CASIMIR ENERGY OF ROTATING STRING — INDIRECT APPROACH*

Leszek Hadasz

M. Smoluchowski Institute of Physics, Jagiellonian University
Reymonta 4, 30–069 Cracow, Poland
e–mail address: hadasz@thrisc.if.uj.edu.pl

(Received March 3, 1998)

Methods of calculating the Casimir energy which do not require the explicit knowledge of the oscillation frequencies are developed and applied to the model of the Nambu–Goto string with the Gauss–Bonnet term in the action.

PACS numbers: 11.25.Pm

1.

It is commonly believed that the construction of a string model which is equivalent to QCD (the hypothetical "QCD string"), or even the approximate description of QCD in the low momentum regime in terms of some sort of strings, would be extremely helpful in understanding non-perturbative properties of quantum chromodynamics, such as the nature of the ground state or mechanism of confinement. The conjecture of existence of such a description is supported by a number of facts [1, 2], to mention only the nature of the $1/N_{\rm c}$ expansion [3], success of the dual models in description of Regge phenomenology, area confinement law found in the strong coupling lattice expansion [4] or the existence of flux–line solutions in confining gauge theories [5, 6] and the analytical results concerning two–dimensional QCD [7]. The results obtained recently in the framework of M theory (see, for instance, [8]), stimulated by the Maldacena conjecture [9], are also very promising.

* Work supported by the KBN grant 2 P03B 095 13.

One of the simplest models which may have some of the properties of the QCD string is the Nambu–Goto string with the boundary, Gauss–Bonnet term added (for details see [10, 11]). It is defined by the action:

$$S = \int d^2\xi \sqrt{-g}\left(-\gamma - \frac{\alpha}{2}R\right) , \tag{1}$$

where

$$g_{ab} = \partial_a X_\mu \partial_b X^\mu , \qquad a, b = \tau, \sigma , \qquad g = \det(g_{ab}) ,$$

X_μ gives immersion of the two-dimensional string world–sheet parameterized by (τ, σ) into the four-dimensional Minkowski spacetime, γ and α are constants and R is the inner curvature of the string world–sheet.

Distinguished class of solutions of the equations of motion following from (1) consists of strings, which rotate rigidly in a plane,

$$(X^\mu) = \frac{q}{\lambda^2}\Big(\lambda\tau,\, \cos\lambda\tau \sin\lambda\sigma,\, \sin\lambda\tau \sin\lambda\sigma\, , 0\Big) , \tag{2}$$

where the parameters of the model α, γ and the parameters of the solution λ, q are connected by the condition:

$$\frac{\cos^2 \frac{\pi\lambda}{2}}{\lambda^2} = \frac{1}{q}\sqrt{\frac{\alpha}{\gamma}} . \tag{3}$$

The length of the rotating string (2) is given by the formula

$$L = 2\sqrt{\frac{\alpha}{\gamma}} \frac{\tan\left(\frac{\pi\lambda}{2}\right)}{\cos\left(\frac{\pi\lambda}{2}\right)} . \tag{4}$$

Its semiclassical energy consists of the classical part,

$$E_0 = \frac{\gamma q \pi}{2\lambda}\left(1 + \frac{\sin\pi\lambda}{\pi\lambda}\right) , \tag{5}$$

and the Casimir energy, being the (regularized and renormalized) sum of the zero–point energies of the string oscillation modes $\frac{1}{2}\hbar\nu_n$ (in the following we shall adopt the natural system of units $\hbar = c = 1$.)

The frequencies of the oscillation modes are obtained by solving the eigenvalue problem

$$\hat{H}\ \Sigma_n(x) \equiv \left(-\frac{d^2}{dx^2} + V_\lambda(x)\right) \Sigma_n(x) = \omega_n^2\, \Sigma_n(x)$$

with the potential

$$V_\lambda(x) = \frac{2\lambda^2}{\cos^2\left[\lambda\left(x - \frac{\pi}{2}\right)\right]} \tag{6}$$

and the boundary conditions

$$\Sigma(0) = \Sigma(\pi) = 0\,. \tag{7}$$

The goal of this letter is to develop the techniques of calculating the Casimir energy which does not require the form of the oscillation frequencies to be known explicitly (the result of the direct calculation can be found in [12]). For the case of a field which interacts with some non–trivial background the explicit knowledge of the oscillation frequencies is an exception rather then a rule, and we expect the results of this paper to be useful in such a generic case.

2.

The formal sum over mode frequencies which gives the value of the Casimir energy is divergent and in order to give it a physical meaning we have to adopt some regularization scheme. In the ζ function regularization method [13] one defines:

$$E_{\mathrm{C}} \stackrel{\mathrm{def}}{=} \frac{\lambda}{2q} \lim_{s\to -1} \mu^{s+1}\zeta(s)\,, \tag{8}$$

where, for $\Re\, s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \omega_n^{-s} \tag{9}$$

and the parameter μ with dimension of mass is introduced for dimensional reasons. The Casimir energy is obtained by analytically continuing $\zeta_H(s)$ to the vicinity of $s = -1$. To achieve this, we need to take some technical steps.

Let us first make use of the Mellin transform and write

$$\begin{aligned}\sum_{n=1}^{\infty} \omega_n^{-s} &= \sum_{n=1}^{\infty} \left(\omega_n^2\right)^{-\frac{s}{2}} = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^{\infty} dt\, t^{\frac{s}{2}-1} \sum_{n=1}^{\infty} \mathrm{e}^{-t\omega_n^2} \\ &\equiv \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^{\infty} dt\, t^{\frac{s}{2}-1}\, \mathrm{Tr}\, \mathrm{e}^{-t\hat{H}}\,. \end{aligned} \tag{10}$$

Due to the homogeneous boundary conditions (7) the trace of the heat kernel operator $\mathrm{e}^{-t\hat{H}}$ can be computed in the (orthonormal) Fourier basis [14],

$$\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)\,, \qquad n \in I\!N,$$

with the result

$$\begin{aligned}\mathrm{Tr}\,\mathrm{e}^{-t\hat{H}} &= \frac{2}{\pi}\int_0^\pi dx\,\mathrm{e}^{-tV_\lambda(x)}\sum_{n=1}^{\infty}\sin^2(nx)\mathrm{e}^{-tn^2} \\ &= \frac{1}{\pi}\int_0^\pi dx\,\mathrm{e}^{-tV_\lambda(x)}\sum_{n=-\infty}^{\infty}\sin^2(nx)\mathrm{e}^{-tn^2}. \end{aligned} \tag{11}$$

Application of the Poisson summation formula,

$$\sum_{n=-\infty}^{\infty}\sin^2(nx)\mathrm{e}^{-tn^2} = \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} dz\ \sin^2(zx)\mathrm{e}^{-tz^2+2\pi ikz}, \tag{12}$$

then gives

$$\mathrm{Tr}\,\mathrm{e}^{-t\hat{H}} = \frac{1}{\sqrt{\pi t}}\int_0^\pi dx\,\mathrm{e}^{-tV_\lambda(x)}\left\{\frac{1}{2}-\mathrm{e}^{-\frac{x^2}{t}}+\sum_{k=1}^{\infty}\left[\mathrm{e}^{-\frac{\pi^2k^2}{t}}-\mathrm{e}^{-\frac{(\pi k+x)^2}{t}}\right]\right\}. \tag{13}$$

Performing the integration over t one arrives at the expression

$$\begin{aligned}\zeta_H(-1+2\varepsilon) &= \frac{\Gamma\left(-1+\varepsilon\right)}{\sqrt{\pi}\Gamma\left(-\frac{1}{2}+\varepsilon\right)}\int_0^\pi dx\ \left(V_\lambda(x)\right)^{1-\varepsilon} \\ &\quad -\frac{2}{\sqrt{\pi}\Gamma\left(-\frac{1}{2}+\varepsilon\right)}\left(\int_0^\pi dx\ \left(\frac{U_\lambda(x)}{x}\right)^{1-\varepsilon} K_{1-\varepsilon}(2xU_\lambda(x))\right. \\ &\quad +\int_0^\pi dx\sum_{k=1}^{\infty}\left\{\left(\frac{U_\lambda(x)}{\pi k}\right)^{1-\varepsilon} K_{1-\varepsilon}\left(2\pi kU_\lambda(x)\right)\right. \\ &\quad \left.\left. -\left(\frac{U_\lambda(x)}{\pi k+x}\right)^{1-\varepsilon} K_{1-\varepsilon}\left(2(\pi k+x)U_\lambda(x)\right)\right\}\right), \end{aligned} \tag{14}$$

where $U_\lambda(x)=\sqrt{|V_\lambda(x)|}$ and $\varepsilon=\frac{1+s}{2}$.

Under the assumption that $\lim_{x\to 0}U_\lambda(x)$ exists we can rewrite the expression in the second line of (14) in the form

$$\int_0^\pi dx \quad \left(\frac{U_\lambda(x)}{x}\right)^{1-\varepsilon} K_{1-\varepsilon}(2xU_\lambda(x))$$
$$= \int_0^\pi dx \left\{ \left(\frac{U_\lambda(x)}{x}\right)^{1-\varepsilon} K_{1-\varepsilon}\left(2xU_\lambda(x)\right) - \frac{\pi\cot(\pi\varepsilon)}{2\Gamma(\varepsilon)} x^{2\varepsilon-2} \right\}$$
$$+ \frac{\pi\cot(\pi\varepsilon)}{2\Gamma(\varepsilon)} \frac{\pi^{2\varepsilon-1}}{2\varepsilon-1}\,. \tag{15}$$

Inserting (15) into (14) we finally get

$$\zeta_H(-1+2\varepsilon) = \frac{1}{2\pi}\left[\frac{1}{\varepsilon} + 2\log 2 - 1\right] \int_0^\pi dx\, V_\lambda(x) + \Sigma_H + \mathcal{O}(\varepsilon) \tag{16}$$

with

$$\begin{aligned}
\Sigma_H \;=\; & -\frac{1}{2\pi^2} - \frac{1}{2\pi}\int_0^\pi dx\, V_\lambda(x)\log V_\lambda(x) \\
& -\frac{1}{2\pi}\int_0^\pi \frac{dx}{2x^2}\,\left[2xU_\lambda(x)K_1\left(2xU_\lambda(x)\right) - 1\right] \\
& +\frac{1}{\pi}\int_0^\pi dx\, U_\lambda(x)\sum_{k=1}^{\infty}\left\{\frac{K_1\left(2(\pi k + x)U_\lambda(x)\right)}{\pi k + x} - \frac{K_1\left(2\pi k U_\lambda(x)\right)}{\pi k}\right\}.
\end{aligned}$$

The pole in Eq. (16) for $\varepsilon \to 0$ leads to the ambiguity in the definition of the finite part of the Casimir energy,

$$\left(\sum_{n=1}^{\infty} \omega_n\right)_\zeta^{\text{finite}} = \frac{\log\mu}{2\pi}\int_0^\pi dx\, V_\lambda(x) + \Sigma_H. \tag{17}$$

3.

In the heat–kernel regularization method one introduces into the sum over frequencies an exponential cut–off,

$$\left(\sum_{n=1}^{\infty}\omega_n\right)_\varepsilon = \sum_{n=1}^{\infty}\omega_n \mathrm{e}^{-\varepsilon\omega_n} \equiv \mathrm{Tr}\left\{\sqrt{\hat{H}}\mathrm{e}^{-\varepsilon\sqrt{\hat{H}}}\right\}, \tag{18}$$

with $\varepsilon \to 0$. In order to separate in (18) terms which diverge in this limit let us first evaluate the trace in the same basis as in (11),

$$\begin{aligned} \mathrm{Tr}\left\{\sqrt{\hat{H}}\mathrm{e}^{-\varepsilon\sqrt{\hat{H}}}\right\} &= -\frac{\partial}{\partial\varepsilon}\mathrm{Tr}\left\{\mathrm{e}^{-\varepsilon\sqrt{\hat{H}}}\right\} \\ &= \frac{1}{\pi}\frac{\partial^2}{\partial\varepsilon^2}\int_0^\pi dx \sum_{n=-\infty}^{\infty}\frac{\sin^2(nx)}{\sqrt{n^2+V_\lambda(x)}}\mathrm{e}^{-\varepsilon\sqrt{n^2+V_\lambda(x)}}\,. \end{aligned} \tag{19}$$

Using the Poisson summation formula,

$$\begin{aligned} &\sum_{n=-\infty}^{\infty}\frac{\sin^2(nx)}{\sqrt{n^2+V_\lambda(x)}}\mathrm{e}^{-\varepsilon\sqrt{n^2+V_\lambda(x)}} \\ &= \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} dz\,\frac{\sin^2(zx)}{\sqrt{z^2+V_\lambda(x)}}\mathrm{e}^{2\pi ikz-\varepsilon\sqrt{z^2+V_\lambda(x)}}, \end{aligned} \tag{20}$$

we arrive at the expression

$$\begin{aligned} \left(\sum_{n=1}^{\infty}\omega_n\right)_\varepsilon &= \frac{2}{\pi}\frac{\partial^2}{\partial\varepsilon^2}\int_0^\pi dx\,\left\{\frac{1}{2}K_0\left(\varepsilon U_\lambda(x)\right) - K_0\left(U_\lambda(x)\sqrt{\varepsilon^2+4x^2}\right)\right. \\ &+\left.\sum_{k=1}^{\infty}\left[K_0\left(U_\lambda(x)\sqrt{\varepsilon^2+4\pi^2k^2}\right) - K_0\left(U_\lambda(x)\sqrt{\varepsilon^2+4(\pi k+x)^2}\right)\right]\right\}. \end{aligned} \tag{21}$$

Rewriting the second term in the Eq. (21) in the form

$$\begin{aligned} &\frac{2}{\pi}\frac{\partial^2}{\partial\varepsilon^2}\int_0^\pi dx\,K_0\left(U_\lambda(x)\sqrt{\varepsilon^2+4x^2}\right) \\ &= \frac{2}{\pi}\frac{\partial^2}{\partial\varepsilon^2}\int_0^\pi dx\,\left[K_0\left(U_\lambda(x)\sqrt{\varepsilon^2+4x^2}\right) + \log\sqrt{\varepsilon^2+4x^2}\right] + \frac{\pi}{4\pi^2+\varepsilon^2}, \end{aligned} \tag{22}$$

and using the resulting formula in (21) one finally gets

$$\left(\sum_{n=1}^{\infty}\omega_n\right)_\varepsilon = \frac{1}{\varepsilon^2} - \frac{1}{2\pi}\int_0^\pi dx\,V_\lambda(x)\left[\log\varepsilon + \gamma + \frac{1}{2} - \log 2\right] + \Sigma_H + \mathcal{O}(\varepsilon), \tag{23}$$

where Σ_H is given by Eq. (17).

The finite part of this expression cannot be, as in the zeta function regularization method, defined uniquely. In taking the limit $\varepsilon \to 0$ one can equally well use instead of ε a combination $\mu\varepsilon$ with arbitrary μ. This gives

$$\left(\sum_{n=1}^{\infty}\omega_n\right)_{\varepsilon}^{\text{finite}} = \frac{\log\mu}{2\pi}\int_0^{\pi} dx\, V_\lambda(x) + \Sigma_H, \tag{24}$$

i.e. precisely the same expression as obtained in the zeta function method.

4.

After calculating the finite part of a quantity such as a sum of the mode frequencies we are facing the final task of fixing the ambiguities which arise due to the subtraction of the pole term. This can be achieved by imposing on the calculated quantity appropriately chosen physical conditions [15]. In the considered case we expect that the Casimir energy for long strings should behave like $\sim L^{-1}$, where L is the string length. Terms which do not vanish in this limit can be treated [16] as renormalizing the classical string mass (let us note that the ambiguous term in (17) also belongs to this class). In order to fix the coefficient of the L^{-1} term let us note that in the Nambu–Goto limit (vanishing coefficient α in front of the Gauss–Bonnet term in the action (1)) the oscillation eigenfrequencies tend to

$$\nu_n^{\mathrm{NG}} = \frac{2n}{L}\,,$$

and we expect the Casimir energy to be equal to [12, 17]

$$E_{\mathrm{C}}^{\mathrm{NG}} = -\frac{1}{12}\frac{1}{L}\,.$$

This finally gives

$$\begin{aligned} E_{\mathrm{C}} &= \frac{\lambda}{2q}\Big[-\frac{1}{12} + \frac{1}{\pi}\int_0^{\pi} dx U_\lambda(x) \\ &\quad\times\sum_{k=1}^{\infty}\left\{\frac{K_1\left(2(\pi k + x)U_\lambda(x)\right)}{\pi k + x} - \frac{K_1\left(2\pi k U_\lambda(x)\right)}{\pi k}\right\}\Big]. \end{aligned} \tag{25}$$

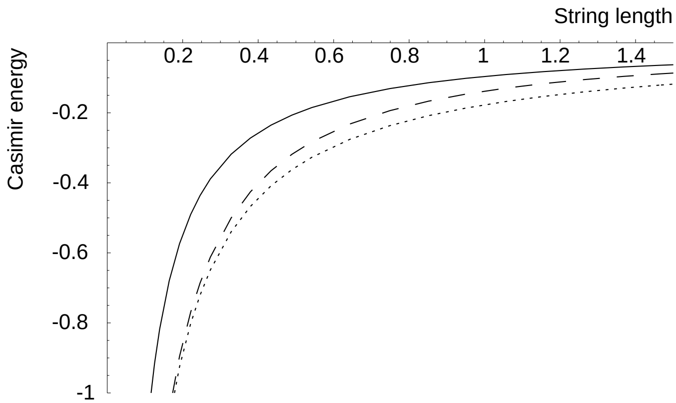

Fig. 1. Casimir energy versus string length for various values of the parameter α : $\alpha = 10^{-3}$ (solid line), $\alpha = 1$ (dashed line) and $\alpha = 10^3$ (dotted line). All dimensionful quantities in the system of units $\gamma = 1$.

REFERENCES

[1] J. Polchinski, Strings and QCD?, preprint UTTG-92-16; hep-th/9210045.

[2] A.M. Polyakov, The wall of the cave, preprint PUPT-1812; hep-th/9809057.

[3] G. 't Hooft, *Nucl. Phys.* **B72**, 461 (1974).

[4] K. Wilson, *Phys. Rev.* **D8**, 2445 (1974).

[5] H.B. Nielsen, P. Olesen, *Nucl. Phys.* **B61**, 45 (1973).

[6] M. Baker, J.S. Ball, F. Zachariasen, *Phys. Rev.* **D41**, 2612 (1990).

[7] David J. Gross, *Nucl. Phys.* **B400**, 161 (1993); David J. Gross, Washinghton Taylor IV, *Nucl. Phys.* **B400**, 181 (1993); David J. Gross, Washinghton Taylor IV, *Nucl. Phys.* **B403**, 395 (1993).

[8] E. Witten, *Nucl. Phys.* (Proc. Suppl.) **B68**, (1998) 216.

[9] J. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998).

[10] P. Węgrzyn, *Phys. Rev.* **D50**, 2769 (1994).

[11] L. Hadasz, P. Węgrzyn, *Phys. Rev.* **D51**, 2891 (1995).

[12] L. Hadasz, *Mod. Phys. Lett.* **A13**, 605 (1998).

[13] S.K. Blau, M. Viser, A. Wipf, *Nucl. Phys.* **B310**, 163 (1988).

[14] R.D. Ball, *Phys. Rep.* **182**, 1 (1989).

[15] N.D. Birrell, P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, 1982.

[16] V.V. Nesterenko, I.G. Pirozhenko, *J. Math. Phys.* **38**, 6265 (1997); V.V. Nesterenko, I.G. Pirozhenko, *Mod. Phys. Lett.* **A13**, 2513 (1998).

[17] M. Lüsher, *Nucl. Phys.* **B180**, 317 (1981); O. Alvarez, *Phys. Rev.* **D24**, 440 (1981).