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SEARCHING FOR A UNIVERSAL INTEGRABLE SYSTEM

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It is argued for the hypothesis that the SDYM equations for the Moyal bracket algebra (the *master equation*) is in a sense a universal integrable system. We show how the su(N) SDYM equations, the KP equation and the integrable equations in two dimensions can be encoded in the master equation.

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1. Introduction

In an interesting paper by Mason [1] it has been suggested that the heavenly equation of self-dual gravity may be a universal integrable system. Here we intend to argue for the hypothesis that the universal integrable system is, perhaps, defined by the Moyal deformation of the heavenly equation lifted to *six* dimensions (the *master equation*).The master equation can be also considered to be the SDYM system for the Moyal bracket algebra. It is known that the Moyal bracket algebra appears to be the most general 2-index infinite Lie algebra [2] and it contains the Poisson algebra as well as the su(N) algebra [3]. To find the master equation we first deal with the su(N) SDYM equations in R^4 of the metric $(++--)$

$$ds_o^2 = 2(dx \otimes_s d\tilde{x} + dy \otimes_s d\tilde{y}) \,. \tag{1.1}$$

These equations read [4]

$$F_{xy} = 0 \,, \quad F_{\tilde{x}\tilde{y}} = 0 \quad \text{and} \quad F_{x\tilde{x}} + F_{y\tilde{y}} = 0 \,, \tag{1.2}$$

where $F_{\mu\nu}\varepsilon$ su(N) $\otimes C^\infty(R^4)$, $\mu,\nu\varepsilon\{x,y,\tilde{x},\tilde{y}\}$, denotes the Yang–Mills field tensor. Then, as

$$F_{\mu\nu} = [\partial_\mu + A_\mu, \partial_\nu + A_\nu] \tag{1.3}$$

($A_\mu \,\varepsilon\,$ su(N) $\otimes C^\infty(R^4)$ stands for the Yang–Mills potential) one finds (1.2) to read

$$\partial_x A_y - \partial_y A_x + [A_x, A_y] = 0\,, \tag{1.4}$$

$$\partial_{\tilde{x}} A_{\tilde{y}} - \partial_{\tilde{y}} A_{\tilde{x}} + [A_{\tilde{x}}, A_{\tilde{y}}] = 0\,, \tag{1.5}$$

$$\partial_x A_{\tilde{x}} - \partial_{\tilde{x}} A_x + \partial_y A_{\tilde{y}} - \partial_{\tilde{y}} A_y + [A_x, A_{\tilde{x}}] + [A_y, A_{\tilde{y}}] = 0\,. \tag{1.6}$$

From (1.4) it follows that there exists the gauge such that

$$A_x = 0 = A_y\,. \tag{1.7}$$

Thus we get

$$\partial_{\tilde{x}} A_{\tilde{y}} - \partial_{\tilde{y}} A_{\tilde{x}} + [A_{\tilde{x}}, A_{\tilde{y}}] = 0\,, \tag{1.8}$$

$$\partial_x A_{\tilde{x}} + \partial_y A_{\tilde{y}} = 0\,. \tag{1.9}$$

From (1.9) one infers that

$$A_{\tilde{x}} = -\partial_y \theta \quad \text{and} \quad A_{\tilde{y}} = \partial_x \theta\,, \tag{1.10}$$

$\theta = \theta(x, y, \tilde{x}, \tilde{y})\,\varepsilon\,\mathrm{su(N)} \otimes C^\infty(R^4)\,.$

Inserting (1.10) into (1.8) we obtain [4–9]

$$\partial_x \partial_{\tilde{x}} \theta + \partial_y \partial_{\tilde{y}} \theta + [\partial_x \theta, \partial_y \theta] = 0\,. \tag{1.11}$$

It is also well known that Eqs. (1.8), (1.9) can be derived from the following Lax pair [10]

$$\begin{aligned} (\lambda \partial_y + \partial_{\tilde{x}})\psi_\lambda &= -A_{\tilde{x}} \psi_\lambda\,, \\ (-\lambda \partial_x + \partial_{\tilde{y}})\psi_\lambda &= -A_{\tilde{y}} \psi_\lambda\,, \quad \lambda\,\varepsilon\,CP^1\,. \end{aligned} \tag{1.12}$$

We generalize (1.12) to the *Moyal $*$-product algebra* . Thus we write

$$\begin{aligned} i\hbar(\lambda \partial_y + \partial_{\tilde{x}})\varphi_\lambda &= -a_{\tilde{x}} * \varphi_\lambda\,, \\ i\hbar(-\lambda \partial_x + \partial_{\tilde{y}})\varphi_\lambda &= -a_{\tilde{y}} * \varphi_\lambda\,, \quad \lambda\,\varepsilon\,CP^1\,, \end{aligned} \tag{1.13}$$

where $\varphi_\lambda,\ a_{\tilde{x}}\,and\,a_{\tilde{y}}$ are functions of $(\hbar, x, y, \tilde{x}, \tilde{y}, p, q)$; the Moyal $*$ -product is defined by [11–13]

$$\begin{aligned} f_1 * f_2 :&= f_1 \exp\left(\frac{i\hbar}{2}\overleftrightarrow{P}\right) f_2\,, \\ \overleftrightarrow{P} :&= \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\,, \end{aligned} \tag{1.14}$$

($\hbar$ is the deformation parameter). The integrability conditions of the system (1.13) read

$$\partial_{\tilde{x}} a_{\tilde{y}} - \partial_{\tilde{y}} a_{\tilde{x}} + \{a_{\tilde{x}}, a_{\tilde{y}}\}_{\mathrm{M}} = 0\,, \tag{1.15}$$

$$\partial_x a_{\tilde{x}} + \partial_y a_{\tilde{y}} = 0\,, \tag{1.16}$$

where $\{\cdot,\cdot\}_{\mathrm{M}}$ denotes the *Moyal bracket* [11–13]

$$\begin{aligned}\{f_1, f_2\}_{\mathrm{M}} &:= \frac{1}{i\hbar}(f_1 * f_2 - f_2 * f_1)\\ &= f_1 \frac{2}{\hbar} \sin\left(\frac{\hbar}{2} \overset{\leftrightarrow}{P}\right) f_2\,.\end{aligned} \tag{1.17}$$

From (1.16) we get

$$\begin{aligned} a_{\tilde{x}} = -\partial_y \Theta\,, \quad &\text{and} \quad a_{\tilde{y}} = \partial_x \Theta\,,\\ \Theta = \Theta(&\hbar; x, y, \tilde{x}, \tilde{y}, p, q)\,. \end{aligned} \tag{1.18}$$

Consequently, inserting (1.18) into (1.15) one finds the *master equation* [9,14]

$$\partial_x \partial_{\tilde{x}} \Theta + \partial_y \partial_{\tilde{y}} \Theta + \{\partial_x \Theta, \partial_y \Theta\}_{\mathrm{M}} = 0\,. \tag{1.19}$$

From the Lax pair (1.13) and from (1.18) it follows that under the assumption that the functions

$$\begin{aligned} a_{\tilde{x}} &= \varphi_\lambda * i\hbar(\lambda\partial_y + \partial_{\tilde{x}})\varphi_\lambda^{\overset{*}{-1}}\,,\\ a_{\tilde{y}} &= \varphi_\lambda * i\hbar(-\lambda\partial_x + \partial_{\tilde{y}})\varphi_\lambda^{\overset{*}{-1}}\\ &(\varphi_\lambda * \varphi_\lambda^{\overset{*}{-1}} = 1) \end{aligned} \tag{1.20}$$

are independent of λ, the solution φ_λ and Θ are related by

$$\begin{aligned} -\partial_y \Theta &= i\hbar\varphi_1 * (\partial_y + \partial_{\tilde{x}})\varphi_1^{\overset{*}{-1}}\,,\\ \partial_x \Theta &= i\hbar\varphi_1 * (-\partial_x + \partial_{\tilde{y}})\varphi_1^{\overset{*}{-1}}\,. \end{aligned} \tag{1.21}$$

Now we recall how the master equation (1.19) can be reduced to the heavenly equations (for details see [9, 14]). Assume first the following symmetry

$$(\partial_x - \partial_{\tilde{x}})\Theta = 0 = (\partial_y - \partial_{\tilde{y}})\Theta\,. \tag{1.22}$$

Consequently, equation (1.19) takes the form of the *Moyal deformation of the Husain–Park equation*

$$\partial_x^2 \Theta + \partial_y^2 \Theta + \{\partial_x \Theta, \partial_y \Theta\}_{\mathrm{M}} = 0\,. \tag{1.23}$$

Hence, if Θ is analytic in $\hbar$, *i.e.*,

$$\begin{aligned} \Theta &= \sum_{n=0}^{\infty} \hbar^n \Theta_n \,, \\ \Theta_n &= \Theta_n(x+\tilde{x}, y+\tilde{y}, p, q) \end{aligned} \tag{1.24}$$

then

$$\lim_{\hbar \to 0} \{\partial_x \Theta, \partial_y \Theta\}_{\mathrm{M}} = \{\partial_x \Theta_0, \partial_y \Theta_0\}_{\mathrm{P}} \,, \tag{1.25}$$

where $\{\cdot,\cdot\}_{\mathrm{P}}$ stands for the Poisson bracket, and (1.23) yields

$$\partial_x^2 \Theta_0 + \partial_y^2 \Theta_0 + \{\partial_x \Theta_0, \partial_y \Theta_0\}_{\mathrm{P}} = 0 \,. \tag{1.26}$$

This is exactly the *Husain–Park heavenly equation* [15, 16].
The symmetry

$$(\partial_x - \partial_{\tilde{x}})\Theta = 0 = \partial_{\tilde{y}} \Theta \,. \tag{1.27}$$

leads to the *Moyal deformation of Grant's equation*

$$\partial_x^2 \Theta + \{\partial_x \Theta, \partial_y \Theta\}_{\mathrm{M}} = 0 \,. \tag{1.28}$$

Analogously, if

$$(\partial_x - \partial_q)\Theta = 0 = (\partial_y - \partial_p)\Theta \tag{1.29}$$

then one gets the *Moyal deformation of Plebański's second heavenly equation* and, under (1.24), the *second heavenly equation* [18]

$$\partial_x \partial_{\tilde{x}} \Theta_0 + \partial_y \partial_{\tilde{y}} \Theta_0 + \partial_x^2 \Theta_0 \partial_y^2 \Theta_0 - (\partial_x \partial_y \Theta_0)^2 = 0 \,. \tag{1.30}$$

Similar considerations lead to the *first heavenly equation* or to the *Cauchy–Kovalevskaya form of the second heavenly equation* (see [9]). Concluding, we have found the reduction of the master equation (1.19) to the heavenly equations.

In Section 2 we show how the Lie algebra representation of the Moyal bracket algebra onto su(N) algebra leads from the master equation to the su(N) SDYM equations or to the chiral model equations.

Section 3 is devoted to the reduction of the master equation to the KP equation. In Section 4 we show that the Lax equation for the function dependent on *two* variables can be written in the form of our master equation. Finally, some concluding remarks close the paper.

2. From the master equation to the su(N) SDYM and chiral models

To start with we recall some results of [3] where the basis of the su(N) algebra, which appears to be useful for our purpose, has been investigated.

Define two $N \times N$ matrices

$$S := \sqrt{\omega} \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & .0 & \dots & \omega^{N-1} \end{pmatrix},$$

$$\omega := \exp\left(\frac{2\pi i}{N}\right), \quad \sqrt{\omega} = \exp\left(\frac{\pi i}{N}\right),$$

$$T := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 \\ -1 & 0 & .0 & \dots & 0 \end{pmatrix},$$

$$S^N = T^N = -1\,, \quad T \cdot S = \omega S \cdot T\,. \tag{2.1}$$

Define

$$L_{\vec{m}} := \frac{iN}{2\pi} \omega^{\frac{m_1 m_2}{2}} S^{m_1} T^{m_2}; \quad \vec{m} := (m_1, m_2) \varepsilon Z \times Z\,. \tag{2.2}$$

The matrices $L_{\vec{m}}$ have the following properties

$$\begin{aligned} L_{\vec{m}+N\vec{r}} &= (-1)^{(m_1+1)r_2+(m_2+1)r_1+Nr_1r_2} L_{\vec{m}}\,, \\ \mathrm{Tr}\, L_{\vec{m}} &= 0 \text{ except for } m_1 = m_2 = 0 \bmod N\,. \end{aligned} \tag{2.3}$$

$$\mathrm{Tr}\, L_{N\vec{r}} := (-1)^{r_2+r_1+Nr_1r_2} \frac{iN^2}{2\pi}\,, \tag{2.4}$$

$$\begin{aligned} L_{\vec{m}} L_{\vec{n}} &= \frac{iN}{2\pi} \omega^{\frac{\vec{n}\times\vec{m}}{2}} L_{\vec{m}+\vec{n}}\,, \\ \vec{n} \times \vec{m} :&= n_1 m_2 - n_2 m_1\,, \end{aligned} \tag{2.5}$$

$$[L_{\vec{m}}, L_{\vec{n}}] = \frac{N}{\pi} \sin\left(\frac{\pi}{N} \vec{m} \times \vec{n}\right) L_{\vec{m}+\vec{n}}\,. \tag{2.6}$$

Moreover, as $S^\dagger = S^{-1}$ and $T^\dagger = T^{-1}$ we get (see also (2.5))

$$L_{\vec{m}}^\dagger = -L_{-\vec{m}} = \left(\frac{N}{2\pi}\right)^2 L_{\vec{m}}^{-1}\,. \tag{2.7}$$

Finally, from formulae det$S$=det$T = (-1)^N$ it follows that

$$\det L_{\vec{m}} := (-1)^{N(m_1+m_2+m_1m_2)} \left(\frac{iN}{2\pi}\right)^N . \tag{2.8}$$

It has been shown in [3] that N^2-1 *matrices* $L_{\vec{\mu}}$, $0 \leq \mu_1 \leq N-1$, $0 \leq \mu_1 \leq N-1$ *and* $\vec{\mu} \neq (0,0)$, *span the su(N) algebra.* (Henceforth, the Greek indices $\vec{\mu}, \vec{\nu}, \ldots$, *etc.* are assumed to satisfy the above conditions).

Now we deal with the basis of smooth functions on the 2-torus T^2

$$E_{\vec{m}} := \exp[i(m_1 p + m_2 q)] , \tag{2.9}$$

$$\vec{m} := (m_1, m_2) \varepsilon Z \times Z \quad \text{and} \quad (p,q) \varepsilon T^2 .$$

Employing (1.14) and (1.17) one quickly finds the relations

$$E_{\vec{m}} * E_{\vec{n}} = \exp\left(\frac{i\hbar}{2} \vec{m} \times \vec{n}\right) E_{\vec{m}+\vec{n}} , \tag{2.10}$$

$$\{E_{\vec{m}}, E_{\vec{n}}\}_{\mathrm{M}} = \frac{2}{\hbar} \sin\left(\frac{\hbar}{2} \vec{m} \times \vec{n}\right) E_{\vec{m}+\vec{n}} . \tag{2.11}$$

Take the deformation parameter $\hbar$ to be

$$\hbar = \frac{2\pi}{N} . \tag{2.12}$$

Then (2.11) reads

$$\{E_{\vec{m}}, E_{\vec{n}}\}_{\mathrm{M}} = \frac{N}{\pi} \sin\left(\frac{\pi}{N} \vec{m} \times \vec{n}\right) E_{\vec{m}+\vec{n}} . \tag{2.13}$$

Comparing (2.13) with (2.6) we are led to the linear mapping of smooth function on T^2 onto su(N) defined by the linear extension of the following mapping

$$\chi : \begin{array}{ccc} E_{\vec{\mu}+N\vec{r}} & \longmapsto & L_{\vec{m}+N\vec{r}} = (-1)^{(\mu_1+1)r_2+(\mu_2+1)r_1+Nr_1r_2} L_{\vec{\mu}} \\ E_{N\vec{r}} & \longmapsto & 0 , \end{array} \tag{2.14}$$

where, as before, $\vec{\mu} := (\mu_1, \mu_2)$, $0 \leq \mu_1 \leq N-1$, $0 \leq \mu_2 \leq N-1$, and $\vec{\mu} \neq (0,0)$, and $\vec{r} := (r_1, r_2) \varepsilon Z \times Z$.

Using (2.3), (2.6) and (2.13) one easily finds that the mapping (2.14) defines *the Lie algebra homomorphism of the Moyal bracket algebra on* T^2 *with* $\hbar = \frac{2\pi}{N}$ *onto su(N).*

Let $\Theta = \Theta(\hbar; x, y, \tilde{x}, \tilde{y}, p, q)$ be a solution of the master equation (1.19) on $V \times T^2$, $V \subset R^4$, and let

$$\Theta = \Theta(\hbar; x, y, \tilde{x}, \tilde{y}, p, q) = \sum_{\vec{m}} \Theta_{\vec{m}}(\hbar; x, y, \tilde{x}, \tilde{y}) E_{\vec{m}} \tag{2.15}$$

be the Fourier expansion of Θ. Substituting $\hbar = \frac{2\pi}{N}$ into (2.15) and employing (2.14) we get

$$\theta = \theta(N; x, y, \tilde{x}, \tilde{y}) := \chi\left(\Theta\left(\frac{2\pi}{N}; x, y, \tilde{x}, \tilde{y}, p, q\right)\right) = \sum_{\vec{\mu}} \theta_{\vec{\mu}}(N; x, y, \tilde{x}, \tilde{y}) L_{\vec{\mu}}\,, \tag{2.16}$$

where

$$\theta_{\vec{\mu}} = \theta_{\vec{\mu}}(N; x, y, \tilde{x}, \tilde{y}) := \sum_{\vec{r}} (-1)^{(\mu_1+1)r_2 + (\mu_2+1)r_1 + N r_1 r_2} \Theta_{\vec{\mu}+N\vec{r}}\left(\frac{2\pi}{N}; x, y, \tilde{x}, \tilde{y}\right). \tag{2.17}$$

As χ is the Lie algebra homomorphism the su(N)-valued function defined by (2.16) and (2.17) fulfills the su(N) SDYM equation (1.11) on V.

Conversely, every analytic solution of Eq. (1.11) can be obtained in this way. Indeed, let the su(N)-valued function

$$\theta = \theta(N; x, y, \tilde{x}, \tilde{y}) = \sum_{\vec{\mu}} \theta_{\vec{\mu}}(N; x, y, \tilde{x}, \tilde{y}) L_{\vec{\mu}} \tag{2.18}$$

be the analytic solution of Eq. (1.11) for the following analytic Cauchy data

$$\begin{aligned} \tilde{\rho} &= \tilde{\rho}(x, y, \tilde{y}) = \sum_{\vec{\mu}} \tilde{\rho}_{\vec{\mu}}(x, y, \tilde{y}) L_{\vec{\mu}} := \theta(N; x, y, \tilde{x}, \tilde{y})\mid_{\tilde{x}=x}\,, \\ \tilde{\sigma} &= \tilde{\sigma}(x, y, \tilde{y}) = \sum_{\vec{\mu}} \tilde{\sigma}_{\vec{\mu}}(x, y, \tilde{y}) L_{\vec{\mu}} := (\partial_x + \partial_{\tilde{x}})\theta(N; x, y, \tilde{x}, \tilde{y})\mid_{\tilde{x}=x}\;. \end{aligned} \tag{2.19}$$

Define the functions $\rho_{\vec{\mu}+N\vec{r}} = \rho_{\vec{\mu}+N\vec{r}}(x, y, \tilde{y})$ and $\sigma_{\vec{\mu}+N\vec{r}} = \sigma_{\vec{\mu}+N\vec{r}}(x, y, \tilde{y})$ by (compare with (2.17))

$$\begin{aligned} \tilde{\rho}_{\vec{\mu}} &= \sum_{\vec{r}} (-1)^{(\mu_1+1)r_2 + (\mu_2+1)r_1 + N r_1 r_2} \rho_{\vec{m}+N\vec{r}}\,, \\ \tilde{\sigma}_{\vec{\mu}} &= \sum_{\vec{r}} (-1)^{(\mu_1+1)r_2 + (\mu_2+1)r_1 + N r_1 r_2} \sigma_{\vec{m}+N\vec{r}}\,. \end{aligned} \tag{2.20}$$

Let $\Theta(\frac{2\pi}{N};x,y,\tilde{x},\tilde{y},p,q)$ be the unique solution of the master equation (1.19) in $R^4 \times T^2$ for $\hbar = \frac{2\pi}{N}$ and for the Cauchy data

$$\begin{aligned} \Theta(\frac{2\pi}{N};x,y,\tilde{x},\tilde{y},p,q)\mid_{\tilde{x}=x} &= \sum_{\vec{\mu},\vec{r}} \rho_{\vec{\mu}+N\vec{r}} E_{\vec{\mu}+N\vec{r}}\,, \\ (\partial_x + \partial_{\tilde{x}})\theta(N;x,y,\tilde{x},\tilde{y})\mid_{\tilde{x}=x} &= \sum_{\vec{\mu},\vec{r}} \sigma_{\vec{\mu}+N\vec{r}} E_{\vec{\mu}+N\vec{r}}\,. \end{aligned} \tag{2.21}$$

From our previous considerations it follows that the su(N)-valued function $\chi(\Theta)$ fulfills the su(N) SDYM equation (1.11) for the Cauchy data given by (2.19). Consequently, by the uniqueness of the solution of the Cauchy problem for Eq. (1.11) we conclude that $\chi(\Theta) = \theta$, where θ is defined by (2.18). Gathering, one arrives at the theorem

Theorem 2.1
Let

$$\Theta = \Theta(\hbar;x,y,\tilde{x},\tilde{y},p,q) = \sum_{\vec{m}} \Theta_{\vec{m}}(\hbar;x,y,\tilde{x},\tilde{y}) \exp[i(m_1p + m_2q)] \tag{2.22}$$

be a solution of the master equation (1.19) on $V \times T^2, V \subset R^4$. Then, the su$(N)$-valued function

$$\begin{aligned} \theta &= \theta(N;x,y,\tilde{x},\tilde{y}) \\ &= \sum_{\vec{\mu}} \left(\sum_{\vec{r}} (-1)^{(\mu_1+1)r_2+(\mu_2+1)r_1+Nr_1r_2} \Theta_{\vec{\mu}+N\vec{r}} \left(\frac{2\pi}{N};x,y,\tilde{x},\tilde{y} \right) \right) L_{\vec{\mu}} \end{aligned} \tag{2.23}$$

is the solution of the su(N) SDYM equation (1.11) on V. Conversely, every analytic solution of Eq. (1.11) can be obtained in this way. □

Consider now an especially interesting case when the solution Θ has the symmetry (1.22) *i.e.*, it is of the form

$$\Theta = \Theta(\hbar;x+\tilde{x},y+\tilde{y},p,q)\,. \tag{2.24}$$

In this case, as we know the function Θ satisfies the Moyal deformation of the Husain–Park equation (1.23) and then the su(N)-valued function $\theta = \theta(N;x+\tilde{x},y+\tilde{y})$ defined by (2.23) appears to be the solution of the su(N) principal chiral equation in two dimensions

$$\partial_x^2\theta + \partial_y^2\theta + [\partial_x\theta, \partial_y\theta] = 0 \tag{2.25}$$

(see [19, 20]). Moreover, if $N \to \infty$ then $\hbar = \frac{2\pi}{N} \to 0$. Assuming that Θ given by (2.24) is analytic in $\hbar$ we find that the function

$$\Theta_0 = \Theta_0(x+\tilde{x}, y+\tilde{y}, p, q) := \Theta(0; x+\tilde{x}, y+\tilde{y}, p, q) \tag{2.26}$$

fulfills the Husain–Park heavenly equation (1.26). From (2.23) one concludes that

$$\lim_{N\to\infty} \theta(N; x+\tilde{x}, y+\tilde{y}) = \sum_{\vec{\mu}\neq(0,0)} \Theta_{\vec{\mu}}(0; x+\tilde{x}, y+\tilde{y}) L_{\vec{\mu}}\,. \tag{2.27}$$

These results can be interpreted as follows;

If $\Theta(\hbar; x+\tilde{x}, y+\tilde{y}, p, q)$ is a solution of the master equation on $V \times T^2$, $V \subset R^4$, analytic in $\hbar$, then the sequence of su(N) chiral fields $\theta(N; x+\tilde{x}, y+\tilde{y}) := \chi(\Theta(\frac{2\pi}{N}; x+\tilde{x}, y+\tilde{y}, p, q))$, for $N = 2, 3, \dots$ tends (in general) to a curved space in the limit.

This is a partial answer to Ward's question [21] (see also [22]).

Another interesting case corresponds to the integrable chiral equations in 2+1 dimensions [23–26]. Here the following symmetry is assumed

$$(\partial_x - \partial_{\tilde{x}})\Theta = 0 \tag{2.28}$$

i.e., Θ is of the form $\Theta = \Theta(\hbar; x+\tilde{x}, y, \tilde{y}, p, q)$. Consequently, by Theorem 2.1, the su(N)-valued function θ defined by (2.23), $\theta = \theta(N; x+\tilde{x}, y, \tilde{y})$ fulfills the chiral equation in 2+1 dimension

$$\partial_x^2\theta + \partial_y\partial_{\tilde{y}}\theta + [\partial_x\theta,\, \partial_y\theta] = 0\,. \tag{2.29}$$

Remark: The proof of the second part of Theorem 2.1 is based on the Cauchy–Kovalevskaya theorem. Consequently, this part has been proved under the assumption that the solutions are *analytic.* However, we suppose that it holds for the general case.

3. From the master equation to the KP equation

Consider the pseudo-differential operator [27–29]

$$\overset{\wedge}{\mathcal{L}} := \partial + \sum_{n=1} u_n(t_1, t_2, \dots)\partial^{-n}\,, \qquad \partial := \partial_{t_1}\,. \tag{3.1}$$

Define the following operators

$$\overset{\wedge}{\mathcal{B}}_n := \left[\, \overset{\wedge}{\mathcal{L}}{}^n \,\right]_+, \qquad n = 1, 2, \dots\,, \tag{3.2}$$

where $[\overset{\wedge}{\mathcal{R}}]_+$ stands for the projection of the pseudo-differential operator $\overset{\wedge}{\mathcal{R}}$ onto differential part, *i.e.*, $[\overset{\wedge}{\mathcal{R}}]_+$ denotes the part of $\overset{\wedge}{\mathcal{R}}$ containing non negative powers of ∂. Then the KP hierarchy is given by the Lax equations

$$\frac{\partial \overset{\wedge}{\mathcal{L}}}{\partial t_n} = \left[\overset{\wedge}{\mathcal{B}}_n, \overset{\wedge}{\mathcal{L}} \right], \qquad n = 1, 2, \ldots \tag{3.3}$$

(About KP hierarchy see also [5]).

Alternatively, the KP hierarchy can be defined by the zero-curvature conditions

$$\partial_{t_m} \overset{\wedge}{\mathcal{B}}_n - \partial_{t_n} \overset{\wedge}{\mathcal{B}}_m + \left[\overset{\wedge}{\mathcal{B}}_n, \overset{\wedge}{\mathcal{B}}_m \right] = 0 \,, \qquad m, n = 1, 2, \ldots \tag{3.4}$$

One quickly finds that for $n = 1$ or $m = 1$ Eq. (3.4) appears to be trivial.

Now, for the further convenience we put

$$q := t_1 \,. \tag{3.5}$$

Then the operators $\overset{\wedge}{\mathcal{L}}$ and $\overset{\wedge}{\mathcal{B}}_n$, $n = 1, 2, \ldots$ can be considered to be the operators acting in the Hilbert space $L^2(R^1)$. Therefore one can use the machinery of the Weyl–Wigner–Moyal formalism [11–13, 30–33].

In particular the Weyl correspondence W^{-1} gives

$$\begin{aligned} W^{-1} &: \overset{\wedge}{\mathcal{B}}_n \mapsto \mathcal{B}_n = \mathcal{B}_n(\hbar; p, q, t_2, t_3, \ldots) \\ &:= \int_{-\infty}^{+\infty} \left\langle q - \frac{\xi}{2} \right| \overset{\wedge}{\mathcal{B}}_n \left| q + \frac{\xi}{2} \right\rangle \exp\left(\frac{ip\xi}{\hbar} \right) d\xi \,, \qquad n = 1, 2, \ldots \end{aligned} \tag{3.6}$$

Define

$$b_n := -i\hbar \mathcal{B}_n, \qquad n = 1, 2, \ldots \tag{3.7}$$

Thus Eq. (3.4) can be equivalently written in the following form

$$\partial_{t_m} b_n - \partial_{t_n} b_m + \{b_m, b_n\}_{\mathrm{M}} = 0 \,, \qquad m, n = 1, 2, \ldots \tag{3.8}$$

An especially interesting case is when $m = 2$ and $n = 3$. Straightforward calculations give

$$\begin{aligned} b_2 &= \frac{i}{\hbar} p^2 - 2i\hbar u_1 \,, \\ b_3 &= -\frac{1}{\hbar^2} p^3 + 3pu_1 - 3i\hbar \left(\frac{1}{2} \partial_q u_1 + u_2 \right) . \end{aligned} \tag{3.9}$$

Then Eq. (3.8) for $m = 2$ and $n = 3$ reads

$$\partial_{t_2} b_3 - \partial_{t_3} b_2 + \{b_2, b_3\}_{\mathrm{M}} = 0\,. \tag{3.10}$$

Substituting (3.9) into (3.10) one gets

$$4\partial_{t_3} u_1 - 3\partial_q \partial_{t_2} u_1 - 6\partial_{t_2} u_2 - \partial_q^3 u_1 - 12 u_1 \partial_q u_1 = 0\,, \tag{3.11}$$

$$-3\partial_{t_2} u_1 + 3\partial_q^2 u_1 + 6\partial_q u_2 = 0\,. \tag{3.12}$$

Differentiating (3.11) with respect to q and (3.12) with respect to t_2 and comparing the results one gets the well known KP equation

$$\partial_q (4\partial_{t_3} u_1 - 12 u_1 \partial_q u_1 - \partial_q^3 u_1) = 3\partial_{t_2}^2 u_1\,. \tag{3.13}$$

Eq. (3.10) resembles very much Eq. (1.15). Therefore we put

$$\begin{aligned} \tilde{x} :&= t_2, \qquad\qquad \tilde{y} := t_3\,, \\ a_{\tilde{x}} &= a_{\tilde{x}}(\hbar; \tilde{x}, \tilde{y}, p, q) := \frac{i}{\hbar} p^2 - 2i\hbar u\,, \\ a_{\tilde{y}} &= a_{\tilde{y}}(\hbar; \tilde{x}\,, \tilde{y}, p, q) = -\frac{1}{\hbar^2} p^3 + 3pu - 3i\hbar v\,, \\ u &= u(q, \tilde{x}, \tilde{y}) \qquad \text{and} \qquad v = v(q, \tilde{x}, \tilde{y})\,. \end{aligned} \tag{3.14}$$

Then by (3.9), (3.10), (3.11) and (3.12) one infers that $a_{\tilde{x}}$ and $a_{\tilde{y}}$ defined by (3.14) fulfill Eq. (1.15) iff,

$$4\partial_{\tilde{y}} u - 6\partial_{\tilde{x}} v - \partial_q^3 u - 12 u \partial_q u = 0\,, \tag{3.15}$$

and

$$-3\partial_{\tilde{x}} u + 6\partial_q v = 0\,. \tag{3.16}$$

Consequently, as before, we are led to the KP equation

$$\partial_q (4\partial_{\tilde{y}} u - 12 u \partial_q u - \partial_q^3 u) = 3\partial_{\tilde{x}}^2 u\,, \qquad u = u(q, \tilde{x}, \tilde{y})\,. \tag{3.17}$$

It is evident that $a_{\tilde{x}}$ and $a_{\tilde{y}}$ given by(3.14) fulfill also Eq. (1.16). Gathering, one arrives at the lemma

Lemma 3.1

The functions $a_{\tilde{x}}$ and $a_{\tilde{y}}$ given by (3.14) satisfy Eqs. (1.15)and (1.16) iff the function u is a solution of the KP equation (3.17) and the function v is determined by u according to (3.15) and (3.16). $\Box$

Let $a_{\tilde{x}}$ and $a_{\tilde{y}}$ be given by (3.14). Then by (1.18) we find Θ to be

$$\begin{aligned} \Theta = \Theta(\hbar; x, y, \tilde{x}, \tilde{y}, p, q) &= -\frac{1}{\hbar^2} x p^3 - i\frac{1}{\hbar} y p^2 + 3xpu + i\hbar(2yu - 3xv)\,, \\ u &= u(q, \tilde{x}, \tilde{y}), \qquad v = v(q, \tilde{x}, \tilde{y})\,. \end{aligned} \tag{3.18}$$

Finally, employing Lemma 3.1, one concludes that the KP equation is encoded in the master equation according to the following theorem

Theorem 3.1

The function $\Theta = \Theta(\hbar; x, y, \tilde{x}, \tilde{y}, p, q)$ given by (3.18) is the solution of the master equation (1.19) iff the function u fulfills the KP equation (3.17) and v is defined by u according to (3.15) and (3.16). □

The above results suggest that other integrable equations can be encoded in the master equation. We consider this question in the next section.

4. From the master equation to the Lax equation

Let $\hat{L}$ be a pseudo-differential operator constructed by $\partial := \partial_q$ and by the functions $u_i(q, \tilde{y})$, $i = 1, 2, \ldots$. Then, let $\hat{B}_+$ be a differential operator defined by ∂ and u_i, $i = 1, 2, \ldots$.

The *Lax equation* reads

$$\partial_{\tilde{y}} \hat{L} = \left[\hat{B}_+, \hat{L} \right] . \tag{4.1}$$

By means of the Weyl correspondence (see (3.6)) one gets

$$\begin{aligned} L &= L(\hbar; p, q, \tilde{y}) := W^{-1}\left(\hat{L} \right) \\ &= \int_{-\infty}^{+\infty} \left\langle q - \frac{\xi}{2} \right| \hat{L} \left| q + \frac{\xi}{2} \right\rangle \exp\left(\frac{ip\xi}{\hbar} \right) d\xi \, , \end{aligned} \tag{4.2}$$

$$\begin{aligned} B_+ &= B_+(\hbar; p, q, \tilde{y}) := W^{-1}\left(\hat{B}_+ \right) \\ &= \int_{-\infty}^{+\infty} \left\langle q - \frac{\xi}{2} \right| \hat{B}_+ \left| q + \frac{\xi}{2} \right\rangle \exp\left(\frac{ip\xi}{\hbar} \right) d\xi \, . \end{aligned} \tag{4.3}$$

Define

$$l := -i\hbar L \qquad \text{and} \qquad b_+ := -i\hbar B_+ \, . \tag{4.4}$$

In terms of l and b_+ the equation (4.1) reads

$$-\partial_{\tilde{y}} l + \{l, b_+\}_{\mathrm{M}} = 0 \, . \tag{4.5}$$

Finally, defining

$$\Theta = \Theta(\hbar; x, y, \tilde{y}, p, q) := -yl + xb_+ = i\hbar(yL - xB_+) \tag{4.6}$$

one concludes that the master equation (1.19) with Θ defined by (4.6) is equivalent to the Lax equation (4.1).

Examples

(i) The KdV equation

If we put in Eqs. (3.15) and (3.16)

$$v = 0 \tag{4.7}$$

then we get

$$\partial_{\tilde{x}} u = 0 \quad i.e., \qquad u = u(q, \tilde{y}) \tag{4.8}$$

and

$$4\partial_{\tilde{y}} u - \partial_q^3 u - 12u\partial_q u = 0\,. \tag{4.9}$$

This last equation appears to be the rescaled KdV equation. Indeed, substituting into (4.9)

$$Q := 2^{\frac{1}{5}} q, \qquad \tilde{Y} := 2^{\frac{3}{5}} \tilde{y} \qquad \text{and} \qquad U := 2^{\frac{3}{5}} u \tag{4.10}$$

one gets the KdV equation [5]

$$4\partial_{\tilde{Y}} U - \partial_Q^3 U - 6U\partial_Q U = 0\,. \tag{4.11}$$

Consequently, inserting $v = 0$ into (3.18) and assuming that $u = u(q, \tilde{y})$ we find that the function Θ

$$\Theta = \Theta(\hbar; x, y, \tilde{y}, p, q) = -\hbar^{-2} x p^3 - \hbar^{-1} i y p^2 + 3xpu + \hbar 2iyu, \tag{4.12}$$

fulfills the master equation (1.19) iff u satisfies the rescaled KdV equation (4.9).

(ii) The Toda lattice equation and the Boyer–Finley–Plebański equation

Consider the operators $\hat{L}$ and $\hat{B}_+$ to be [29]

$$\begin{aligned} \hat{L} &= \mathrm{e}^{\partial} - u_0 - u_1 \mathrm{e}^{-\partial}, \\ \hat{B}_+ &= \mathrm{e}^{\partial} - u_0\,, \\ u_0 &= u_0(q, \tilde{y}), \qquad u_1 = u_1(q, \tilde{y}), \qquad \partial := \partial_q\,. \end{aligned} \tag{4.13}$$

Then the Lax equation (4.1) gives

$$\begin{aligned} \partial_{\tilde{y}} u_0(q, \tilde{y}) &= u_1(q+1, \tilde{y}) - u_1(q, \tilde{y})\,, \\ \partial_{\tilde{y}} u_1(q, \tilde{y}) &= -u_1(q, \tilde{y})\{u_0(q, \tilde{y}) - u_0(q-1, \tilde{y})\}\,. \end{aligned} \tag{4.14}$$

Eliminating u_0 in (4.14) and substituting $u := ln\, u_1$ one gets the Toda lattice equation

$$\partial^2_{\tilde{y}} u(q,\tilde{y}) = -\mathrm{e}^{u(q+1,\tilde{y})} + 2e^{u(q,\tilde{y})} - \mathrm{e}^{u(q-1,\tilde{y})}\,. \tag{4.15}$$

From (4.2) with (4.13) we obtain

$$\begin{aligned} L &= \mathrm{e}^{-\frac{1}{i\hbar}p} - u_0(q,\tilde{y}) - \mathrm{e}^{\frac{1}{i\hbar}p}\, u_1\left(q+\frac{1}{2},\tilde{y}\right), \\ B_+ &= \mathrm{e}^{-\frac{1}{i\hbar}p} - u_0(q,\tilde{y})\,. \end{aligned} \tag{4.16}$$

Finally, by (4.6) and (4.16) we find Θ to be

$$\Theta = \Theta(\hbar;x,y,\tilde{y},p,q) = i\hbar\left\{(y-x)(\mathrm{e}^{-\frac{1}{i\hbar}p} - u_0(q,\tilde{y})) - y\mathrm{e}^{\frac{1}{i\hbar}p}u_1\left(q+\frac{1}{2},\tilde{y}\right)\right\}. \tag{4.17}$$

Now one can deform Eq. (4.15) as follows. First, we define (compare with [29])

$$\begin{aligned} L &= \frac{1}{i\hbar}\{\mathrm{e}^{-p} - u_0(\hbar;q,\tilde{y}) - \mathrm{e}^{p}\, u_1(\hbar;q,\tilde{y})\}, \\ B_+ &= \frac{1}{i\hbar}\{\mathrm{e}^{-p} - u_0(\hbar;q,\tilde{y}\}\,. \end{aligned} \tag{4.18}$$

Then according to (4.4) one finds

$$\begin{aligned} l &= -i\hbar L = -\mathrm{e}^{-p} + u_0(\hbar;q,\tilde{y}) + \mathrm{e}^{p}\, u_1(\hbar;q,\tilde{y})\,, \\ b_+ &= -i\hbar B_+ = -\mathrm{e}^{-p} + u_0(\hbar;q,\tilde{y}\}\,. \end{aligned} \tag{4.19}$$

Consequently, the Lax equation (4.5) gives

$$\begin{aligned} \partial_{\tilde{y}} u_0(\hbar;q,\tilde{y}) &= \frac{1}{i\hbar}\left\{u_1\left(\hbar;q+\frac{i\hbar}{2},\tilde{y}\right) - u_1\left(\hbar;q-\frac{i\hbar}{2},\tilde{y}\right)\right\}, \\ \partial_{\tilde{y}} u_1(\hbar;q,\tilde{y}) &= -\frac{1}{i\hbar}u_1(\hbar;q,\tilde{y})\left\{u_0\left(\hbar;q+\frac{i\hbar}{2},\tilde{y}\right) - u_0\left(\hbar;q-\frac{i\hbar}{2},\tilde{y}\right)\right\}. \end{aligned} \tag{4.20}$$

Finally, eliminating u_0 in (4.20) and substituting $u = \ln u_1$ we get the deformation of the Toda lattice equation

$$\partial_{\tilde{y}} u(\hbar;q,\tilde{y}) = \frac{1}{(i\hbar)^2}\left\{\mathrm{e}^{-u(\hbar;q+\frac{i\hbar}{2},\tilde{y})} + 2\mathrm{e}^{u(\hbar;q,\tilde{y})} - \mathrm{e}^{u(\hbar;q-\frac{i\hbar}{2},\tilde{y})}\right\}. \tag{4.21}$$

For $i\hbar = 1$ one gets the Toda lattice equation (4.15), for $\hbar \to 0$ we obtain the reduced Boyer–Finley–Plebański equation [34–35]

$$\partial^2_{\tilde{y}}\Phi(q,\tilde{y}) + \partial^2_q\,\mathrm{e}^{\Phi(q,\tilde{y})} = 0\,, \tag{4.22}$$

where

$$\Phi(q,\tilde{y}) = \lim_{\hbar\to 0} u(\hbar; q, \tilde{y})\,.$$

From (4.6) and (4.19) it follows that Θ leading to (4.21) reads

$$\Theta = \Theta(\hbar; x, y, \tilde{y}, p, q) = (y-x)\{\mathrm{e}^{-p} - u_0(\hbar; q, \tilde{y})\} - y\,\mathrm{e}^{p}\,u_1(\hbar; q, \tilde{y})\,. \quad (4.23)$$

As $\lim_{\hbar\to 0}\{\cdot,\cdot\}_{\mathrm{M}} = \{\cdot,\cdot\}_{\mathrm{P}}$ one concludes that the function

$$\Theta_0 = \Theta_o(x, y, \tilde{y}, p, q) = (y-x)\{\mathrm{e}^{-p} - U_0(q, \tilde{y})\} - y\,\mathrm{e}^{p}\,\mathrm{e}^{\Phi(q,\tilde{y})} \qquad (4.24)$$

satisfies the equation

$$\partial_y\partial_{\tilde{y}}\Theta_0 + \{\partial_x\Theta_0, \partial_y\Theta_0\}_{\mathrm{P}} = 0 \qquad (4.25)$$

iff

$$\begin{aligned} \partial_{\tilde{y}} U_0 &= \partial_q\,\mathrm{e}^{\Phi}\,, \\ \partial_{\tilde{y}}\Phi &= -\partial_q U_0 \end{aligned} \qquad (4.26)$$

(see Eqs (4.20) with $\hbar \to 0$). Eliminating U_0 we get the reduced Boyer–Finley–Plebański equation (4.22).

5. Conclusions

The main purpose of our paper has been to show that the master equation (1.19) is in a sense a universal integrable system. We have proved (Theorem 2.1) that any solution of the master equation on $V \times T^2$, $V \subset R^4$, defines by the Lie algebra representation a solution to the su(N) SDYM equations on V.

Conversely, every analytic solution to the su(N) SDYM equation can be obtained in this manner.

The fundamental problem which should be considered is to generalize Theorem 2.1 on the cases when one deals with the solutions of the master equation on $V \times \Sigma$, where $V \subset R^4$ and Σ is an arbitrary two dimensional surface.

Such a generalization applied to the results of the Sections 3 and 4 would enable us to encode the KP equation and other integrable equations in the SDYM equations (compare with Ward's hypothesis [5]). We intend to consider this question soon.

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