QUANTUM MECHANICS OF THE ELECTRIC CHARGE

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The electric charge is certainly the most mysterious physical quantity. There are two things about the electric charge which are particularly hard to understand: its universality and its quantization. By universality I mean the well known fact that electric charges of all elementary particles seem to be exactly the same. In the case of the electron and the proton the equality of absolute values of their electric charges has been established experimentally with accuracy like $1 : 10^{-20}$. This accuracy exceeds by ten orders of magnitude the accuracy with which the absolute value of the electron's charge is known. There is no doubt that the electron's charge and the proton's charge are — just like their spins — mathematically equal. The mathematical equality of spins of various fermions follows from the elementary quantum mechanics of angular momentum. One feels that there should be a comparable argument for electric charges.

Let me elaborate on the analogy between electric charge and spin. Spins of all fermions are mathematically equal; we know it from the group theory of angular momentum. Since there is no comparable argument for electric charges, everyone is free to speculate on the physical origin of degeneracy which holds always and with the fantastic accuracy $1 : 10^{-20}$. Many authors — I will not quote them because I think that they are misguided — speculate that this degeneracy is of dynamical origin. The relevant ideology says that at some very high energy of interaction all forces of Nature have approximately the same strength. Moreover, this strength can be determined from some physical principle, unfortunately unknown at present. I wish every success to people thinking along these lines but I am not able to believe in it. It is impossible to have accidental degeneracy of dynamical origin which is preserved by all sorts of perturbations, for example by "perturbations" which make the proton different from the electron. Do you really believe that there exists a spectrum with degeneracy which cannot be removed by a skilfully contrived perturbation? This simply cannot be the case, if you give me a Hamiltonian which produces a degenerate spectrum, I will certainly invent a perturbation which removes this degeneracy.

It follows then that the universality of electric charge must be of kinematical origin. I will formulate the relevant principle in a moment but I have to comment first on the second mystery associated with the electric charge, namely its quantization. The Coulomb field is by far the most classical object in Nature, it is much more classical than the desks you are sitting at or the blackboard you are looking at. This follows from the criterion of applicability of the classical field concept which Berestetsky, Lifshitz, and Pitaevsky give on page 30 of their excellent book [1]: the electromagnetic field $F_{\mu\nu}$ is approximately classical if ($\hbar = 1 = c$)

$$\left(\Delta x^0\right)^2 \sqrt{F_{01}^2 + F_{02}^2 + F_{03}^2} \gg 1\,,$$

where Δx^0 is a time interval over which the field can be averaged without being significantly changed. For a static field this time is obviously infinite and therefore, conclude Berestetsky, Lifshitz, and Pitaevsky, *a static field is always classical.*

I have always wondered why the illustrious authors say what they say without any comment at all because experimental facts are crying for such a comment: the amplitude of the Coulomb field is quantized, which means that the Coulomb field is a classical object with quantized amplitude, a monstrosity unknown in the rest of physics. I have indicated some time ago [2] that there is a way to bypass the inequality of Berestetsky, Lifshitz, and Pitaevsky: one has to note that the total electric charge, as determined from the Gauss law, "lives" at the spatial infinity, where the eternity of available time is limited by the opening of the light cone,

$$\left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2 < 0\,.$$

This means that Δx^0 cannot exceed $2r$, where $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, and the inequality of Berestetsky, Lifshitz, and Pitaevsky for the Coulomb field with the total charge Q takes on the form

$$\frac{|Q|}{r^2}(2r)^2 \gg 1$$

i.e.

$$|Q| \gg \frac{1}{4} = \frac{1}{4}\sqrt{137}e \approx 3e\,.$$

This eminently sensible inequality has been obtained from the experimental value of the fine structure constant which is sometimes found to be mysteriously small. This argument resolves the problem of quantization: only sufficiently large charges are classical. The problem of universality can be solved as follows.

All charged particles are massive. We do not know why this should be the case; it is simply another unexplained but indubitable experimental fact. I assume that there is a law of Nature which prevents charged particles to be massless. The argument due to Schwinger makes this assumption extremely plausible, even if it does not prove that the assumption is actually true. Wave functions of massive particles are exponentially damped by mass at the spatial infinity. This means that at the spatial infinity the electromagnetic field is free. Since no length scale survives at the spatial infinity, the field $F_{\mu\nu}(x)$ must be homogeneous of degree -2:

$$F_{\mu\nu}(\lambda x) = \lambda^{-2} F_{\mu\nu}(x) \text{ for each } \lambda > 0\,.$$

It is easy to show that if the tensor $F_{\mu\nu}(x)$ fulfills Maxwell's equations and is homogeneous of degree -2, then there exist two functions $e(x)$ and $m(x)$ such that

$$F_{\mu\nu}(x)x^\nu = \partial_\mu e(x)\,, \qquad \tfrac{1}{2} \in^{\mu\nu\rho\sigma} x_\nu F_{\rho\sigma}(x) = \partial^\mu m(x)\,.$$

Since $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ and $A_\mu(x)$ is homogeneous of degree $-1, F_{\mu\nu}(x)x^\nu = \partial_\mu[x^\nu A_\nu(x)]$, which means that $e(x) = x^\mu A_\mu(x)$ up to an irrelevant additive constant. Moreover $\Box\, e(x) = 0$ *i.e.* the function $e(x) = x^\mu A_\mu(x)$ is a homogeneous of degree zero solution of the wave equation. A simple argument [2] shows that it is prudent to put $m(x) = 0$. In this way the following statement is seen to be true: the electromagnetic field at the spatial infinity is completely determined by a single, homogeneous of degree zero solution of the wave equation $e(x) = x^\mu A_\mu(x)$. This function is gauge invariant because in the gauge transformed potential $A_\mu(x) + \partial_\mu f(x)$ the "arbitrary" function $f(x)$ must be homogeneous of degree zero, which means that $x^\mu \partial_\mu f(x) = 0$ on the strength of Euler's theorem on homogeneous functions. I make now the following argument: since, as I have shown previously, the total electric charge is not a classical object, there must exist at the spatial infinity its canonically conjugate partner called phase. In the usual quantum electrodynamics the phase of a charged field fulfills a complicated set of nonlinear equations. At the spatial infinity, however, every charged system is described by a single function, namely $e(x) = x^\mu A_\mu(x)$, hence the phase $S(x)$ must be proportional to the function $e(x)$. I assume that $S(x) = -ex^\mu A_\mu(x)$, where e is the constant which enters the canonical commutation relation $[Q, S(x)] = ie$. I have at least five independent arguments which support this assumption. One should note, that the assumption consists in the identification of phase as $S(x) = -ex^\mu A_\mu(x)$. The equation $[Q, S(x)] = ie$ is a theorem in Q.E.D.; in the present context it is simply an implicit definition of the constant e.

The two equations

$$\boxed{\begin{array}{l} [Q, S(x)] = ie \\ S(x) = -ex^\mu A_\mu(x) \end{array}}$$

form together *a closed kinematical scheme* akin to the quantum mechanics of angular momentum; I believe that they form the true quantum mechanics of the electric charge. You may observe that in this scheme, unlike in the usual quantum electrodynamics, there is a place for a single constant e only.

REFERENCES

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[2] A. Staruszkiewicz, *Banach Center Publications*, Vol. 41, Part II, page 257, Warsaw 1997.