ACOUSTIC PHASE LENSES IN SUPERFLUID HELIUM AS MODELS OF COMPOSITE SPACE-TIMES IN GENERAL RELATIVITY: CLASSICAL AND QUANTUM FEATURES

Konstantin G. Zloshchastiev

Department of Theoretical Physics, Dnepropetrovsk State University
Nauchniy lane 13, Dnepropetrovsk 320625, Ukraine
e-mail: zlosh@usa.net

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In the spirit of the well-known analogy between inviscid fluids and pseudo-Riemannian manifolds we study spherical singular hypersurfaces in the static superfluid. Such hypersurfaces turn out to be the interfaces dividing superfluid into the pairs of spherical domains, examples of which are "superfluid A–superfluid B" or "impurity–superfluid" phases. It is shown that these shells form the acoustic lenses which are the sonic counterparts of the usual optical lenses. The exact equations of motion for the lens interfaces are obtained. Also some quantum aspects of the theory are considered. We calculate energy spectra for bound states of acoustic lenses in dynamical equilibrium taking into account the analogy to a material shell model of a black hole (we consider the cases of spatial topology of a black hole and a wormhole type).

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It was shown in numerous works that the superfluid phases of ^{3}He (and perhaps ^{4}He) can simulate phenomena encountered in gravitation and the standard model of elementary particles. Physics of superfluid ^{3}He illustrates concepts in quantum field theory and gravity such as: black holes, surface gravity, Hawking radiation, horizons, ergoregions, trapped surfaces [1–3] (see Ref. [4] for an introduction into recent developments), baryogenesis, vortices, strings, textures, standard electroweak model (see [5, 6, 11], and references therein), and so on. This turns out to be possible due to the certain analogy between inviscid fluids and pseudo-Riemannian manifolds. The simplest way to show this correspondence is as follows.

The fundamental equations of dynamics of an inviscid fluid are the Euler equations

$$\rho\left[\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right]=-\nabla p-\rho\nabla\Phi\,, \tag{1}$$

and the equation of continuity

$$\frac{\partial\vec{v}}{\partial t}+\nabla\cdot(\rho\vec{v})=0\,, \tag{2}$$

where Φ is the potential of an external force (including gravity), $\vec{v}$ is the flow velocity, ρ and p are, respectively, the fluid density and pressure. If one assumes the flow to be locally irrotational then we can introduce the velocity potential ψ, $\vec{v}=-\nabla\psi$. Hence, assuming the barotropic equation of state $\rho(p)$, the Euler equation can be rewritten in the form of the Bernoulli equation [4]

$$-\frac{\partial\psi}{\partial t}+\frac{1}{2}(\nabla\psi)^2+\int\limits_0^p\frac{\mathrm{d}p'}{\rho(p')}+\Phi=0\,. \tag{3}$$

We can linearize these equations around some background $\{\rho_0,p_0,\psi_0\}$ in order to consider the propagation of small fluctuations (sound waves). We assume, $\rho=\rho_0+\varepsilon\rho_1+o(\varepsilon)$, $p=p_0+\varepsilon p_1+o(\varepsilon)$, $\psi=\psi_0+\varepsilon\psi_1+o(\varepsilon)$, with the external potential fixed. Linearizing the Euler equation and taking into account the linearized continuity equation, we finally obtain the wave equation describing the propagation of the fluctuation ψ_1

$$\begin{aligned}&-\frac{\partial}{\partial t}\left[\rho_0\frac{\partial\rho}{\partial p}\left(\frac{\partial\psi_1}{\partial t}+\vec{v}_0\cdot\nabla\psi_1\right)\right]\\&+\nabla\cdot\left[\rho_0\nabla\psi_1-\rho_0\vec{v}_0\frac{\partial\rho}{\partial p}\left(\frac{\partial\psi_1}{\partial t}+\vec{v}_0\cdot\nabla\psi_1\right)\right]=0\,.\end{aligned} \tag{4}$$

This equation can be rewritten as the d'Alembert equation in the curved background space-time

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\;g^{\mu\nu}\frac{\partial\psi_1}{\partial x^\nu}\right)=0\,, \tag{5}$$

where $\mu=\{0,i\}$, $x^\mu=\{t,\vec{x}\}$, $g=\det\left(g_{\mu\nu}\right)$,and the acoustic background metric is:

$$\mathrm{d}s^2=\frac{\rho_0}{c}\left[-c^2\mathrm{d}t^2+\delta_{ij}(\mathrm{d}x^i-v_0^i\mathrm{d}t)(\mathrm{d}x^j-v_0^j\mathrm{d}t)\right]\,, \tag{6}$$

where $c=\sqrt{\partial p/\partial\rho}$ is the local speed of sound. Thus, the vorticity-free flow of a zero viscosity fluid can be seen to define a Lorentzian signature metric on a curved space-time. Of course, the physical space-time is just the usual flat Minkowski space-time.

The aim of this paper is to study infinitely thin shells in such acoustic space-times as models of physical objects whose thickness is negligible in comparison with a circumference radius (*e.g.*, surfaces of phase domains). A thin shell is thought to be a discontinuity of the second kind (the density has the delta-like singularity on the shell). Its dynamics is determined by the Lichnerowicz–Darmois–Israel junction conditions: the first quadratic form (the metric) is continuous while the second quadratic form (the extrinsic curvature) has a finite jump across a shell. Geometrically a shell is described by a three-dimensional closed singular hypersurface embedded in the four-dimensional space-time which is this way divided into two domains: the external (Σ^+) and the internal (Σ^-) regions of space-time. Since the classic works [7, 8] the theory of singular hypersurfaces has been widely considered in the literature (see Ref. [9] for details). We describe only some basic properties of timelike hypersurfaces corresponding to dynamical evolution of thin shells now. One considers a singular matter layer Σ described by the three-dimensional space-time with the surface stress-energy tensor of a perfect fluid in the general case

$$S_{ab} = \sigma u_a u_b + \frac{p}{c_\Sigma^2}(u_a u_b + \ {}^{(3)}g_{ab}) \,, \tag{7}$$

where σ and p are, respectively, the surface mass-energy density and pressure, u^a is the timelike unit tangent vector, ${}^{(3)}g_{ab}$ is the 3-metric of shell's hypersurface (in the acoustic sense (6)), c_Σ is the speed of sound in the shell. We assume the metrics of the fluid space-times outside (Σ^+) and inside (Σ^-) a spherical shell to be flat:

$$\mathrm{d}s_\pm^2 = -c_\pm^2 \mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \,, \tag{8}$$

where $\mathrm{d}\Omega^2$ is the metric of the unit 2-sphere, $c_\pm$ are the constants of the speed of sound in the space-times $\Sigma^\pm$. These metrics correspond to a spherical shell dividing different phase domains inside the motionless homogeneous superfluid (6). It is possible to show that if one uses the shell's proper time τ then the 3-metric on the shell's space-time history is

$$^{(3)}ds^2 = -c_\Sigma^2 d\tau^2 + R^2 d\Omega^2 \,, \tag{9}$$

where $R(\tau)$ is the shell's radius. It can be seen from Eqs. (8), (9) that we obtain the composite space-time consisting of three regions, Σ_+, Σ_-, and Σ, characterized by proper fundamental constants. The space-time domains inside and outside a shell (8) are flat and are characterized only by the fundamental constants of the speed of sound, whereas the three-dimensional hypersurface of shell's history can be curved and in addition to the constant c_Σ, the "gravitational" constant γ_Σ may appear as well. The energy conservation law for a shell (which is the shell's interpretation of the integrability condition $S^a_{b;a} = 0$) can be written as

$$c_\Sigma^2\, d\left(\sigma\ {}^{(3)}g\right) + pd\left({}^{(3)}g\right) = 0\,, \tag{10}$$

where ${}^{(3)}g = \sqrt{-\det\left({}^{(3)}g_{ab}\right)} = c_\Sigma R^2 \sin\theta$. In this equation, the first term corresponds to a change in the shell's internal energy, the second term corresponds to the work done by the shell's internal forces.

It is important to note that the analogy between inviscid fluids and pseudo-Riemannian manifolds appears to be justified on the kinematical level only. Thus, the Einstein equations as such have no evident physical sense within the framework of inviscid fluid dynamics. Only the (acoustic) metric, the manifold topology, and equations of motion (which are the consequence of the Bianchi identity) have direct physical interpretation. However, the above-mentioned junction conditions, strictly speaking, are connected rigidly neither with general relativity nor with the Einstein equations, despite the fact that historically they were first derived in the context of general relativity. They simply represent the procedure of geometrical matching of two Riemannian manifolds across a surface of discontinuity of the second kind, and thus can be assumed independently as equations describing behaviour of an interface between two liquid phases. In this connection the words of the famous mathematician Kolmogorov that "the whole mathematics (and, therefore, physics, too) can be reformulated as geometry" become quite relevant. By imposing the junction conditions

$$(K^a_b)^+ - (K^a_b)^- = 4\pi\sigma(2u^a u_b + {}^{(3)}\delta^a_b)\,,$$

where $(K_{ab})^\pm$ are the extrinsic curvatures of the spherically symmetric singular hypersurface [10] with respect to the external and internal acoustic manifolds $\Sigma^\pm$, we obtain the equation of motion for a shell in the form

$$\varepsilon_+\sqrt{1+(\dot{R}/c_+)^2} - \varepsilon_-\sqrt{1+(\dot{R}/c_-)^2} = -4\pi\zeta\sigma R\,, \tag{11}$$

where $\dot{R} = \mathrm{d}R/\mathrm{d}\tau$ is the velocity of a shell, $\varepsilon_\pm = \operatorname{sign}\left[\sqrt{1+(\dot{R}/c_\pm)^2}\right]$ (see below), $\zeta = \gamma_\Sigma/c_\Sigma^2$ is the fundamental constant characterizing shell's space-time Σ [11] with the dimensionality $[\zeta] = \mathrm{cm\,g^{-1}}$.

From Eq. (11) one can see that we obtain a simple but nontrivial object. Such a shell is characterized by the proper velocity, the tension and the mass-energy density (therefore, an equation of state). One can consider the 'matter' on the shell to be highly exotic or ordinary. Once the function $\sigma(R)$ is known then by means of the conservation law (10) we can obtain the equation of state $p = p(\sigma)$.

It should also be noted that a sound, passing through the shell-interface (8) will be refracted, as it happens for light rays in a usual (spherical) lens. Other analogous phenomena, *e.g.*, the spectral factorization or focusing of sound, can appear as well.

The equation of motion (11) together with the equation of state (or, equivalently, with the known function $\sigma(R)$) and the choice of signs $\varepsilon_\pm$ completely determines the motion of superfluid shells (interfaces of acoustic lenses). Therefore, first of all we must specify $\varepsilon_\pm$ and $\sigma(R)$.

Let us say few words about the topological features of the theory. In general relativity it is well-known [12, 13] that $\varepsilon = +1$ if R increases in the outward normal direction to a shell, and $\varepsilon = -1$ if R decreases. Thus, under the condition $\varepsilon_+ = \varepsilon_-$ we have the ordinary (black hole type) shell, and when $\varepsilon_+ = -\varepsilon_-$ we have the traversable wormhole type shell [14]. The appropriate cases are represented in the Table I (we assume the surface density σ to be positive), where the shells (*i.e.*, surfaces of the second kind) corresponding to ordinary lenses are the sonic analogs of the black hole type shells, and shells corresponding to anomalous lenses are counterparts of the wormhole type shells [15]. The superscript "$\dagger$" denotes the case of ordinary lenses when the notions "outside the shell" and "inside the shell" are reversed (for anomalous, wormhole, lenses such notions are absent *ab initio*).

TABLE I

The classification of acoustic lenses into the ordinary (OL) and anomalous (AL) ones, the sign "$\star$" denotes the impossibility of the Lichnerowicz–Darmois–Israel's junction.

$\sigma > 0$	$\varepsilon_+ = \varepsilon_-$		$\varepsilon_+ = -\varepsilon_-$	
	$\varepsilon_+=1$, $\varepsilon_-=1$	$\varepsilon_+=-1$, $\varepsilon_-=-1$	$\varepsilon_+=1$, $\varepsilon_-=-1$	$\varepsilon_+=-1$, $\varepsilon_-=1$
$c_+ > c_-$	OL	$\star$	$\star$	AL
$c_+ = c_-$	$\star$	$\star$	$\star$	AL
$c_+ < c_-$	$\star$	OL†	$\star$	AL

Below we assume the rate of change of the lens size to be small, $\dot{R} \ll c_\pm$. Otherwise, the disturbances which may occur could be incompatible with the assumed flatness of the superfluid space-times (8). Following (11), we obtain the equation of motion for the lenses in the form

$$\frac{\mu \dot{R}^2}{2} = \varXi(R)\,, \tag{12}$$

where

$$\begin{aligned} \varXi(R) &= 4\pi\zeta\sigma R - 2\delta\,, \\ \mu &= c_-^{-2} + (2\delta - 1)c_+^{-2} > 0\,, \\ \delta &= \left\{ \begin{smallmatrix} 0 & \quad (\mathrm{OL}) \\ 1 & \quad (\mathrm{AL}) \end{smallmatrix} \right., \end{aligned} \tag{13}$$

and we call δ the *parameter of lens anomaly.*

We can specify the class of lenses in the dynamical equilibrium (the other lenses will be either growing or decreasing in size and this eventually leads to the vanishing of either of the two phases $\varSigma^{\pm}$). The Taylor expansion of the function $\varXi(R)$ in a small neighborhood of the equilibrium point R_0 yields

$$\varXi(R) = -2\delta + \varepsilon - \frac{k^2}{2}(R - R_0)^2 + o\left((R - R_0)^2\right), \tag{14}$$

where ε and k are the following constants,

$$\varepsilon = 4\pi\zeta\sigma|_{R=R_0} R_0,\ k^2 = -4\pi\zeta(\sigma R)''|_{R=R_0}\,.$$

Eq. (12) can be rewritten as the energy conservation law for the harmonic oscillator. Performing the shift $x = R - R_0$, we obtain

$$E = \frac{P^2}{2m} + \frac{2\delta}{\zeta\mu k} + \frac{m\omega^2 x^2}{2}, \tag{15}$$

where

$$P = m\dot{R} = m\dot{x}, \qquad E = \frac{\varepsilon}{\zeta\mu k}, \qquad m = \frac{1}{\zeta k}, \qquad \omega = \frac{k}{\sqrt{\mu}}\,.$$

It should be noted that $R \in [0, +\infty)$ and $x \in [-R_0, +\infty)$. This circumstance is very important for further studies, first of all for analysis of quantum aspects of the theory.

Below we study the quantum mechanical properties of our lenses. One can perform the standard procedure of quantization. Then the conservation law (15) gives us the stationary Schrödinger equation for the spatial wave function $\varPsi(x)$

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\varPsi}{\mathrm{d}x^2} + \left[-E + \frac{2\delta}{\zeta\mu k} + \frac{m\omega^2}{2}x^2\right]\varPsi = 0, \tag{16}$$

or, in the dimensionless form,

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}y^2} + \left(\varrho - y^2\right)\Psi = 0, \tag{17}$$

where

$$y = \sqrt{\frac{m\omega}{\hbar}}x, \quad \varrho = \frac{2}{\hbar\omega}\left(E - \frac{2\delta}{\zeta\mu k}\right).$$

Although Eq. (17) is the wave equation for the quantum harmonic oscillator the oscillator's wave functions are not defined on the whole line $(-\infty, +\infty)$. We have to consider two cases, one of the whole line $y \in (-\infty, +\infty)$, and the second case of the half-line $y \in [-R_0\sqrt{m\omega/\hbar}, +\infty)$. The analytic continuation of y to the whole line $(-\infty, +\infty)$ can be correctly explained only for the AL type shells because they are acoustic wormholes as it was mentioned above. Such a continuation of the spatial coordinate appears to be a somewhere artificial but necessary technique. Indeed, in the wormhole case we have matched the two non-embedded spacetimes, which both have their own infinitely distant points. Then after the continuation one can explicitly discriminate these spatial infinities from each other by virtue of a sign. Thus, besides the parameter δ, the ordinary and anomalous lenses have different topological properties. Below we distinguish these cases.

(i) Anomalous lenses. In this case physics admits the analytic continuation to $y \in (-\infty, +\infty)$. For bound states the quantum boundary conditions, corresponding to the singular Stourm-Liouville problem, require $\Psi(+\infty) = \Psi(-\infty) = 0$, and the normalized solution of Eq. (17) can be expressed in terms of the Hermite polynomials $H_n(y)$ [17]

$$\Psi(y) = \left(2^n\sqrt{\pi}n!\right)^{-1/2}\exp\left(-y^2/2\right)H_n(y)\,, \tag{18}$$

where $n = 0, 1, 2, \dots$. The discrete values of energy are $\varrho = 2n + 1$. Taking into account Eq. (13) we obtain,

$$E_n = \frac{2}{\zeta k(c_-^{-2} + c_+^{-2})} + \frac{\hbar k}{2\sqrt{c_-^{-2} + c_+^{-2}}}(2n+1)\,, \tag{19}$$

which is the quantized energy of the equilibrium state of the wormhole type lens.

(ii) Ordinary lenses. In this case it is necessary to solve the Schröedinger equation (16) on the half-line

$$R \in [0, +\infty) \;\Rightarrow\; y \in [-R_0\sqrt{m\omega/\hbar}, +\infty)\,.$$

Performing the transformation $z = y^2$ (which works like the Baker transformation [16]), we obtain that $z \in [0, +\infty)$. Then Eq. (17) can be written as the confluent hypergeometric equation

$$z\,\frac{d^2\varphi}{dz^2} + \left(\frac{3}{2} - z\right)\frac{d\varphi}{dz} + \frac{\varrho - 3}{4}\varphi = 0\,, \tag{20}$$

where $\Psi(z) = z^{1/2}\mathrm{e}^{-z/2}\,\varphi(z)$. For bound states the quantum boundary conditions require $\Psi(0) = \Psi(+\infty) = 0$. The confluent hypergeometric functions $\varphi(z)$ compatible with the boundary conditions are the Laguerre polynomials $L_n^{(\alpha)}(z)$, $\alpha > -1$ [15, 17]. Finally we obtain the normalized wave functions

$$\Psi(y) = \sqrt{\frac{n!}{\Gamma(n+1/2)}}\,y\exp\left(-y^2/2\right)L_n^{(1/2)}(y^2)\,, \tag{21}$$

where $\varrho = 4n + 3$ and $n = 0,\, 1,\, 2, \ldots$. The spectrum of energy is given by the expression (after taking into account Eq. (13)),

$$E_n = \frac{\hbar k}{2\sqrt{c_-^{-2} - c_+^{-2}}}(4n+3)\,. \tag{22}$$

It does not depend explicitly on the shell's fundamental constant ζ, as it can be easily seen, but it involves the constant k related to the specific matter on a shell. Comparing expressions (19) and (22) we conclude that

$$E_{2n+1}^{(\mathrm{AL})} - \frac{2}{\zeta k(c_-^{-2} + c_+^{-2})} = E_n^{(\mathrm{OL})}\,. \tag{23}$$

This can be also proved using the relation between the Laguerre and Hermite polynomials.

In the present paper the classical and the quantum aspects of the spherically symmetric thin shells in the motionless homogeneous superfluid helium were studied. We have considered such singular hypersurfaces as the traversable interfaces between the pairs of domains, examples of which are the phases "^{3}He A–^{3}He B", the mixtures "^{4}He–^{3}He", or "the inviscid impurity - He". It was shown that these shells can give rise to the acoustic lenses which have to be sonic models of the composite space-times (*i.e.* the patched space-time regions characterized by different fundamental constants) in general relativity.

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