PSEUDO-SU(2) SYMMETRY AND A LOW ENERGY LIMIT OF THE DIRAC EQUATION WITH THE WOODS–SAXON POTENTIALS* **

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A structure of the Woods–Saxon deformed mean-field Hamiltonian adapted to the nuclear Dirac equation is discussed. The underlying SU(2) symmetries of the Hamiltonian are reviewed together with their consequences for the behavior of the single-nucleon spectra. This brings up a strong motivation for introducing the realistic Woods–Saxon parametrisation in such a way that the position-dependent effective mass, an extra linear momentum potential and other features are directly modeled. It is demonstrated that the relatively low values of the effective mass in the nuclear interior do *not* cause any incorrect level-density problem close to the Fermi level. Moreover, an over-all correct spreading of the single-particle levels over a large energy scale and a very good single-particle level order can be obtained. The corresponding preliminary calculation results are illustrated.

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1. Introduction and historical remarks

Mean-field nuclear structure calculations played and play until now the most important role in understanding numerous forms of nuclear behavior, for instance at high spins or those in the domain of physics of exotic- and halo-nuclei. A recent article [1] brings an important new understanding of the relativistic origin of the fundamental SU(2) symmetry that appears in the Dirac mean-field Hamiltonian, $\hat{H}_D$,

$$\hat{H}_D \stackrel{\text{def}}{=} \{c\vec{\alpha}\cdot\hat{\vec{p}} + \hat{V}(\vec{r})\;\mathbb{I}_4 + \beta\;[m_0c^2 + \hat{S}(\vec{r})]\} \;\rightarrow\; \hat{H}_D\,\psi_n = \mathcal{E}_n\,\psi_n, \tag{1}$$

in which $\{\vec{\alpha},\beta\}$ denote an ensemble of the usual Dirac matrices, $\mathbb{I}_4$ is a four-dimensional unit matrix, and $\hat{\vec{p}} = -i\hbar\nabla$. The forms of the $\vec{r}$-dependent interaction potentials $\hat{S}(\vec{r})$ and $\hat{V}(\vec{r})$ can be provided by the Relativistic Mean Field (RMF) theories in terms of couplings of the nucleon fields to the scalar- and the vector-meson fields, respectively (*cf. e.g.* Ref. [2] for a review). Paradoxically, the *consequences* of the SU(2) symmetry mentioned above have been discovered on the phenomenological grounds before the discovery of the symmetry *itself* already in 1969, Ref. [3]; slightly more formal framework has been given in Ref. [4]. Those early papers predicted an existence, and suggested a way to verify in experiment, of approximate degeneracies among certain well defined single-nucleonic levels. The degeneracies in question and other related features will be discussed in a considerable detail below in view of a more modern understanding of their origin.

The symmetry that underlies the existence of those degeneracies was called "pseudo-spin symmetry" or pseudo-SU(2) symmetry, below denoted $\mathrm{SU}_{\tilde{s}}(2)$. Despite a tendency towards a pejorative interpretation of the adjective "pseudo" this symmetry is probably one of the most fundamental (although approximate) symmetries that the nuclear mean-field should carry. This is so because, firstly, the existence of this symmetry relies on a general universal fact: the relatively weak nucleonic binding in all nuclei as a result of a partial cancellation of a strong nuclear attraction caused by an exchange of the scalar mesons and of a strong nuclear repulsion caused by an exchange of the vector mesons. Secondly, because up to a good an approximation this symmetry should depend only weakly (if at all) on the nuclear deformation and thus it should remain a common feature of all nuclei for which the mean-field can be introduced — *i.e.* for all, except for a very few very light nuclei.

In 1973, Ref. [5], it has been suggested that the pseudo-spin double degeneracies whose presence is manifested in deformed nuclei are involved in a larger symmetry structures, described in terms of a larger group, SU(3), called by the authors pseudo-SU(3) and below denoted $\mathrm{SU}(\tilde{3})$. Following the existence of this larger symmetry the pseudo-spin doublets are merely simple building blocks of a richer multiplet structure. Those "rich" multiplets involve an increasing number of the pseudo-spin doublets when the shell number, N, increases.

It is interesting to note that the formal understanding of the solution to the nuclear pseudo-spin symmetry problem in the framework of the Dirac formalism has been liyed out already in 1976, Ref. [6]. However, the authors of this article were unaware of the importance of their result for the nuclear structure domain. Reciprocally, the paper was seemingly unknown to the nuclear structure community who continued to investigate the pseudo-spin problem using various other means. In particular, Ref. [7], it has been demonstrated on the basis of the Nilsson model that the pseudo-spin doublets can be viewed upon as manifestations of a certain specific representation of the nucleonic-shell structure, *i.e.* the one in terms of a pseudo-orbital pseudo-spin interaction, $\delta\hat{V}_{\tilde{\ell}\cdot\tilde{s}} \sim \tilde{\ell}\cdot\tilde{s}$, rather than the spin-orbit $\delta\hat{V}_{\ell\cdot s} \sim \ell\cdot s$, interaction. According to such an approach, the "traditional" spin-orbit interaction leads to a strong spin-orbit splitting between orbitals corresponding to the spin-parallel, $\ell \uparrow\uparrow s$, *vs.* spin-antiparallel, $\ell \uparrow\downarrow s$, coupling of the intrinsic spin to the orbital angular momentum. Within a proposed picture in terms of pseudo-spin pseudo-orbital angular momenta, very small energy differences between the "parallel vs. antiparallel", $\tilde{\ell} \uparrow\uparrow \tilde{s}$ and $\tilde{\ell} \uparrow\downarrow \tilde{s}$ configurations are obtained, while an overall comparison with experiment remains qualitatively good in both cases.

The structure of the Nilsson-model Hamiltonian, quite particular from the pseudo-spin symmetry point of view, has become a central discussion element in various studies that addressed the problem of $\mathrm{SU}_{\tilde{s}}(2)$ doublets in the nucleonic spectra after 1982. In particular, a concept of a pseudo-oscillator as a harmonic oscillator model of the related algebraic $\mathrm{SU}(\tilde{3})$ properties has become a fashionable topic and a number of elegant mathematical ways of transforming the so-called normal-parity sets of the harmonic oscillator Hamiltonian to the corresponding pseudo-oscillator space has been found (*cf.* Ref. [8] and references therein, and for a review on earlier formulations: Refs. [9] and [10]). For the first time a possible relativistic origin of a weak pseudo-spin pseudo-orbital momentum coupling within the Nilsson model has been suggested in [11]; a discussion of the possible mathematical forms of the pseudo-spin transformation, again within the the Nilsson model structure, can be found in [12].

In the late 80'ies the consequences of the $\mathrm{SU}_{\tilde{s}}(2)$- and of the larger, $\mathrm{SU}(\tilde{3})$-symmetries for the domain of the very strongly deformed nuclei have been explored. In particular in 1987 — at an early stage of the nuclear superdeformation (SD) studies, when only two SD rotational bands were known experimentally, a general abundance scheme of the nuclear superdeformation as a large-scale nuclear phenomenon, [13], has been predicted theoretically. It has been later on confirmed *up to a detail* by the experiments with the multidetector systems on over a hundred of SD bands. In the mean time a possible influence of the pseudo-spin degeneracies on the existence of unexpected similarities among the SD bands ("sameness" among the SD bands) has been brought up in [14].

The most recent discovery of a connection between spin and pseudo-spin within the Dirac equation, symmetries of the latter and in particular an interrelation in terms of the large and small components of the Dirac bispinors bring more light on the relativistic dynamics of the problem [15,16]; these aspects will be overviewed and discussed in the following sections.

The new ways to understanding the mean-field Hamiltonian's SU(2) symmetries are slightly hidden. One needs to consider a four-component Dirac solutions rather than a "traditional" form of the nucleonic wave functions. One needs to explicitly use the fact that for the parity-conserving nuclear Dirac Hamiltonian the parity of the grand component must be opposite to that of the small component yet giving a good total parity of the Dirac bispinor, *etc.* The question of how to combine these facts is one of the issues underlying the pseudo-spin symmetry and goes deeper into the relativistic physics, in particular a decomposition of the Lorentz group in terms of two constituent SU(2) groups.

2. Motivations and relation to experiment

At the first glance one may think that the experimental results, for instance on the single particle levels in spherical nuclei, clarify in a rather unambigous manner the degree of the (weak) pseudo-spin symmetry breaking. The theoretical considerations provide the spectroscopic labels of the states that should be the pseudo-spin degenerate (see also below), the corresponding states have often been identified in experiment and their energy differences can be compared to zero — the larger the difference the stronger the symmetry breaking.

The spherical symmetry case, although in some sense the simplest, the most "academic" one, is by far not the only one of interest. The pseudo-spin doublets are obtained in any realistic deformed mean-field potentials as *e.g.* Woods–Saxon, Nilsson or Folded–Yukawa, as well as in the Hartree–Fock approaches, and are known to depend very little on the nuclear deformation over the very large variation ranges of the deformation parameters. The close-lying doublet states then propagate also together in terms of the rotational (cranking) frequency and possibly contribute to the similarities in behavior of certain rotational bands.

One may be tempted to say that the verification of the size of the pseudo-spin symmetry breaking is a direct matter and can be done more or less "automatically" in terms of the existing simple experimental information.

There are, however, several mechanism that make the above "straightforward" comparison biased, difficult — sometimes perhaps strongly contaminated with the quantum mechanisms that have nothing to do with the one under consideration. To start with let us recall that the energy differences among the levels that in the exact symmetry limit are expected to be zero — are going to be small (*cf.* illustrations in the following section). Comparison with the existing data (although the quality of this comparison will be criticized just a few lines below) gives numbers in the range between a couple of hundreds of keV and about 1.5 MeV, roughly, for the spherical nuclei.

In the case of the spherical shape pseudo-spin degeneracies (strictly speaking, in the spherical-symmetry case, what is referred to as doublets in terms of the pseudo-spin quantum number are rather rich multiplets in terms of the orbital angular momentum quantum number, ℓ and $\ell+2$ of the degree of degeneracy that goes like $\sim 4\ell$) the contributing orbitals differ in ℓ by two units. Coupling of the corresponding states with the relatively low-lying collective vibrations may be significantly different for the two ℓ-members in a doublet thus contaminating, perhaps considerably, the comparison with experiment related primarily to the "naked", pure mean-field states — the ones that are expected to obey the pseudo-spin symmetry.

In deformed nuclei in which the pseudo-spin doublets seem to have a similar degree of the symmetry breaking, the pairing correlations mix the single-particle degrees of freedom strongly while the theoretical predictions based on the present formulation of the nuclear Dirac formalism again address pure single-particle properties that result from considering the deformed mean-field alone.

All these aspects will need to be considered when addressing the problem of the nuclear pseudo-spin symmetry, perhaps not so much *in principle* but certainly in the case of the *real life*. What seems to be a dangerous possibility, the present time comparisons with experiment may be strongly biased through the presence of various mechanisms that are not directly related to the mean-field structure. In other words: the actual knowledge about the experimental verification of the pseudo-spin symmetry breaking could be much poorer than it is felt today.

Extending a simple and very well studied in the past deformed mean-field parametrisations in terms of the Woods–Saxon potentials to the Dirac type formalism in the nuclear structure context may turn out to be very useful in overcoming several of the difficulties mentioned above through advances in the realistic calculations. This can be done by modeling and parametrizing the above mechanisms in connection with the Woods–Saxon technique within the nuclear Dirac Hamiltonian. The following presentation gives an introduction to such an approach.

3. Dirac equation and SU(2)-type symmetries

It is a matter of a straightforward transformation to demonstrate that within a standard representation of the Dirac matrices, relation (1) is equivalent to

$$\hat{\mathcal{H}}_D = \begin{pmatrix} +\{m_0c^2 + [\hat{S}(\vec{r}\,) + \hat{V}(\vec{r}\,)]\}, & c\,(\vec{\sigma}\cdot\hat{\vec{p}}\,) \\ c\,(\vec{\sigma}\cdot\hat{\vec{p}}\,), & -\{m_0c^2 + [\hat{S}(\vec{r}\,) - \hat{V}(\vec{r}\,)]\} \end{pmatrix}, \quad (2)$$

where $\vec{\sigma}$ denotes an ensemble of the three Pauli matrices. The solutions in the form of the Dirac bi-spinors $\psi = \binom{\xi}{\eta}$ contain ξ an η, two two-dimensional spinors called respectively grand and small components. The above form is better adapted, as compared to that in Eq. (1), for the discussion which follows. In Eq. (2) the operators $\hat{S} = \hat{S}(\vec{r}\,)$ and $\hat{V} = \hat{V}(\vec{r}\,)$ can be viewed as simple functions of all the three Cartesian variables. In principle, within a phenomenological treatment both these functions can be modeled in terms of the Woods–Saxon type potentials. However, it will turn out to be of more advantage to parametrize the sum $(\hat{S}(\vec{r}\,) + \hat{V}(\vec{r}\,))$ and the difference $(\hat{S}(\vec{r}\,) - \hat{V}(\vec{r}\,))$ in terms of another set of the Woods–Saxon potentials since these combinations enter directly the equations of the motion, *cf.* Ref. [17], and see also below. Consequently, one may write

$$\hat{U}_c(\vec{r}\,) \equiv \hat{S}(\vec{r}\,) + \hat{V}(\vec{r}\,) = \frac{[1+\kappa_c I]\,U_c^0}{1+\exp[\mathrm{dist}_\Sigma(\vec{r}\,)/a_c]}\,, \quad (3)$$

and similarly

$$\hat{U}_{so}(\vec{r}\,) \equiv \hat{S}(\vec{r}\,) - \hat{V}(\vec{r}\,) = \frac{[1+\kappa_{so} I]\,U_c^{so}}{1+\exp[\mathrm{dist}_\Sigma(\vec{r}\,)/a_{so}]}\,, \quad (4)$$

where $I = (N-Z)/(N+Z)$ is the usual isospin factor, while U_c (U_{so}), κ_o (κ_{so}) and a_0 (a_{so}) are adjustable constants. The nuclear surface is represented by the symbol Σ; function $\mathrm{dist}_\Sigma(\vec{r}\,)$ denotes the distance of a given point $\vec{r}$ in space from the nuclear surface. (Anticipating the interpretation of the corresponding potentials as the central and spin-orbit interactions the indices "c" and "so" have been introduced.)

In the following we would like to address the problem of the symmetries first; the possible practical and simple parametrisations will come next.

An Approximate SU(2) Symmetry of the Nuclear Dirac Hamiltonian. Let us introduce, by slightly shortening the reasoning of Ref. [6], operators[1]

$$\hat{\mathcal{S}}_j \equiv \begin{pmatrix} \hat{\tilde{s}}_j & 0 \\ 0 & \hat{s}_j \end{pmatrix}; \ \hat{\tilde{s}}_j \equiv (\vec{\sigma}\cdot\hat{p})\,\hat{s}_j\,(\vec{\sigma}\cdot\hat{p}),\ \hat{h} \equiv \vec{\sigma}\cdot\hat{p} = \left(\frac{2\vec{s}}{\hbar}\right)\cdot\hat{p}. \tag{5}$$

The latter object is the usual momentum-helicity operator that satisfies

$$\hat{h}^\dagger = (\vec{\sigma}\cdot\hat{p})^\dagger = \hat{p}^\dagger\cdot\vec{\sigma}^\dagger = \hat{p}\cdot\vec{\sigma} = \vec{\sigma}\cdot\hat{p} = \hat{h} \leftrightarrow \hat{h}^\dagger\hat{h} = (\vec{\sigma}\cdot\hat{p})^2 = \underbrace{\hat{p}^2}_{1}\,\mathbb{1}_2 = \mathbb{1}_2. \tag{6}$$

Since obviously $\hat{h}^\dagger = \hat{h}^{-1}$ and $\hat{h} = \hat{h}^{-1}$ it becomes clear from Eq. (5) that the definition of $\hat{\tilde{s}}_j$ is equivalent to a similarity transformation that preserves the commutation relations. Consequently the commutation relations for the spin operators $\{\hat{s}_j;\ j=1,2,3\}$ imply

$$[\hat{s}_j,\hat{s}_k] = i\hbar\,\epsilon_{jk\ell}\hat{s}_\ell \ \rightarrow\ [\hat{\tilde{s}}_j,\hat{\tilde{s}}_k] = i\hbar\,\epsilon_{jk\ell}\hat{\tilde{s}}_\ell \ \leftrightarrow\ [\hat{\mathcal{S}}_j,\hat{\mathcal{S}}_k] = i\hbar\,\epsilon_{jk\ell}\hat{\mathcal{S}}_\ell, \tag{7}$$

and we see that all the three ensembles of operators above generate three SU(2) groups: ensemble $\{\hat{s}_j;\ j=1,2,3\} \rightarrow \mathrm{SU}_s(2)$; ensemble $\{\hat{\tilde{s}}_j;\ j=1,2,3\} \rightarrow \mathrm{SU}_{\tilde{s}}(2)$, in two dimensional spaces of spinors η and ξ, respectively, and the ensemble $\{\hat{\mathcal{S}}_j;\ j=1,2,3\} \rightarrow \mathrm{SU}_{\mathcal{S}}(2)$ in a four-dimensional space of the Dirac bi-spinors.

The main reason for introducing the $\{\hat{\mathcal{S}}_j\}$ operators is that, as one can easily verify,

$$[\hat{\mathcal{H}}_D\,,\,\hat{\mathcal{S}}_j] = \begin{pmatrix} [\hat{S}(\vec{r}) + \hat{V}(\vec{r})\,,\,\hat{\tilde{s}}_j] \sim 0 & ,0 \\ 0 & ,0 \end{pmatrix}. \tag{8}$$

The commutator above is strictly speaking not zero. Since $\hat{\tilde{s}}_j$ are differential operators, *cf.* Eq. (5), we find that $[\hat{S}(\vec{r}) + \hat{V}(\vec{r})\,,\,\hat{\tilde{s}}_j] \neq 0$ unless $\hat{S}(\vec{r}) + \hat{V}(\vec{r}) = 0$. However, the breaking of this exact symmetry corresponding to the exact commutation relations $[\hat{S}+\hat{V},\hat{\tilde{s}}_j] = 0$ is expected to be weak. This is so because as it has been found out earlier, see *e.g.* Ref. [17], on the average the discussed potentials satisfy

$$\left.\begin{array}{lcl} \langle\hat{S}(\vec{r})\rangle & \sim & -\,400\,\mathrm{MeV} \\ \langle\hat{V}(\vec{r})\rangle & \sim & +\,350\,\mathrm{MeV} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \langle\hat{S}(\vec{r}) + \hat{V}(\vec{r})\rangle \sim -50\,\mathrm{MeV} \\ \langle\hat{S}(\vec{r}) - \hat{V}(\vec{r})\rangle \sim -750\,\mathrm{MeV} \end{array}\right\}. \tag{9}$$

[1] Formally the momentum-helicity operators introduced here can be viewed as complicated differential operator expressions of the type $\hat{p} = \hat{\vec{p}}/(\hat{\vec{p}}\cdot\hat{\vec{p}})$; both momentum related operators, *i.e.* $\hat{\vec{p}}$ and $(\hat{\vec{p}}\cdot\hat{\vec{p}})^{-1}$ are mathematically well defined objects as *e.g.* matrices calculated with respect to a certain basis. In particular, $(\hat{\vec{p}}\cdot\hat{\vec{p}})^{-1}$ is an inverse matrix with respect to $\hat{\vec{p}}\cdot\hat{\vec{p}} = -\hbar^2\Delta$ where Δ denotes the Laplace operator.

Denoting $\hat{U}(\vec{r}) \equiv \hat{S}(\vec{r}) + \hat{V}(\vec{r})$ we compare the average absolute values of the depths (or heights) of the potentials, (averages denoted with the symbol "$\langle\rangle$"). We find that $\langle\hat{U}\rangle$ is much smaller than the difference $\langle(\hat{S} - \hat{V})\rangle$; also $\langle\hat{U}\rangle$ is much smaller than $\langle\hat{S}\rangle$ and at the same time much smaller than $\langle\hat{V}\rangle$. Moreover, as mentioned above, the $\tilde{s}_j$ is a differential operator and it becomes clear that for the flat bottom potentials the non-vanishing of the commutator takes place mainly at the nuclear surface. Since the nuclear surface-to-volume ratio decreases as $A^{-\frac{1}{3}}$ the influence of the symmetry breaking should be a decreasing function of the nuclear mass number or, in other words, the symmetry should become better the heavier the nucleus.

Conclusion. The Dirac equation with the average nuclear interactions represented by potentials $\hat{S}(\vec{r})$ and $\hat{V}(\vec{r})$ obeys approximately an SU(2) symmetry in the four-dimensional space of the Dirac bi-spinors, with the related group of transformations spanned by the generators $\{\hat{\mathcal{S}}_j;\ j = 1, 2, 3\}$. The corresponding approximate symmetry should, to a far an extent, be deformation independent as long as the non-vanishing of the commutator in Eq. (7) can be neglected.

Groups $\mathbf{SU}_{s}(2)$ and $\mathbf{SU}_{\tilde{s}}(2)$ as Symmetry Groups for Spinors $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$. It is an easy exercise to show that Eqs. (1) or (2) can be written down in terms of two Schrödinger-like equations for the Dirac bi-spinor's grand component ξ and small component η:

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}; \quad \left\{ \begin{array}{ll} \xi: & \hat{\mathcal{H}}_D^{\xi}\, \xi = \mathcal{E}\xi; \\ \eta: & \hat{\mathcal{H}}_D^{\eta}\, \eta = \mathcal{E}\eta, \end{array} \right. \tag{10}$$

where the two Dirac operators are:

$$\begin{aligned} \hat{\mathcal{H}}_D^{\xi} \;\equiv\; & (c\vec{\sigma} \cdot \hat{\vec{p}}) \frac{1}{[\mathcal{E} + m_0c^2 + (\hat{S}(\vec{r}) - \hat{V}(\vec{r}))]} (c\vec{\sigma} \cdot \hat{\vec{p}}) \\ & + [m_0c^2 + (\hat{S}(\vec{r}) + \hat{V}(\vec{r}))] \end{aligned} \tag{11}$$

and

$$\begin{aligned} \hat{\mathcal{H}}_D^{\eta} \;\equiv\; & (c\vec{\sigma} \cdot \hat{\vec{p}}) \frac{1}{[\mathcal{E} - m_0c^2 - (\hat{S}(\vec{r}) + \hat{V}(\vec{r}))]} (c\vec{\sigma} \cdot \hat{\vec{p}}) \\ & - [m_0c^2 + (\hat{S}(\vec{r}) - \hat{V}(\vec{r}))] \,. \end{aligned} \tag{12}$$

In the above two-dimensional representation a characteristic symmetry in appearances of the sum $(\hat{S} + \hat{V})$ and of the difference $(\hat{S} - \hat{V})$ in the two operators deserves noticing. It is also worth emphasizing that the above eigen-equations, Eq. (10), are *not* eigen-energy problems in the usual sens, since the energy dependence there is not linear, *cf.* Eqs. (11) and (12).

Consider, similarly as in Ref. [1], an ideal limiting case $\hat{S}+\hat{V}\to 0$. Strictly speaking such a limit is a non-physical one since it corresponds to a vanishing of the nucleonic binding through the vanishing of the central potential $\hat{U}$, *cf.* Eqs. (11) and (3). However, this limit can be used to model the situation of the weak symmetry breaking as presented below. In such a limit

$$[\hat{\mathcal{H}}_D^-,\hat{s}_i]=0;\;\to\;\hat{\mathcal{H}}_D^-\eta_n=\mathcal{E}_n\eta_n;\;\hat{s}^2\eta_{n,s}\;=\;s(s+1)\eta_{n,s};\;\hat{s}_z\eta_{n,s,s_z}\;=\;s_z\eta_{n,s,s_z}\,, \tag{13}$$

i.e. the small components in the Dirac equation can be labeled with the help of the spin quantum numbers s and s_z, and similarly

$$[\hat{\mathcal{H}}_D^+,\hat{\tilde{s}}_i]=0;\;\to\;\hat{\mathcal{H}}^+\xi_n=\mathcal{E}_n\xi_n;\;\hat{\tilde{s}}^2\xi_{n,\tilde{s}}\;=\;\tilde{s}(\tilde{s}+1)\xi_{n,\tilde{s}};\;\hat{\tilde{s}}_z\xi_{n,\tilde{s},\tilde{s}_z}\;=\;\tilde{s}_z\xi_{n,\tilde{s},\tilde{s}_z}\,; \tag{14}$$

i.e. the grand components of the Dirac bi-spinors can be labeled with the pseudo-spin quantum numbers $\tilde{s}$ and $\tilde{s}_z$. (To show that the commutation relation in Eq. (14) is valid is a matter of an easy exercise for $\hat{S}+\hat{V}\to 0$.)

Observation. Let us emphasize that at the $\hat{S}+\hat{V}\to 0$ limit, the dependence of functions η on spin factorizes out exactly; similarly the dependence of spinors ξ on the pseudo-spin factorizes out and we may look for the corresponding solutions in the form of products depending on the $\vec{r}$ and s_z ($\tilde{s}_z$) variables separately

$$\xi_{n,\tilde{s},\tilde{s}_z}(\vec{r}\,)=\Xi_n(\vec{r}\,)\,\chi_{\tilde{s},\tilde{s}_z};\qquad \eta_{n,s,s_z}(\vec{r}\,)=\Psi_n(\vec{r}\,)\,\chi_{s,s_z}. \tag{15}$$

Prediction of Degeneracies: Fingerprints of the SU$_{\mathcal{S}}$(2) Symmetry. The fact that $[\hat{\mathcal{H}}_D^+,\hat{\tilde{s}}_j]=0$ for $j=1,2,3$, signifies among others that the eigen-energies characterized by "pseudospin-up" and "pseudospin-down" condition, *cf.* commutation relation in Eq. (14), must be exactly equal in the considered limit:

$$\mathcal{E}_{n,\tilde{s},+\tilde{s}_z}=\mathcal{E}_{n,\tilde{s},-\tilde{s}_z},\quad\leftrightarrow\quad \hat{S}(\vec{r}\,)+\hat{V}(\vec{r}\,)\;\to\;0,\;\forall\,\vec{r}, \tag{16}$$

or, that there should exist double degeneracies. These should take place irrespectively of the deformation of the nuclear system since none of the arguments evoked so far was related to the particular dependence of $\hat{S}(\vec{r}\,)$ or $\hat{V}(\vec{r}\,)$ on $\vec{r}$. Moreover, the above condition can be now related to the fact that the nuclear Dirac Hamiltonian does not depend on time wherefrom it follows that ($\hat{\mathcal{T}}$ denoting the time-reversal operator)

$$[\hat{\mathcal{H}}_D,\hat{\mathcal{T}}]=0\quad\leftrightarrow\quad\hat{\mathcal{T}}\,\hat{\mathcal{H}}_D\,\hat{\mathcal{T}}^{-1}=\hat{\mathcal{H}}_D. \tag{17}$$

The above result implies that the wave functions ψ and $\hat{\mathcal{T}}\psi$ satisfy the respective equations

$$\hat{\mathcal{H}}_D\,\psi = \mathcal{E}\,\psi \quad \leftrightarrow \quad \underbrace{[\hat{\mathcal{T}}\,\hat{\mathcal{H}}_D\,\hat{\mathcal{T}}^{-1}]}_{\hat{\mathcal{H}}_D}(\hat{\mathcal{T}}\,\psi) = \mathcal{E}\,(\hat{\mathcal{T}}\,\psi), \tag{18}$$

i.e. the two linearly independent solutions, ψ and $\hat{\mathcal{T}}\psi$ are degenerate with the common energy eigenvalue $\mathcal{E}$ (the well known Kramers degeneracies). In other words: any energy eigenvalue is double degenerate and the related wave functions correspond to two opposite directions of time. However, the above argument can be repeated to each eigenenergy in Eq. (16) and it follows that the nucleonic states in the limit $\hat{S}(\vec{r}) + \hat{V}(\vec{r}) \;\to\; 0$ must be *quadruply* degenerate:

$$\left.\begin{array}{lcl} \mathcal{E}_{n,\tilde{s},+\tilde{s}_z} & \to & \psi_{n,\tilde{s},+\tilde{s}_z} \text{ and } \mathcal{T}\psi_{n,\tilde{s},+\tilde{s}_z} \\ \mathcal{E}_{n,\tilde{s},-\tilde{s}_z}, & \to & \psi_{n,\tilde{s},-\tilde{s}_z} \text{ and } \mathcal{T}\psi_{n,\tilde{s},-\tilde{s}_z} \end{array}\right\}. \tag{19}$$

Conclusion. In the exact pseudo-spin symmetry limit *all* the single nucleon states split into the groups of four-fold degenerate states with the four wave-functions specified above, in particular for the deformed nuclei.

A Special Case: The Spherical Symmetry. The relation above will be particularly instructive to study in the case of the spherical symmetry where several properties can be deduced analytically. Most of the considerations related to the implications of the pseudo-spin symmetry can be repeated after having introduced the pseudo-orbital angular momentum operator and the related symmetry operator $\hat{\mathcal{L}}_j$, an analog to the $\hat{\mathcal{S}}_j$, through the definition

$$\hat{\mathcal{L}}_j \equiv \begin{pmatrix} \hat{\tilde{\ell}}_j & 0 \\ 0 & \hat{\ell}_j \end{pmatrix}; \quad \hat{\tilde{\ell}}_j \equiv (\vec{\sigma}\cdot\hat{p}\,)\,\hat{\ell}_j\,(\vec{\sigma}\cdot\hat{p}\,). \tag{20}$$

Similarly as before we demonstrate that in the commutator below

$$[\hat{\mathcal{H}}_D\,,\,\hat{\mathcal{L}}_j] = \begin{pmatrix} [\hat{S}(\vec{r}\,) + \hat{V}(\vec{r}\,)\,,\,\hat{\tilde{\mathcal{L}}}_j] & ,0 \\ 0 & ,0 \end{pmatrix} \tag{21}$$

the approximate commutation relation $[\hat{S}(\vec{r}\,) + \hat{V}(\vec{r}\,), \hat{\tilde{\mathcal{L}}}_j] \sim 0$ holds for the same reasons as those discussed above. The last relation implies immediately that for the new operator defined as the following sum

$$\hat{\mathcal{J}}_j \stackrel{\text{def}}{=} \hat{\mathcal{L}}_j + \hat{\mathcal{S}}_j \quad \leftrightarrow \quad [\hat{\mathcal{H}}_D, \hat{\mathcal{J}}_j] = 0 \quad \leftrightarrow \quad [\hat{\mathcal{H}}_D, \hat{\vec{\mathcal{J}}}^{\,2}\,] = 0\,, \tag{22}$$

i.e. that the conservation of a new observable, $\hat{\mathcal{J}}_j$, follows. More precisely, this new observable is generated by the operator

$$\hat{\mathcal{J}}_j \equiv \begin{pmatrix} \hat{\tilde{\ell}}_j + \hat{\tilde{s}}_j, & 0 \\ 0, & \hat{\ell}_j + \hat{s}_j \end{pmatrix} \equiv \begin{pmatrix} \hat{\tilde{j}}_j, & 0 \\ 0, & \hat{j}_j \end{pmatrix}; \quad j = 1, 2, 3, \tag{23}$$

where a new symbol, $\hat{\tilde{j}}_j$, is defined as a sum of the pseudo orbital and pseudo intrinsic angular momenta

$$\hat{\tilde{j}}_j \stackrel{\text{def}}{=} \tilde{\ell}_j + \tilde{s}_j = (\vec{\sigma} \cdot \hat{p})\, \hat{\ell}_j\, (\vec{\sigma} \cdot \hat{p}) + (\vec{\sigma} \cdot \hat{p})\, \hat{s}_j\, (\vec{\sigma} \cdot \hat{p}). \tag{24}$$

Using now the commutators between $\hat{\ell}_j$ and $\hat{p}_j$ operators we find that $\hat{\tilde{j}}_j = \hat{\tilde{\ell}}_j + \hat{\tilde{s}}_j = \hat{\ell}_j + \hat{s}_j = \hat{j}_j$ and thus that the total pseudo angular momentum is equal to the total angular momentum although $\hat{\tilde{\ell}}_j \neq \hat{\ell}_j$ and $\hat{\tilde{s}}_j \neq \hat{s}_j$.

The above considerations can be summarized as follows. At the limit $\hat{S} + \hat{V} \to 0$ the nuclear Dirac Hamiltonian of spherical symmetry commutes with the following operators

$$[\hat{\mathcal{H}}_D, \hat{J}_k] = 0 \leftrightarrow [\hat{\mathcal{H}}_D, \hat{\vec{J}} \cdot \hat{\vec{J}}\,] = 0 \text{ and } [\hat{\mathcal{H}}_D, \hat{\mathcal{J}}_k] = 0 \leftrightarrow [\hat{\mathcal{H}}_D, \hat{\vec{\mathcal{J}}} \cdot \hat{\vec{\mathcal{J}}}\,] = 0; \tag{25}$$

as well as

$$[\hat{\mathcal{H}}_D, \hat{\mathcal{L}}_k] = 0 \leftrightarrow [\hat{\mathcal{H}}_D, \hat{\vec{\mathcal{L}}} \cdot \hat{\vec{\mathcal{L}}}\,] = 0 \text{ and } [\hat{\mathcal{H}}_D, \hat{\mathcal{S}}_k] = 0 \leftrightarrow [\hat{\mathcal{H}}_D, \hat{\vec{\mathcal{S}}} \cdot \hat{\vec{\mathcal{S}}}] = 0. \tag{26}$$

At that limit the solutions to the corresponding relativistic problem (let us remind the reader that all these commutation relations hold generally and not just at the non-relativistic reduction case) can be labeled with the following quantum numbers

$$\Phi_{n;\,\mathcal{J},\mathcal{J}_z;\,\mathcal{L},\mathcal{L}_z;\,\mathcal{S},\mathcal{S}_z}; \quad \xi_{n;\,j,j_z;\,\tilde{\ell},\tilde{\ell}_z;\,\tilde{s},\tilde{s}_z} \quad \text{and} \quad \eta_{n;\,j,j_z;\,\ell,\ell_z;\,s,s_z}. \tag{27}$$

In the last relations we took into account that since $\hat{\mathcal{J}}_k = \hat{J}_k$ only one of these vector quantities can be treated as independent and that according to the definitions in Eqs. (5) and (20) the operators acting on the ξ and η spinors differ in structure as it is indicated by the differences in the labels.

Let us now recall that within the representations of the Dirac matrices used above the parity operator $\hat{\mathcal{P}}$ satisfies (up to a phase factor that is unimportant for us here):

$$\hat{\mathcal{P}} = \gamma^0 = \begin{pmatrix} +\mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \leftrightarrow \hat{\mathcal{P}} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} +\xi \\ -\eta \end{pmatrix}, \tag{28}$$

while at the same time we may write

$$\hat{\pi}:\begin{pmatrix}\xi(\vec{r})\\ \eta(\vec{r})\end{pmatrix}\xrightarrow{\vec{r}\to-\vec{r}}\begin{pmatrix}\xi(-\vec{r})\\ \eta(-\vec{r})\end{pmatrix}=\begin{pmatrix}\pi_\xi\xi(\vec{r})\\ \pi_\eta\eta(\vec{r})\end{pmatrix};\pi_\xi=(-1)^{\tilde{\ell}}\text{ and }\pi_\eta=(-1)^{\ell}. \quad (29)$$

Since for a parity invariant Hamiltonian, as it is the case here, we have in addition

$$\hat{\mathcal{P}}\Phi=\pi_\Phi\Phi;\quad \pi_\Phi=\pm 1,$$

it then follows that

$$\pi_\eta=-\pi_\xi\ \leftrightarrow\ \tilde{\ell}=\ell\pm 1,\pm 3,\pm 5\dots\quad, \quad (30)$$

but since we must always have $\tilde{j}=j$ the only possible combinations among the quantum numbers in question are:

$$s=\tilde{s}=\tfrac{1}{2};\ j=\tilde{j}:\ j=\ell-s\quad\leftrightarrow\quad\tilde{j}=\tilde{\ell}+\tilde{s}\ \to\ \tilde{\ell}=\ell-1, \quad (31)$$

and

$$s=\tilde{s}=\tfrac{1}{2};\ j=\tilde{j}:\ j=\ell+s\ \leftrightarrow\ \tilde{j}=\tilde{\ell}-\tilde{s}\ \to\ \tilde{\ell}=\ell+1. \quad (32)$$

Simultaneously, the states corresponding to a given j quantum number must have the energies of the pseudo-spin "up" configuration equal to that of the pseudo-spin "down" configuration and we arrive at an ideal pseudo-spin symmetric spectrum of a spherically symmetric Hamiltonian that is represented schematically in Fig. 1. To the left: for a given principal quantum number N, the possible ℓ quantum numbers form a sequence as indicated. The levels marked with the label *"No spin orbit"* correspond to the spin-up vs. spin-down degeneracy — in such a case the Hamiltonian does not depend on spin. Introducing gradually the spin-orbit potential will split the levels corresponding to the orbital angular momenta coupled with spins in either parallel or antiparallel configurations. In such a fictitious pseudo-spin symmetry obeying potential, the Hamiltonian will produce a spectrum marked to the right, in which one orbital coming from below and one orbital coming from above form eventually a degenerate pseudo-spin doublet. Keeping $\ell_{\max}\equiv\ell=N$ as a reference value and proceeding upwards we obtain all possible $\tilde{\ell}$ values. They are equal $\tilde{\ell}=N-1$, $\tilde{\ell}=N-3$, $\tilde{\ell}=N-5$ *etc.*

The schematic illustration in Fig. 1 is confronted with the experimental results on the neutron single particle energies in ^{208}Pb nucleus, the levels *above* the $Z=126$ gap, Fig. 2 and *below* the $Z=126$ gap, Fig. 3. The pseudo-spin degeneracies are marked explicitly. It becomes clear from these figures that the splitting of the levels that in the exact symmetry limit should coincide, does not exceed ~ 1 MeV; these splittings should be compared to

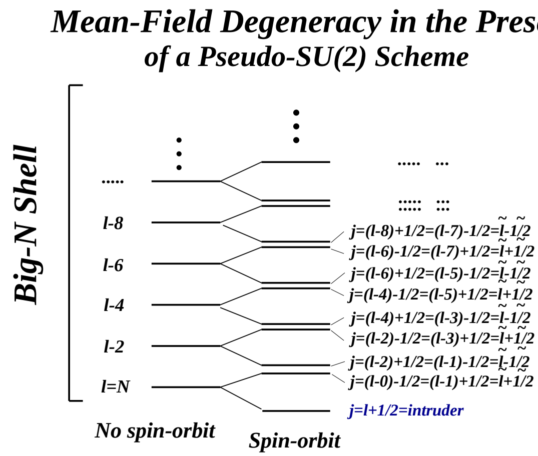

Fig. 1. A schematic representation of a single-nucleon spectrum for a heavy nucleus illustrating a possible scenario of the pseudo-spin symmetry. This Figure should be interpreted as an artist's view rather than any numerical simulation result, recalling that the exact symmetry limit arises when $\hat{S} + \hat{V} \to 0$ *i.e.* at the limit of the disappearing nucleonic binding.

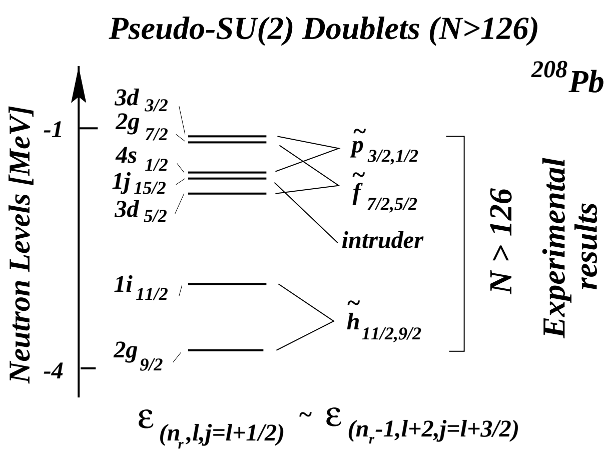

Fig. 2. Experimental results for the single-particle neutron-levels in the ^{208}Pb nucleus above the $N = 208$ gap. The usual spectroscopic labels are placed to the left; the analogous notation interms of the pseudo angular momentum is given to the right. In particular, the notation like $\tilde{p}_{3/2,1/2}$ means: p-type level in the pseudo angular momentum sense, $\tilde{\ell} = 1$, *cf.* Eqs. (31) and (32), originating from "traditional" $d_{3/2}$ and $s_{1/2}$ levels of $\ell = \tilde{\ell} + 1$ and $\ell = \tilde{\ell} - 1$, *etc.*

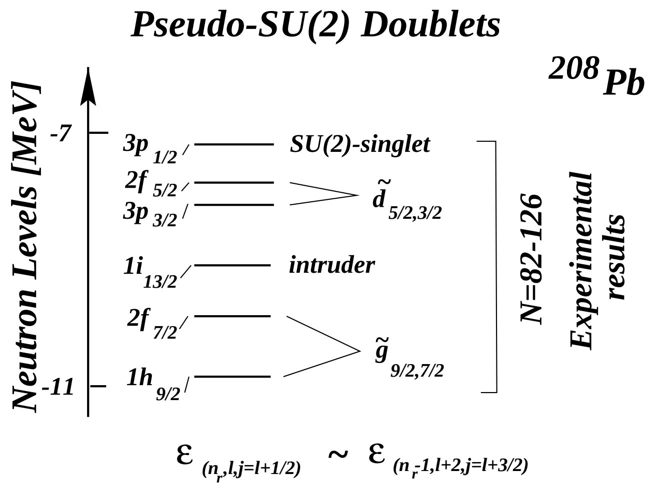

Fig. 3. Similar to Fig. 2 but for the neutron levels below the $N = 208$ gap (for more details see text).

$\langle U(\vec{r}\,)\rangle \sim -50$ MeV or to the other averages that represent directly either the attractive nucleon-nucleon forces, $\langle S(\vec{r}\,)\rangle \sim -400$ MeV or the repulsive forces $\langle V(\vec{r}\,)\rangle \sim +350$ MeV. Independently of the scale used the breaking of the pseudo-spin symmetry could be considered small since the corresponding *relative* deviations defined in the above sense are of the order of 2 % to a few per mille. Very similar picture is obtained for the protons (not shown). One observes here systematically an increase of the pseudo-spin splitting with an increase in ℓ (or $\tilde{\ell}$).

4. The nuclear Dirac equation at low-energy

Despite the fact that we were able to present above a few illustrations related to the actual experimental situation with respect to the pseudo-spin symmetry the conclusions could be drawn mainly on the qualitative level. The symmetry discussed so far was formulated for the "naked" nucleons in the Dirac formalism. In a real nucleus the nucleon coupling to, for instance, surface vibrations or their participation in the pairing interactions make an adequate comparison in terms of small quantities as those seen in Figs. (2) and (3) more difficult as it may seem. To prepare the grounds for the more adequate comparisons, taking into account couplings of the type mentioned, one will need the single-particle Hamiltonian that is conform with the Dirac formalism presented above. For this purpose one will need a reasonable model potentials that replace those used in many microscopic model calculations in the past.

We will use Eq. (11) as a starting point; we apply the non-relativistic reduction $\mathcal{E} \sim m_0c^2 + \varepsilon$ and follow a suggestion of Ref. [17], to introduce a position dependent effective mass as follows

$$\begin{aligned}\frac{1}{2m_0c^2 + \varepsilon + [\hat{S}(\vec{r}) - \hat{V}(\vec{r})]} &= \frac{1}{\varepsilon + 2m^*} = \frac{1}{2m^*}\left(\frac{1}{1+\frac{\varepsilon}{2m^*}}\right) \\ &\simeq \frac{1}{2m^*}\left(1 - \frac{\varepsilon}{2m^*}\right) . \end{aligned} \tag{33}$$

The effective mass $m^*(\vec{r})$ is defined by

$$m^*(\vec{r}) = m_0c^2 + \tfrac{1}{2}[\hat{S}(\vec{r}) - \hat{V}(\vec{r})]. \tag{34}$$

By elementary transformations we obtain now the Schrödinger-type equation for the spinor ξ

$$\left\{(c\vec{\sigma}\cdot\hat{\vec{p}})\frac{1}{2m^*(\vec{r})}(c\vec{\sigma}\cdot\hat{\vec{p}}) + [\hat{S}(\vec{r}) + \hat{V}(\vec{r})]\right\}\xi_n = \varepsilon_n\xi_n, \tag{35}$$

from where, by directly applying the fact that $\hat{\vec{p}} = -i\hbar\nabla$ we find

$$\left\{\frac{1}{2m^*(\vec{r})}(c\hat{\vec{p}})^{\,2} + \hat{V}_{\vec{p}}(\vec{r},\hat{\vec{p}}) + \hat{V}_{so}(\vec{r},\hat{\vec{p}},\hat{\vec{s}}) + [\hat{V}(\vec{r}) + \hat{S}(\vec{r})]\right\}\xi_n = \varepsilon_n\xi_n. \tag{36}$$

In the above relation, the term $\hat{V}(\vec{r}) + \hat{S}(\vec{r})$ plays a role of the central potential and will be parametrized with the help of Eq. (3). The spin-orbit potential, following a straightforward but a little longer sequence of transformations, is given by

$$\hat{V}_{so}(\vec{r},\hat{\vec{p}},\hat{\vec{s}}) = \frac{\hbar c^2}{2m_0c^2}\{(\vec{\nabla}V_{\ell s})\wedge\hat{\vec{p}}\}\cdot\hat{\vec{s}} \;\text{ with }\; \hat{V}_{\ell s}(\vec{r}) \equiv \frac{1}{m^*(\vec{r})}[\hat{V}(\vec{r}) - \hat{S}(\vec{r})]. \tag{37}$$

The difference $\hat{V}(\vec{r}) - \hat{S}(\vec{r})$ will be parametrised in terms of another Woods–Saxon type expression — *cf.* Eq. (4). In principle the same difference appears also in the definition of the effective mass, Eq. (34). The same is true for the "linear momentum potential" of Eq. (36):

$$\hat{V}_{\vec{p}}(\vec{r},\hat{\vec{p}}) \equiv -\frac{i\hbar c^2}{(2m^*)^2}\left[\vec{\nabla}\Big(\hat{V}(\vec{r}) - \hat{S}(\vec{r})\Big)\right]\cdot\hat{\vec{p}} \tag{38}$$

that satisfies

$$(c\hat{\vec{p}})\frac{1}{2m^*(\vec{r})}(c\hat{\vec{p}}) = \frac{c^2}{2m^*(\vec{r})}\hat{\vec{p}}^{\,2} + \hat{V}_p(\vec{r},\vec{p}). \tag{39}$$

There are three terms in the above Hamiltonian, Eq. (36), that depend on the difference $\hat{V}(\vec{r}) - \hat{S}(\vec{r})$: these are the effective mass, the linear momentum potential and the spin-orbit potential. This difference is going to be parametrised with the Woods–Saxon type forms. In principle one may look for the maximum parametric freedom that will be contained in the final parametrization of the Hamiltonian. In particular it will be possible to parametrize the spin-orbit term with the help of parameters that differ from those in the effective mass and in the linear momentum term. However those contained in the effective mass and the linear momentum term *must* be the same in order that the operator (39) remains hermitian (as an element of the Hamiltonian).

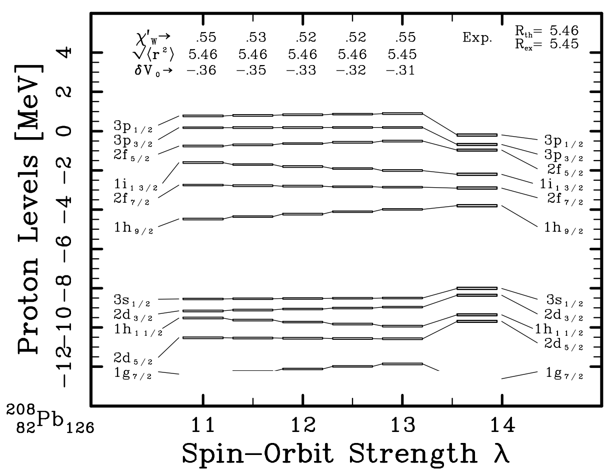

Fig. 4. An example of the single proton-level spectrum calculated with the new (preliminary) parametrization of the Woods–Saxon potentials within the low-energy limit of the nuclear Dirac formalism. The optimal central potential depth is larger than those usually found in the literature — here: $U_c^0 = 71.5$ MeV, *cf.* Eq. (3). The quality of the fit is comparable to- or better than most of the Woods-Saxon type parametrisations that can be found in the literature. The other parameter values are: Radii (central, spin-orbit and effective mass: 1.15 fm, 0.92 fm and 0.81 fm, respectively); diffuseness (in the same order, 1.22 fm, 0.6 fm and 0.42 fm); isospin strengths, *cf.* Eq. (3) (0.5, 0.6, 0.5). The strengths of the Woods-Saxon factors in the spin-orbit and the effective mass terms are expressed as multiples of a "standard unit" equal 50 MeV. For the effective mass $\lambda_{\text{eff}} = 14.4$ while λ_{so} is varied. On top of the figure: root-mean-square deviation for the single particle energies (experiment vs. theory), χ'_W, root-mean-square radii, $\sqrt{\langle r^2 \rangle}$, and the estimated error of the binding energy of the last nucleon, δV_0. Experimental value of the proton radius is given in the upper-right corner.

It is not our purpose to discuss in details the properties of the new parametrization used here — this will be done elsewhere. Here we will limit ourselves to presenting a typical illustration of the new fit calculated with the help of the Hamiltonian (39) where, in addition to the nuclear potentials discussed the usual Coulomb potential corresponding to a uniformly charged sphere has been added, *cf.* Fig. 4. Despite the fact that the effective mass term is of the order of 60 % of the rest mass, the single particle level density around the Fermi level is comparable to the one obtained with the best among the older parametrizations.

5. Summary and conclusions

After a short historical overview of the problem of the nuclear SU(2) symmetries and in particular of the pseudo-spin symmetry, we have discussed in some more detail a recent formulation of the problem. This recent formulation is based on the properties of the nuclear Dirac equation, [1]. The latter contains two potentials that differ in sign: an attractive one contributed by the exchange of the scalar mesons and a repulsive one coming from the mechanism of the exchange of the vector mesons. After presenting the commutation relations of the related Dirac Hamiltonian with appropriately chosen operators involving nucleonic spin and helicity, the relativistic pseudo-spin formalism has been discussed together with the existence of the approximate pseudo-spin symmetry. In view of the microscopic calculations that use the above concepts in realistic situations a new set of the Woods–Saxon type potentials is introduced and a preliminary set of parameters found that assure a good description of the single nucleonic states within the low energy limit of the Dirac formalism. The potentials are found to differ considerably from those usually used in the literature; here however the position dependent effective mass is explicitly introduced. As expected, the effective mass that differs markedly from the nucleonic rest-mass, only when combined with the new potential parametrisations brings very reasonable fits that take into account at the same time the binding energies, the single-particle level order as well as the nucleon (in the first place the proton) spatial mass distributions.

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