cross-product of three conformal spins. A complete set of commuting operators: $\hat{h}^2$ and $\hat{q}$ is diagonalized and quantization conditions for eigenvalues of the odderon charge $\hat{q}$ are solved for arbitrary conformal weight h.

PACS numbers: 12.38.Cy, 11.55.Jy

In 1980, following Lipatov and collaborators [1, 2], the integral equation for the exchange of 3 or more reggeized gluons has been formulated [3–5]. This equation describes both the leading contribution to the odderon (odd C parity) exchange, as well as the unitarity correction to the Pomeron. Although originally the odderon equation has been written in the momentum space, it turns out, that it is convenient to rewrite it in the 2 dimensional configuration space of the impact parameters $b_i = (x_i, y_i)$, where $i = 1, 2$ or 3 (or n for amplitudes with more Reggeon exchange) [6]. It has been observed that the odderon problem is equivalent to the system of 3 conformal spins [7–9] associated with each Reggeon i:

$$S_+^{(i)} = z_i^2\partial_i\,, \qquad S_3^{(i)} = z_i\partial_i \quad \text{and} \quad S_-^{(i)} = -\partial_i\,. \tag{1}$$

Here $z_i = x_i + iy_i$. The odderon intercept

$$\alpha = 1 - \frac{N_c\alpha_s}{\pi}E\,, \tag{2}$$

† A.v. Humboldt fellow at the Institute for Theoretical Physics II, Ruhr-University Bochum, D–44780 Bochum, Germany.

is related to the eigenvalue E of the interaction Hamiltonian of the conformal spins [6, 9]:

$$\mathcal{H} = \text{const.} \sum_{i>j}^{3} \left(H(z_i, z_j) + H(\overline{z}_i, \overline{z}_j)\right) \,, \tag{3}$$

where the explicit form of H can be found *e.g.* in Ref. [9]. Equation (3) exhibits conformal separability into holomorphic and antiholomorphic parts, the latter depending only on $\overline{z}_i = x_i - iy_i$. Therefore $E = \varepsilon + \overline{\varepsilon}$ and the wave function is given as a bilinear form $\Phi(z, \overline{z}) = \overline{\Psi}(\overline{z}) \times \Psi(z)$. There are two conditions for the total wave function: 1) $\Phi(z, \overline{z})$ has to be single-valued and 2) normalizable, which determine the spectrum of E.

There are two scalars which can be constructed from three spins:

$$\begin{aligned} \hat{h}^2 &= -\left(\vec{S}^{(1)} + \vec{S}^{(2)} + \vec{S}^{(3)}\right)^2, \\ \hat{q} &= -\,\vec{S}^{(1)} \cdot \left(\vec{S}^{(2)} \times \vec{S}^{(3)}\right) . \end{aligned} \tag{4}$$

It has been shown in Refs. [7–9] that the Casimir operator $\hat{h}^2$ and the *odderon charge* $\hat{q}$ (denoted often as $\hat{q}_3$) can be simultaneously diagonalized.

There have been many attempts to find either directly values of E [10–13] or spectrum of $\hat{q}$ [14,15]. In the recent paper Janik and Wosiek [16] calculated the spectrum of $\hat{q}$ for *conformal weight* $h = \overline{h} = 1/2$, which corresponds to the lowest representation of the SL(2,C) group, and found the odderon intercept with $E = 0.24717$.

In the present paper we shall concentrate on calculating the spectrum of the odderon charge for arbitrary h. We shall first consider the holomorphic sector only; however the same arguments apply to the antiholomorphic sector as well.

Following Lipatov [6] we shall use conformal Ansatz for Ψ:

$$\Psi(z_1, z_2, z_3) = z^{h/3}\, \psi(x), \tag{5}$$

where

$$z = \frac{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}{(z_1 - z_0)^2 (z_2 - z_0)^2 (z_3 - z_0)^2} \,, \qquad x = \frac{(z_1 - z_3)(z_3 - z_0)}{(z_1 - z_0)(z_3 - z_2)} \tag{6}$$

and z_0 is a reference point. A remarkable feature of Ansatz (5) is, that $\hat{h}^2$ is automatically diagonal:

$$\hat{h}^2\, \Psi(z_1, z_2, z_3) = -h(h-1)\, \Psi(z_1, z_2, z_3)\,. \tag{7}$$

In the representation (5) the eigenvalue equation for q takes the following form:

$$\begin{aligned} i\hat{q}\,\psi(x) \;=\; & -\left(\frac{h}{3}\right)^2\left(\frac{h}{3}-1\right)\frac{(x-2)(x+1)(2x-1)}{x(x-1)}\,\psi(x) \\ & -\left[2x(x-1)-\frac{h}{3}(h-1)\left(x^2-x+1\right)\right]\psi'(x) \\ & -2x(x-1)(2x-1)\,\psi''(x) \quad -x^2(x-1)^2\,\psi'''(x)\,. \end{aligned} \tag{8}$$

This equation has been recently studied by Janik and Wosiek in Ref. [16]. They have formulated quantization conditions for q by imposing singlevaluedness constraints on the whole wave function $\Phi(z,\overline{z})$, and solved them for $h=\overline{h}=1/2$. Discrete, symmetrically distributed values of q have been found on the imaginary, as well as on the real axis in the complex q plane. However, only the imaginary values of q are relevant for the odderon problem; real q's correspond to the wave function which is antisymmetric if the two neighboring Reggeons are exchanged, whereas the odderon wave function should be symmetric under such transformations [15, 16].

It is convenient to rewrite the eigenequation for q in terms of a new variable:

$$\xi = i\,\frac{1}{3\sqrt{3}}\,\frac{(x-2)(x+1)(2x-1)}{x(x-1)}\,, \tag{9}$$

suggested in Ref. [17], where the modular invariance of the odderon equation has been discussed. This mapping sends all singular points of the original equation *i.e.* $x=0$, 1 and ∞ to infinity. The advantage of using variable ξ instead of x consists in the symmetry properties of ξ under the cyclic permutations of the three reggeized gluons, which correspond to:

$$x \to 1-\frac{1}{x}\,, \quad \text{or} \quad x \to \frac{1}{1-x}\,. \tag{10}$$

Under transformations (10) ξ remains unchanged.

In variable ξ we have:

$$\begin{aligned} \Bigg[\frac{1}{2}(\xi^2-1)^2\frac{d^3}{d\xi^3}+2\xi(\xi^2-1)\frac{d^2}{d\xi^2} \;&+\; \left(\frac{4}{9}-\frac{(h+2)(h-3)}{6}(\xi^2-1)\right)\frac{d}{d\xi} \\ &+\; \frac{h^2(h-3)}{27}\xi+\frac{q}{3\sqrt{3}}\Bigg]\,\psi(\xi)=0\,, \end{aligned} \tag{11}$$

where q is an eigenvalue of $\hat{q}$.

Our strategy consists in applying the method of Ref. [16] to Eq. (11). The advantage of using (11) is twofold: 1) because of the symmetry properties

(10) the quantization condition takes a simpler form than in the case of Ref. [16], 2) since Eq. (11) is less singular than the one in Ref. [16], the solutions of the indicial equation do not depend on h. Because of the latter it is easy to find spectrum of $\hat{q}$ for an arbitrary h.

Equation (11) has 3 regular singular points in $\xi = \pm 1$ and in infinity. In what follows we shall consider only solutions around ± 1:

$$u_s^{(\pm 1)}(\xi; q) = (1 \mp \xi)^s \sum_{n=0}^{\infty} u_n^{(\pm 1)} (\xi \mp 1)^n \ . \tag{12}$$

The phases of the two solutions are chosen in such a way, that they are real for real $-1 < \xi < 1$ and real q. The indicial equation for s has the following solutions: $s_1 = 2/3$, $s_2 = 1/3$ and $s_3 = 0$. Introducing notation:

$$\beta_h = \frac{(h+2)(h-3)}{6}\ , \quad \gamma_h = \frac{h(h-1)}{6}\ , \quad \rho_h = \frac{h^2(h-3)}{27}\ , \quad \tilde{q} = \frac{q}{3\sqrt{3}} \tag{13}$$

we can write the recurrence formula:

$$\begin{aligned} u_0^{(\pm 1)} &= 1 \ , \\ u_1^{(\pm 1)} &= \mp \frac{2s(s^2 - 1 - \beta_h) + \rho_h \pm \tilde{q}}{2(1+s)\left[s(1+s) + \frac{2}{9}\right]} \ , \\ u_{n+2}^{(\pm 1)} &= \mp \frac{2(n+1+s)\left[(n+s)(n+2+s) - \beta_h\right] + \rho_h \pm \tilde{q}}{2(n+2+s)\left[(n+2+s)(n+1+s) + \frac{2}{9}\right]} u_{n+1}^{(\pm 1)} \\ &\quad - \frac{(n+s)\left[(n-1+s)(n+2+s) - 2\beta_h\right] + 2\rho_h}{4(n+2+s)\left[(n+2+s)(n+1+s) + \frac{2}{9}\right]} u_n^{(\pm 1)} \ . \end{aligned} \tag{14}$$

These series are convergent in circles of radius 2. Analogously to Eq. (12) one can define solutions in the antiholomorphic sector which in the following will be denoted as $v_s^{(\pm 1)}(\overline{\xi}; \overline{q})$. The three solutions corresponding to 3 different s_i values form a vector:

$$\vec{u}^{(\pm 1)}(\xi; q) = \begin{bmatrix} u_{s_1}^{(\pm 1)}(\xi; q) \\ u_{s_2}^{(\pm 1)}(\xi; q) \\ u_{s_3}^{(\pm 1)}(\xi; q) \end{bmatrix} .$$

The analytical continuation matrix Γ is defined in the intersection of the two convergence circles:

$$\vec{u}^{(-1)}(\xi; q) = \Gamma(q)\, \vec{u}^{(1)}(\xi; q) \ . \tag{15}$$

In order to calculate Γ we construct a Wrońskian:

$$W = \begin{vmatrix} u_1^{(1)}(\xi) & u_2^{(1)}(\xi) & u_3^{(1)}(\xi) \\ {u'}_1^{(1)}(\xi) & {u'}_2^{(1)}(\xi) & {u'}_3^{(1)}(\xi) \\ {u''}_1^{(1)}(\xi) & {u''}_2^{(1)}(\xi) & {u''}_3^{(1)}(\xi) \end{vmatrix} . \tag{16}$$

Next we construct determinants W_{ij}, which are obtained from W by replacing j-th column by the i-th solution around -1. Then:

$$\Gamma_{ij} = \frac{W_{ij}}{W} . \tag{17}$$

Matrix $\Gamma(q)$ does not depend on ξ, but only on q and also on h. We choose to calculate it at $\xi = 0$. Repeating the same steps in the antiholomorphic sector one constructs $\overline{\Gamma}(\overline{q})$ where:

$$\overline{q} = -q^{\star} . \tag{18}$$

Here $\overline{q}$ denotes the odderon charge in the antiholomorphic sector, whereas the *star* over q denotes complex conjugation. In principle two possible choices for $\overline{q}$, namely with $+$ and $-$ signs should be considered. This follows from the fact that both ε and $\overline{\varepsilon}$ are symmetric functions of q (or $\overline{q}$) [9]. However, only the choice of Eq. (18) leads to the non-zero solutions of the quantization conditions[1].

Function Φ is single-valued if:

$$h = \tfrac{1}{2}(\mu + m) + i\nu \quad \text{and} \quad \overline{h} = \tfrac{1}{2}(\mu - m) + i\nu , \tag{19}$$

where μ and ν are real numbers where m is an integer multiple of 3. Further constraint comes from normalizability, which requires that $\mu = 1$ for the physical odderon state [6, 9].

The single-valued wave function can be constructed only if both sectors, holomorphic and antiholomorphic, are considered. In the vicinity of ξ, $\overline{\xi} = \pm 1$ the wave function of the whole system reads:

$$\Phi^{(\pm 1)}_{h\overline{h}\,q\overline{q}}(z, \xi, \overline{z}, \overline{\xi}) \;=\; z^{h/3}\, \overline{z}^{\overline{h}/3}\, \vec{v}^{(\pm 1)\,\mathrm{T}}(\overline{\xi}; \overline{q})\, A^{(\pm 1)}(\overline{q}, q)\, \vec{u}^{(\pm 1)}(\xi; q) . \tag{20}$$

The requirement that the wave function Φ should be single-valued, leads to the observation that matrices $A^{(\pm 1)}$ have to be diagonal: $A^{(-1)} = \mathrm{diag}(\alpha, \beta, \gamma)$ and $A^{(1)} = \mathrm{diag}(\alpha', \beta', \gamma')$. However, because the two solutions (20) are related by Eq. (15), we get the following relation:

$$\overline{\Gamma}^{\mathrm{T}}(\overline{q})\, A^{(-1)}(\overline{q}, q)\, \Gamma(q) = A^{(1)}(\overline{q}, q) . \tag{21}$$

[1] Note, that because of the factor of i in the definition of ξ (9), our sign for $\overline{q}$ is different than the one in Ref. [16].

Introducing $\vec{a} = (\alpha, \beta, \gamma)$ and $\vec{b} = (\alpha', \beta', \gamma')$ we can conveniently rewrite equations corresponding to the zeros of $A^{(1)}$ in the matrix form:

$$C_{\rm up}\, \vec{a} = 0 \quad \text{and} \quad C_{\rm low}\, \vec{a} = 0\,, \tag{22}$$

where matrix $C_{\rm up}$, corresponding to the 3 zeros above the diagonal of matrix $A^{(1)}$, takes the following form:

$$C_{\rm up} = \begin{bmatrix} \overline{\Gamma}_{11}\Gamma_{12} & \overline{\Gamma}_{21}\Gamma_{22} & \overline{\Gamma}_{31}\Gamma_{32} \\ \overline{\Gamma}_{11}\Gamma_{13} & \overline{\Gamma}_{21}\Gamma_{23} & \overline{\Gamma}_{31}\Gamma_{33} \\ \overline{\Gamma}_{12}\Gamma_{13} & \overline{\Gamma}_{22}\Gamma_{23} & \overline{\Gamma}_{32}\Gamma_{33} \end{bmatrix} \tag{23}$$

and matrix $C_{\rm low}$, corresponding to the zeros below the diagonal of $A^{(1)}$, is obtained from $C_{\rm up}$ by interchanging $\overline{\Gamma} \leftrightarrow \Gamma$.

Quantization conditions follow from the requirement that there exist non-zero solutions of Eq. (22) for α, β and γ:

$$\mathrm{Det}\, C_{\rm up} = 0 \quad \text{and} \quad \mathrm{Det}\, C_{\rm low} = 0\,. \tag{24}$$

Moreover the first two equations in (22) should be uniquely solvable for α, β and γ in function of one free parameter (which is not automatic even if Eqs. (24) are satisfied).

Let us now discuss numerical solutions of the quantization conditions (24). We have found that the zero eigenvalue of the odderon charge exists always for arbitrary conformal weight. Korchemsky [9] has argued that these states should be excluded from the spectrum of the odderon Hamiltonian (3), since it is not clear if they are normalizable (see however Ref. [12]). In what follows we shall concentrate only on the imaginary solutions for q which, as already said, are relevant for the odderon system.

We have first looked for the solutions of the quantization conditions (24) corresponding to $m = 0$ and $\nu = 0$, *i.e.* for $h = \overline{h}$. Here solutions are found for arbitrary real μ. With our choice of phases in Eq. (12), both $\mathrm{Det}\, C_{\rm up}$ and $\mathrm{Det}\, C_{\rm low}$ are imaginary along the imaginary axis in the complex q plane. Moreover, for imaginary q: $\mathrm{Det}\, C_{\rm up}$=$\mathrm{Det}\, C_{\rm low} \equiv \mathrm{Det}\, C$. In Fig. 1 we plot Im Det C as a function of Im q for $h = 1/2$, or equivalently $\mu = 1$ (solid line) and for $\mu = 2$ or 0 (long dash line). We see that Im Det C is an antisymmetric, oscillating function of Im q, with amplitude growing with |Im q|. Zeros of the functions in Fig. 1 correspond to the quantized values of q. For $\mu = 1$ two non-zero eigenvalues are visible: $q = \pm 0.2052575 \times i$ and $q = \pm 2.34392 \times i$. We have found two more eigenvalues of Im q: ± 8.326346 and ± 20.080497. For higher q's care must be taken in order not to loose numerical stability. These eigenvalues have been found previously in Ref. [16].

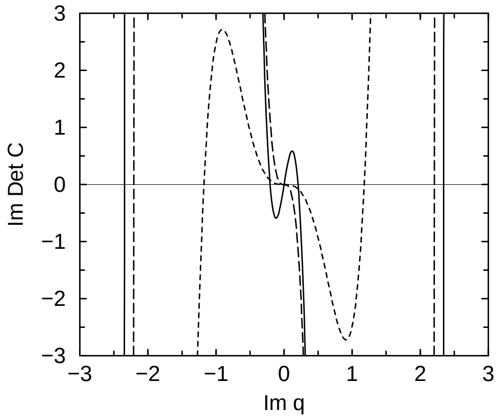

Fig. 1. Im Det C as function of Im q for $h = \overline{h} = 1/2$ — solid line, for $h = \overline{h} = 0$ and 1 — long dash line and for $(h, \overline{h}) = (2, -1)$ or $(-1, 2)$ — short dash line.

It is interesting to follow the flow of q in function of μ (still for $m = 0$ and $\nu = 0$). As soon as we move h away from $1/2$ the little wiggle, seen for $h = 1/2$ (or equivalently $\mu = 1$), straightens up and the two eigenvalues symmetrically drift towards zero. Eventually, for $h = 0$ and for $h = 1$, they reach zero value. This is depicted in Fig. 2 where the drift of the first 2 positive eigenvalues of $\hat{q}$ is plotted in dependence on $h = \overline{h}$. Second eigenvalue reaches zero for $h = 3$ and $h = -2$. This kind of behavior is observed for all imaginary eigenvalues, for which our numerical procedures are stable.

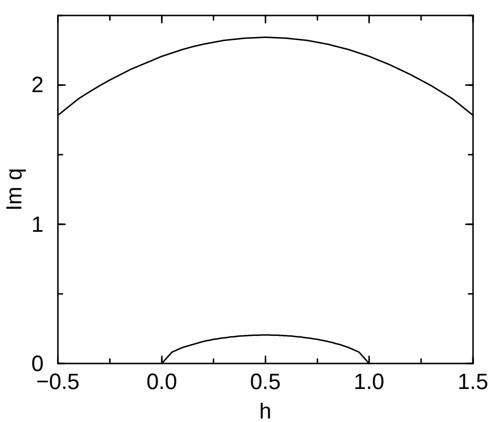

Fig. 2. Positive part of Im q of two first eigenvalues as functions of (real) $h = \overline{h}$.

Physical odderon state, however, corresponds to fixed $\mu = 1$. Therefore we have looked for the solutions of (24) for $\mu = 1$ and $\nu = 0$. Here solutions exist only for discrete values of m. The two lowest values of m for which non-zero, imaginary solutions for q exist are equal $m = 0$ and $|m| = 3$. There also exist solutions for higher $\mid m \mid$, which will not be discussed in this note. In Fig. 1 the short dash line corresponds to Im Det C for $|m| = 3$. Here only one non-zero eigenvalue of q is visible, namely $\pm 1.176667 \times i$. We have also found two next eigenvalues corresponding to Im $q = \pm 6.35591$ and ± 17.69346.

It is also interesting to consider complex $h = 1/2 \pm i\,\nu$. As soon as one varies ν, $\mathrm{Im}\, q$ grows as ν increases [18]. Our results agree with the ones of Ref. [18].

As a cross-check on our method we have also solved quantization conditions for $h = 3$, 4, 5 and 6. For these values of h Korchemsky in Ref. [9] found real spectrum of $\hat{q}$ for the polynomial solutions of the pertinent Baxter equations. Our quantization conditions are more general, so we find more eigenvelues for integer h, among them the ones reported in Ref. [9].

To summarize: in this short note we have solved quantization conditions for the odderon charge $\hat{q}$ which has been identified with the cross-product of three conformal spins. We have proposed to study the odderon equation in terms of a new variable called ξ, which was earlier discussed by Janik in the context of the modular invariance of the odderon system [17]. We have solved quantization conditions for the odderon charge using the method recently proposed by Janik and Wosiek [16]. Our approach can be straightforwardly applied for any conformal weight h. For $h = 1/2$ we have reproduced eigenvalues found in Ref. [16]. For integer $h \geq 3$ we have reproduced eigenvalues of $\hat{q}$ found by Korchemsky in Ref. [9]. To illustrate the possibility of solving the quantization conditions for arbitrary h we have studied the drift of the lowest eigenvalues of $\hat{q}$ for real h. For the physical odderon state: $\mu = 1$, $\nu = 0$ in Eq. (19) the lowest values of h for which Eq. (24) could have been found correspond to $m = 0$ and ± 3. For the unphysical condition $h = \overline{h}$ *i.e.* $m = 0$ continuous sets of solutions exist for real ($\nu = 0$) h. Further results, also for complex h and for real values of q, will be discussed in the forthcoming paper [19].

We thank R. Janik and J. Wosiek for discussions. Special thanks for hospitality are due to K. Goeke and the Institute of Theoretical Physics II of the Ruhr-University in Bochum, Germany, where most of this work was done. Partial support of Polish KBN Grant PB 2 PO3B 044 12 is acknowledged. M.P. acknowledges support of Alexander von Humboldt Stiftung.

REFERENCES

[1] E.A. Kuraev, L.N. Lipatov, V.S. Fadin, *Zh. Eksp. Teor. Fiz.* **71**, 840 (1976) (*JETP* **44**, 433 (1976), **45**, 199 (1977)).

[2] Ya.Ya. Balitzky, L.N. Lipatov, *Yad. Fiz.* **28**, 1597 (1978) (*Sov. J. Nucl. Phys.* **28**, 822 (1978)).

[3] J. Bartels, *Nucl. Phys.* **B175**, 365 (1980).

[4] J. Kwieciński, M.Praszałowicz, *Phys. Lett.* **B94**, 413 (1980).

[5] T. Jaroszewicz, High-energy multi-gluon exchange amplitudes, Trieste preprint IC/80/175, see also *Acta Phys. Pol.* **B11**, 965 (1980).

[6] L.N. Lipatov, *Phys. Lett.* **B251**, 284 (1990); **B309**, 394 (1993).

[7] L.N. Lipatov, *Pisma. Zh. Eksp. Teor. Fiz.* **59**, 569 (1994) (*JETP Lett.* **59**, 571 (1994)).

[8] L.D. Faddeev, G.P. Korchemsky, *Phys. Lett.* **B342**, 311 (1995).

[9] G.P. Korchemsky, *Nucl. Phys.* **B443**, 255 (1995).

[10] P. Gauron, L. Lipatov, B. Nicolescu, *Phys. Lett.* **B260**, 407 (1991); *Phys. Lett.* **B304**, 334 (1993); *Z. Phys.* **C63**, 253 (1994).

[11] N. Armesto, M.A. Braun, The intercept of symmetric multi-gluon configurations in the variational approach, hep-ph/9410411, Santiago de Compostela preprint US-FT-9-94.

[12] N. Armesto, M.A. Braun, *Z. Phys.* **C75**, 709 (1997).

[13] M.A. Braun, On the odderon intercept in the variational approach, hep-ph/9801352, St. Petersburg preprint SPbU-IP-1998/3.

[14] G.P. Korchemsky, *Nucl. Phys.* **B462**, 333 (1996); WKB quantization of reggeon compound states in high-energy QCD, hep-ph/9801377, Orsay preprint LPTHE-97-73.

[15] R. Janik, J. Wosiek, *Phys. Rev. Lett.* **79**, 2935 (1997).

[16] R. Janik, J. Wosiek, Solution of the odderon problem, hep-ph/9802100, Cracow preprint TPJU-2/98.

[17] R. Janik, *Phys. Lett.* **B371**, 293 (1996); *Acta Phys. Pol.* **B27**, 1275 (1996).

[18] M.A. Braun, On the odderon intercept in the perturbative QCD, hep-ph/9804432, St. Petersburg preprint SPbU-IP-1998/8.

[19] M. Praszałowicz, A. Rostworowski in preparation.