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CALOGERO MODEL AND sL(2,$\mathbb{R}$) ALGEBRA*

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The Calogero model with external harmonic oscillator potential is discussed from sL(2,$\mathbb{R}$) algebra point of view. Explicit formulae for functions with exponential time behaviour are given; in particular, the integrals of motion are constructed and their involutivness demonstrated. The superintegrability of the model appears to be a simple consequence of the formalism.

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In memory of our friend Stanislaw Malinowski

The Calogero model [1–4], although introduced more than a quater of century ago, still attracts much attention. It has been show to be related to many branches of theoretical physics like the theory of quantum Hall effect [5], fractional statistics [6], two-dimensional gravity [7], two-dimensional QCD [8] and others. Many advanced techniques has been applied in order to shed light on the structure of the model: inverse scattering method [3, 9], r-matrix methods [10], W-algebra techniques [11] *etc.*

Many aspects of Calogero model can be understood by fairly elementary methods. For instance Barucchi and Regge [12] and Wojciechowski [13] have shown that the sL(2,$\mathbb{R}$) algebra plays an important role in the structure of Calogero model without harmonic external potential. In particular, the superintegrability of the model [14] can be easily shown using elementary group theory.

In the present note we show how the Calogero model with harmonic term can be dealt with in a similar way using sL(2,$\mathbb{R}$) algebra. We con-

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struct explicitly functions defined over phase space with a very simple (exponential) time dependence; in particular, the integrals of motion are given and their involutivness is shown by refering to the pure (*i.e.* without harmonic term) Calogero model. It follows immediately from our results that the model retains the property of superintegrability after including the harmonic potential. This latter result is known [3, 15] but here is shown to be a straightforward consequences of sL(2,$\mathbb{R}$) dynamical symmetry.

Let us recall the construction of sL(2,$\mathbb{R}$) algebra for the pure Calogero model [13]. To any function $f(q,p)$ we ascribe the operator $\hat{F}$ acting in the linear space of functions defined over phase space:

$$\{f,g\} \;=\; \hat{F}g \tag{1}$$

for any function g. Obviously

$$\hat{F} \;=\; \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}\right). \tag{2}$$

Let us define the following three functions

$$\begin{aligned} t_+ \;&=\; -\frac{1}{2}\left(\sum_{i=1}^{N} p_i^2 + g^2 \sum_{i,j=1}^{N}{}' \frac{1}{(q_i-q_j)^2}\right) , \\ t_0 \;&=\; \frac{1}{2}\sum_{i=1}^{N} q_i p_i \, , \\ t_- \;&=\; \frac{1}{2}\sum_{i=1}^{N} q_i^2 \, . \end{aligned} \tag{3}$$

They obey the sL(2,$\mathbb{R}$) algebra rules (with respect to the standard Poisson brackets)

$$\begin{aligned} \{t_0,t_\pm\} \;&=\; \pm t_\pm \, , \\ \{t_+,t_-\} \;&=\; 2t_0 \, , \end{aligned} \tag{4}$$

or, respectively

$$\begin{aligned} \left[\hat{T}_0,\hat{T}_\pm\right] \;&=\; \pm\hat{T}_\pm \, , \\ \left[\hat{T}_+,\hat{T}_0\right] \;&=\; 2\,\hat{T}_0 \, , \end{aligned} \tag{5}$$

where, according to Eq. (1)

$$\begin{aligned}
\hat{T}_+ &= \sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} + \frac{g^2}{2} \sum_{i,j=1}^{N} \frac{1}{(q_i - q_j)^3} \left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_i} \right) , \\
\hat{T}_0 &= \frac{1}{2} \sum_{i=1}^{N} \left(p_i \frac{\partial}{\partial p_i} - q_i \frac{\partial}{\partial q_i} \right) , \\
\hat{T}_- &= \sum_{i=1}^{N} q_i \frac{\partial}{\partial p_i} .
\end{aligned} \tag{6}$$

The equations of motion for the Calogero model can be written as

$$\frac{df}{dt} = \{f, -t_+\} = \hat{T}_+ f . \tag{7}$$

Therefore, the integrals of motion are highest-weight vectors; they can be chosen to be (half-) integer eigenvectors of $\hat{T}_0$ thus providing a finite-dimensional irreducible representations of sL(2,ℝ). Moreover, elementary group theory allows us to give an immediate proof of superintegrability of rational Calogero model (shown, in somewhat different way, in [14]). To this end let us note that it is sufficient to find N independent quantities evolving linearly in time (due to the noncompactness of the system no condition for ratios of frequencies [16] are necessary). But this is rather trivial: for if f is an integral of motion then $\hat{T}_- f$ depends linearly on time provided f is an eigenvector of $\hat{T}_0$.

The sL(2,ℝ) algebra can be slightly extended. Let us add two further functions and corresponding operators

$$\begin{aligned}
s_+ &= \sum_{i=1}^{N} p_i \ , \ \hat{S}_+ = -\sum_{i=1}^{N} \frac{\partial}{\partial q_i} , \\
s_- &= \sum_{i=1}^{N} q_i \ , \ \hat{S}_- = \sum_{i=1}^{N} \frac{\partial}{\partial p_i} .
\end{aligned} \tag{8}$$

The elements $t_0, t_\pm, s_\pm$ obey the following algebra

$$\begin{aligned}
\{t_0, s_\pm\} &= \pm\frac{1}{2} s_\pm , \\
\{t_\pm, s_\pm\} &= 0 , \\
\{t_\pm, s_\mp\} &= s_\pm , \\
\{s_-, s_+\} &= N .
\end{aligned} \tag{9}$$

However, for the operators $\hat{S}_{\pm}$ the last formula is to be replaced by

$$[\hat{S}_-, \hat{S}_+] \; = \; 0 \, . \tag{10}$$

The algebra of operators $\hat{T}_0, \hat{T}_{\pm}, \hat{S}_{\pm}$ is therefore a semidirect product of sL(2,$\mathbb{R}$) with two-dimensional abelian algebra spanned by sL(2,$\mathbb{R}$) dublet $\hat{S}_{\pm}$. The Poisson algebra (9) provides a central extension of the latter, the parameter of extension being the number of particles N.

The representations of the algebra under consideration can be easily obtained. We describe the simplest one containing all independent integrals of motion. Let f_{00} be the highest-weight vector such that $\hat{T}_o f_{00} = \frac{N}{2} f_{00}, \hat{S}_+ f_{00} = 0$. One can take, for example, the translation-invariant integral of motion for the Calogero model given in [17, 18]

$$f_{00} \; = \; e^{\frac{-g}{2} \sum\limits_{i,j=1}^{N}{}' \frac{1}{(q_i - q_j)^2} \frac{\partial^2}{\partial p_i \partial p_j}} \prod_{k=1}^{N} p_k \, . \tag{11}$$

Define the vectors

$$f_{mn} \equiv \hat{T}_-^m \hat{S}_-^n f_{00}, \;\; 0 \leq m \leq N-n, \;\; 0 \leq n \leq N-1; \tag{12}$$

they span a subspace carrying an irreducible representation of our algebra. It reads

$$\begin{aligned}
\hat{T}_- f_{mn} \; &= \; f_{m+1\, n} \, , \\
\hat{S}_- f_{mn} \; &= \; f_{m\, n+1} \, , \\
\hat{T}_0 f_{mn} \; &= \; \left(\frac{N}{2} - m - \frac{n}{2} \right) f_{mn} \, , \\
\hat{S}_+ f_{mn} \; &= \; -m f_m - 1\, n + 1 \, , \\
\hat{T}_+ f_{mn} \; &= \; m \, (N - m - n + 1) \, f_{m-1\, n} \, .
\end{aligned} \tag{13}$$

In particular, it follows from the above formulae that $f_{0n}, n = 0, \ldots, N-1$ are translation-invariant integrals of motion for Calogero model. They are obviously linearly independent; however, their functional independence can be checked only by direct inspection.
Also

$$\hat{T}_+ f_{1n} \; = \; (N-n) f_{0n} \tag{14}$$

implies that

$$(N-k) f_{0k} f_{1n} - (N-n) f_{0n} f_{1k} \tag{15}$$

are again integrals of motion. Obviously, only at most $N-1$ of them can be independent and, also by direct inspection, we verify that this is actually the case.

Let us pass to our main theme — the Calogero model in external harmonic oscillator potential. The hamiltonian of the model reads

$$H = \frac{1}{2}\left(\sum_{i=1}^{N} p_i^2 + g \sum_{i,j=1}^{N}{}' \frac{1}{(q_i-q_j)^2} + \omega^2 \sum_{i=1}^{N} q_i^2\right). \tag{16}$$

Under the redefinition $t_0 \to t_0, t_\pm \to \omega^{\mp 1} t_\pm$ the sL(2,ℝ) algebra remains unchanged. Our hamiltonian can be written as

$$H = \omega(\hat{T}_- - \hat{T}_+) = -2i\omega\hat{T}_2 . \tag{17}$$

The $\omega \neq 0$ case differs qualitatively from the $\omega = 0$ one. On the algebraic level this is reflected in the difference in spectral properties of $\hat{T}_+$ and $\hat{T}_2$. Group theory allows us to find easily the functions having simple time behavior. Let $e_m^s, m = -s, \ldots, s$ be a basis of spin s representation of sL(2,ℝ); the normalization convention adopted is such that

$$\begin{aligned} \hat{T}_+ e_m^s &= (s-m)(s+m+1)e_{m+1}^s , \\ \hat{T}_- e_m^s &= e_{m-1}^s . \end{aligned}$$

Then the equations

$$\begin{aligned} (\hat{T}_- - \hat{T}_+)\phi_k^s &= -2ik\phi_k^s, k = -s, \ldots, s , \\ \hat{T}^2\phi_k^s &= s(s+1)\phi_k^s \end{aligned} \tag{18}$$

imply

$$\phi_k^s = \sum_{m=-s}^{s} \frac{(s+m)!}{(2s)!} c_{s+m}(s,k) e_m^s , \tag{19}$$

where the coefficients $c_n(s,k)$ are defined by

$$(x+i)^{s+k}(x-i)^{s-k} = \sum_{n=0}^{2s} c_n(s,k)x^n . \tag{20}$$

Now, it is easy to solve the Hamilton equations for ϕ_k^s:

$$\frac{d\phi_k^s}{dt} = \{\phi_k^s, \omega(t_- - t_+)\} = -\omega(\hat{T}_- - \hat{T}_+)\phi_k^s \tag{21}$$

give

$$\phi_k^s(t) \;=\; e^{2ik\omega t}\phi_k^s(0)\,. \tag{22}$$

Let us take as e_s^s the integral (11), $N=2s$. Writing equation (19) in the form

$$\phi_k^s \;=\; \sum_{n=0}^{2s}\frac{(2s-n)!}{(2s)!}c_{2s-n}(s,k)\left(\omega\sum_{i=1}^{N}q_i\frac{\partial}{\partial p_i}\right)^n e_s^s\,, \tag{23}$$

and using Eqs. (11) and (20) one finds

$$\phi_k^s = \tfrac{1}{(2s)!}e^{-\frac{g}{2}\sum_{i,j=1}^{N}{}'\frac{1}{(q_i-q_j)^2}\frac{\partial^2}{\partial p_i\partial p_j}}F(q,p;\omega)\,, \tag{24a}$$

$$F(q,p;\omega) = \sum_{\substack{\delta\subset\{1,\ldots,2s\}\\ |\delta|=s+k}}\prod_{i\in\delta}(p_i+i\omega q_i)\prod_{i\notin\delta}(p_i-i\omega q_i)\,. \tag{24b}$$

The function $\phi_k^{s'}, s'<s$, can be obtained by taking, for example, $e_{s'}^{s'}=f_{02(s-s')}$. Let us note that $f_{02(s-s')}$ has the following form: one chooses a subsystem consisting of $N'=2s'$ particles and construct the relevant integral (11); $f_{02(s-s')}$ is the sum of such expressions over all choices of subsystems of N' particles. It is readily seen from our construction that $\phi_{k'}^{s'}, s\leq s'$, have the same structure.

In this way we obtained an explicit representation of functions which have a simple time behaviour under the hamiltonian flow generated by the hamiltonian of Calogero model in external harmonic potential.

In order to show complete integrability of Calogero model with harmonic potential it is more convenient to start with e_s^s given by another well-known formula for pure ($\omega=0$) Calogero model integrals of motion

$$e_s^s \;=\; \mathrm{Tr}(L^{2s})\,, \tag{25}$$

where L is the relevant Lax matrix. Taking $k=0, s=1,\ldots,N$ and inserting e_s^s as given by Eq. (25) into Eq. (23) we obtain N integrals of motion.

$$\phi_0^s \;=\; \sum_{m=0}^{s}\frac{(2s-2m)!}{(2s)!}\binom{s}{m}\omega^{2m}\left(\sum_{i=1}^{N}q_i\frac{\partial}{\partial p_i}\right)^{2m}\mathrm{Tr}(L^{2s})\,. \tag{26}$$

In order to prove that they are in involution, let us first note that

$$e_0^s \simeq \left(\sum_{i=1}^N q_i \frac{\partial}{\partial p_i}\right)^s \mathrm{Tr}(L^{2s}), s = 1, \ldots, N \tag{27}$$

are in involution [19]. Moreover, we have the following identity

$$e^{\frac{i\pi}{4}(\hat{T}_+ + \hat{T}_-)} \hat{T}_0 e^{-\frac{i\pi}{4}(\hat{T}_+ + \hat{T}_-)} = \frac{i}{2}(\hat{T}_- - \hat{T}_+)\,. \tag{28}$$

Therefore

$$\phi_0^s \simeq e^{\frac{i\pi}{4}(\hat{T}_+ + \hat{T}_-)} e_0^s\,. \tag{29}$$

However, $\omega(\hat{T}_+ + \hat{T}_-)$ is the operator representing the function

$$-h = -\left(\frac{1}{2}\sum_{i=1}^N p_i^2 + \frac{g}{2}\sum_{i,j=1}^N{}' \frac{1}{(q_i - q_j)^2} - \frac{\omega^2}{2}\sum_{i=1}^N q_i^2\right), \tag{30}$$

which is the hamiltonian for Calogero model in inverse harmonic potential. Therefore, it follows from Eq. (29) that ϕ_0^s is obtained from e_0^s evolving during the time $\frac{i\pi}{4\omega}$ (which is purely imaginary but this is irrelevant in what follows) according to the hamiltonian flow determined by h. The time evolution is a canonical transformation which proves that ϕ_0^s are also in involution.

In order to show that the integrals (26) are independent for $s = 1, \ldots, N$ it is sufficient to note that they contain only even powers of momenta and have the form

$$\phi_0^s = \sum_{i=1}^N p_i^{2s} + \text{terms of lower degree in p}'\text{s}\,. \tag{31}$$

It is also easy to check that for $g = 0$ they reduce to the following ones

$$\phi_0^s(g=0) = \sum_{i=1}^N (p_i^2 + \omega^2 q_i^2)^s, s = 1, \ldots, N\,. \tag{32}$$

The superintegrability of the model can be also shown by our method. This is fairly obvious — we have constructed a huge set of functions depending harmonically on time, the ratios of frequencies being rational numbers.

To conclude we have shown that (super-)integrability of the Calogero model with harmonic external potential can be easily derived from the properties of pure Calogero model by using elementary group theoretical techniques.

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