ON CALOGERO WAVE FUNCTIONS*

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Two properties of Calogero wave functions for rational Calogero models are studied: *(i)* the representation of the wave functions in terms of the exponential of Lassalle operators, *(ii)* the $\mathrm{sL}(2,\mathbb{R})$ structure of the Calogero–Moser wave functions.

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1. Introduction

It is now well known that the Calogero model [1] is superintegrable, both in classical [2] as well as in quantum version [3]. Superintegrability means here the existence of $2N-1$ independent, regular and globally defined integrals of motion which do not depend explicitly on time (N is a number of degrees of freedom).

A superintegrable system has a number of interesting properties. Its hamiltonian depends on the specific combinations of action variables [4] which, on the quantum level, implies considerable energy degeneracy; angle-action variables are not defined uniquely. The latter property has again its quantum counterpart: there is a freedom in the choice of basis diagonalizing the hamiltonian (this freedom is, of course, closely related to the energy degeneracy). Given a specific choice of angle-action variables one can take a basis spanned by common eigenvectors of the quantum counterparts of action variables.

In the case of Calogero model few bases diagonalizing different sets of commuting integrals of motion have been constructed. First, there exists a basis spanned by the so-called Hi-Jack polynomials [5]; the corresponding integrals are most simply expressible in the Dunkl operator [6] formalism. Second example of basis has been given by Brink *et al.* [7]; again the relevant set of commuting integrals can be easily identified.

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In both cases the basic wave functions can be related to known polynomials [8]÷ [11]. To this end one proceeds as follows [9]÷ [13]. Consider the hamiltonian describing the Calogero model

$$H = \frac{1}{2}\sum_i (p_i^2 + \omega^2 x_i^2) + \frac{a(a-1)}{2}\sum_{i\neq j}\frac{1}{(x_i - x_j)^2}\,. \tag{1}$$

The ground-state wave function and energy read, respectively

$$\Psi_0 = \prod_{i<j}(x_i - x_j)^a \exp\left(-\frac{\omega}{2}\sum_i x_i^2\right), \tag{2}$$

$$E_0 = \frac{\omega N}{2}\left((N-1)a + 1\right); \tag{3}$$

here Ψ_0 is given in the sector $x_1 \geq x_2 \geq \ldots \geq x_N$ — the extension to the other sectors depends on statistics. After gauging out the wave function Ψ_0 one obtains

$$\begin{aligned}\tilde{H} &= \Psi_0^{-1}(H - E_0)\Psi_0 = \omega\sum_i x_i\frac{\partial}{\partial x_i} - \frac{1}{2}\left(\sum_i \frac{\partial^2}{\partial x_i^2}\right.\\ &\left. + a\sum_{i\neq j}\frac{1}{x_i - x_j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\right) \equiv \omega\sum_i x_i\frac{\partial}{\partial x_i} - \frac{1}{2}O_L\,,\end{aligned} \tag{4}$$

$\tilde{H}$ can be further reduced due to the identity

$$e^{\frac{1}{4\omega}O_L}\tilde{H}e^{-\frac{1}{4\omega}O_L} = \omega\sum_i x_i\frac{\partial}{\partial x_i}\,. \tag{5}$$

Therefore, by a similarity transformation $\tilde{H}$ reduces to the dilatation operator. This implies that any energy eigenfunction can be written as a product of ground-state wave function and a polynomial

$$\tilde{W}(x) = e^{-\frac{1}{4\omega}O_L}W(x)\,, \tag{6}$$

where $W(x)$ is a symmetric homogeneous polynomial of degree k.

In order to obtain the basis spanned by the Hi-Jack polynomials one takes $W(x)$ to be Jack polynomial [14]. On the other hand, Brink *et al.* basis corresponds to $W(x)$ being monomial symmetric function [13]

Historically, the first complete set of eigenfunctions was given by Calogero [1]. It can be written in the form (neglecting the ground-state factor)

$$\varphi_{nk} = L_n^b(\omega r^2)P_k(x)\,, \tag{7}$$

where L_n^b is a Laguerre polynomial, $P_k(x)$ is translationally invariant symmetric homogeneous polynomial of degree k (Calogero polynomial) obeying

$$O_L P_k(x) = 0\,, \tag{8}$$

and

$$b \;\equiv\; k + \frac{1}{2}(N-3) + \frac{N}{2}(N-1)a\,, \tag{9}$$

$$r^2 \;\equiv\; \frac{1}{2N}\sum_{i\neq j}(x_i - x_j)^2\,. \tag{10}$$

The structure of wave functions, given by Eqs. (7),(8) results from $\mathrm{sL}(2,\mathbb{R})$ dynamical symmetry inherent in the model [15]. The tower of states obtained by the fixing k and varying n spans an irreducible representation of $\mathrm{sL}(2,\mathbb{R})$, Eq. (8) being the condition for lowest-weight vector.

In the present paper we analyse some formal properties of Calogero basis. First, we construct the representation (6) for Calogero wave functions. We show that, due to the relation (8), $W(x)$ are again expressed in terms of Calogero polynomials. Therefore, we cannot in this way gain much insight into their structure. Second, we extend an old result (see, for example, Ref. [15]) concerning the $\mathrm{sL}(2,\mathbb{R})$ structure of Calogero–Moser model, *i.e.* Calogero model without harmonic term. It appears that this structure can be easily described using the representation theory in the basis diagonalizing noncompact generators [17].

2. Laguerre polynomials

Laguerre polynomial L_n^α is defined to be the polynomial solution to the equation [16]

$$z\frac{d^2u}{dz^2} + (\alpha - z + 1)\frac{du}{dz} + nu \;=\; 0 \tag{11}$$

normalized according to the condition

$$u(z) \;=\; (-1)^n\frac{z^n}{n!} + \quad \text{lower degree terms}\,. \tag{12}$$

Let us rewrite (11) in form

$$\left[z\frac{d}{dz} - \left((\alpha+1)\frac{d}{dz} + z\frac{d^2}{dz^2}\right)\right]u \;=\; nu\,. \tag{13}$$

Put

$$A \equiv (\alpha+1)\frac{d}{dz} + z\frac{d^2}{dz^2}\,, \tag{14}$$

and

$$u = e^{-A}v\,. \tag{15}$$

Simple calculation gives equation for v,

$$z\frac{dv}{dz} = nv\,, \tag{16}$$

so that we get the following representation for Laguerre polynomials

$$L_n^{\alpha}(z) = \frac{(-1)^n}{n!}e^{-(\alpha+1)\frac{d}{dz}-z\frac{d^2}{dz^2}}z^n\,. \tag{17}$$

3. The Calogero wave function

In order to find the representation (6) for Calogero wave functions we have to calculate

$$W(x) = e^{\frac{1}{4\omega}O_L}\varphi_{nk}\,. \tag{18}$$

Following Calogero [1] we introduce spherical coordinates in the space of relative coordinates. Since φ_{nk} depends only on relative coordinates one can neglect the center-of-mass coordinate dependence of O_L; then O_L can be rewritten as

$$O_L = r^{2-N}\frac{\partial}{\partial r}\left(r^{N-2}\frac{\partial}{\partial r}\right) + aN(N-1)\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}(\hat{L}+2a\hat{M}); \tag{19}$$

$\hat{L}+2a\hat{M}$ is an angular part and the following equation holds

$$(\hat{L}+2a\hat{M})(r^{-k}P_k(x)) = -k(k+N-3+aN(N-1))r^{-k}P_k(x)\,. \tag{20}$$

Therefore

$$\begin{aligned} W(x) &= \exp\left(\frac{1}{4\omega}\left(r^{2-N}\frac{\partial}{\partial r}\left(r^{N-2}\frac{\partial}{\partial r}\right) + aN(N-1)\frac{1}{r}\frac{\partial}{\partial r}\right)\right.\\ &\quad \left.+\frac{1}{4\omega r^2}\,(\hat{L}+2a\hat{M})\right)\left(L_n^b(\omega r^2)r^k\right)\left(r^{-k}P_k(x)\right)\\ &= r^{-k}P_k(x)\exp\left(\frac{1}{4\omega}\left(r^{2-N}\frac{\partial}{\partial r}(r^{N-2}\frac{\partial}{\partial r}) + aN(N-1)\frac{1}{r}\frac{\partial}{\partial r}\right.\right.\\ &\quad \left.\left.-\frac{k(k+N-3+aN(N-1))}{r^2}\right)\right)\left(L_n^b(\omega r^2)r^k\right)\,. \end{aligned} \tag{21}$$

Put $z=\omega r^2$; the relevant part of the rhs of Eq. (21) reads

$$
\begin{aligned}
& z^{-\frac{k}{2}} \exp\left\{ z\frac{\partial^2}{\partial z^2} + \left(\frac{N-1+aN(N-1)}{2}\right)\frac{\partial}{\partial z} \right.\\
& \left. - \frac{k(k+N-3+aN(N-1))}{4z} \right\} z^{\frac{k}{2}} \\
& = \exp\left\{ z^{-\frac{k}{2}} \left(z\frac{\partial^2}{\partial z^2} + \left(\frac{N-1+aN(N-1)}{2}\right)\frac{\partial}{\partial z} \right.\right.\\
& \left.\left. - \frac{k(k+N-3+aN(N-1))}{4z} \right) z^{\frac{k}{2}} \right\} \\
& = \exp\left\{ z\frac{\partial}{\partial z^2} + \left(\frac{2k+N-1+aN(N-1)}{2}\right)\frac{\partial}{\partial z} \right\} .
\end{aligned} \tag{22}
$$

Eqs.(9),(17),(21) and (22) imply that, up to a normalization constant,

$$
W(x) \;=\; r^{2n} P_k(x)\,, \tag{23}
$$

and

$$
\varphi_{nk} \;=\; e^{-\frac{1}{4\omega}O_L}\left(r^{2n}P_k(x)\right). \tag{24}
$$

Eq. (24) provides the representation for the wave functions in Calogero basis.

Due to the fact that $P_k(x)$ appears also on the right hand side of Eq. (24) we do not gain much insight into the structure of Calogero polynomials. This is due to the fact that the operators classifying the polynomials $P_k(x)$ commute with $\mathrm{sL}(2,\mathbb{R})$ generators while O_L can be expressed in terms of them.

4. The $\mathbf{sL(2,\mathbb{R})}$ structure of Calogero–Moser eigenfunctions

Let us consider the Calogero–Moser model obtained from Eq. (1) by putting $\omega=0$.

$$
H_{CM} \;=\; \sum_i p_i^2/2 + \frac{a(a-1)}{2}\sum_{i\neq j}\frac{1}{(x_i-x_j)^2}\,. \tag{25}
$$

Obviously, H_{CM} has only continuous spectrum (scattering states). The relevant wave function read [1]

$$
\Psi_{p_k} \;=\; \left(\prod_{i<j}(x_i-x_j)\right)^a r^{-b}J_b(pr)P_k(x) \tag{26}
$$

and corresponds to the energies $E=p^2/2$.

The solutions (26) can be classified according to the representations of sL$(2,\mathbb{R})$ algebra.

Define

$$\begin{aligned} J_+ &= \frac{1}{2}\left(\sum_i p_i^2/2 + \frac{a(a-1)}{2}\sum_{i\neq j}\frac{1}{(x_i-x_j)^2}\right) \equiv \frac{1}{2}H_{CM}\,, \\ J_- &= -\sum_i x_i^2\,, \\ J_2 &= \frac{1}{4}\sum_i (x_i p_i + p_i x_i)\,; \end{aligned} \tag{27}$$

then

$$\begin{aligned} [J_2, J_\pm] &= \pm i J_\pm\,, \\ [J_+, J_-] &= 2iJ_2\,. \end{aligned} \tag{28}$$

Let us gauge out the factor $(\prod_{i<j}(x_i - x_j))^a$ and pass to the center-of-mass $(R \equiv \frac{1}{N}\sum_i x_i)$ and relative coordinates [1]. We get

$$\begin{aligned} J_+ &= -\frac{1}{4N}\frac{\partial^2}{\partial R^2} - \frac{1}{4}\left[r^{2-N}\frac{\partial}{\partial r}\left(r^{N-2}\frac{\partial}{\partial r}\right) + \frac{N(N-1)a}{r}\frac{\partial}{\partial r}\right. \\ &\qquad \left. + \frac{1}{r^2}(\hat{L} + 2a\hat{M})\right]\,, \\ J_- &= -(r^2 + NR^2)\,, \\ J_2 &= \left(-\frac{i}{2}R\frac{\partial}{\partial R} - \frac{i}{4}\right) + \left(-\frac{i}{2}r\frac{\partial}{\partial r} - i(\frac{N-1}{4}) - \frac{iaN(N-1)}{4}\right)\,. \end{aligned} \tag{29}$$

Actually, our algebra is rather diagonal part of sL$(2,\mathbb{R})\oplus$sL$(2,\mathbb{R})$. However, the center-of-mass part may be ignored and we are left with sL$(2,\mathbb{R})$ algebra in relative coordinates.

Let us now recall the structure of $D^{(+)}$ represenations of sL$(2,\mathbb{R})$ in the basis diagonalizing J_2 [17]. The action of generators reads

$$\begin{aligned} J_2|\lambda\rangle &= \lambda|\lambda\rangle\,, \\ J_+|\lambda\rangle &= h(\lambda+i)|\lambda+i\rangle\,, \\ J_-|\lambda\rangle &= -|\lambda-i\rangle\,, \end{aligned} \tag{30}$$

where

$$h(\lambda) = \lambda(\lambda-i) - j(j+1)\,. \tag{31}$$

Contrary to the continuous nonexceptional series for discrete representation $D^{(+)}$ the multiplicity of any λ is one [17]. It is not difficult to see that J_2 is dilatation-type operator and the $D^{(+)}$ representation can be related to harmonic analysis on $\mathbb{R}_+$. In fact, consider the Hilbert space of functions on $\mathbb{R}_+$ equipped with a scalar product

$$(f,g) \; = \; \int_0^\infty drr^c \, \overline{f(r)} \, g(r) \, . \tag{32}$$

Define the action of unitary dilatation group as

$$(U(\alpha)f)\,(r) \; = \; \left(e^{\frac{\alpha}{2}}\right)^{c+1} f(e^{\frac{\alpha}{2}} r) \, . \tag{33}$$

Putting

$$U(\alpha) \; = \; e^{i\alpha D} \tag{34}$$

one gets

$$D \; = \; -\frac{i}{2} r \frac{d}{dr} - \frac{(c+1)i}{4} \, . \tag{35}$$

D has purely continuous spectrum covering the whole real axis. The conditions

$$\begin{aligned} Df_\lambda \; &= \; \lambda f_\lambda \, , \\ (f_\lambda, f_{\lambda'}) \; &= \; \delta(\lambda - \lambda') \end{aligned} \tag{36}$$

imply

$$f_\lambda(r) \; = \; \frac{1}{\sqrt{\pi}} r^{2i\lambda - (\frac{c+1}{2})} \, . \tag{37}$$

Modulo typical mathematical subtleties (cf. the theory of Fourier transform) one can write any element Ψ of the Hilbert space as follows

$$\Psi(r) \; = \; \int_{-\infty}^{\infty} d\lambda \tilde{\Psi}(\lambda) f_\lambda \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\lambda \tilde{\Psi}(\lambda) r^{2i\lambda - (\frac{c+1}{2})} \, , \tag{38}$$

where

$$\tilde{\Psi}(\lambda) \; = \; (f_\lambda, \Psi(r)) = \frac{1}{\sqrt{\pi}} \int_0^\infty drr^{\frac{c-1}{2}} r^{-2i\lambda} \Psi(r) \, . \tag{39}$$

Eqs.(38),(39) represent nothing but Mellin transformation.

Now, identifying

$$|\lambda\rangle \equiv f_\lambda, \quad D \equiv J_2\,, \tag{40}$$

one can find the action of $\mathrm{sL}(2,\mathbb{R})$ generators. First, the action of $J_2, J_\pm$ on wave functions $\tilde{\Psi}(\lambda)$ reads

$$\begin{aligned} J_2\tilde{\Psi}(\lambda) &= \lambda\tilde{\Psi}(\lambda)\,, \\ J_-\tilde{\Psi}(\lambda) &= -\tilde{\Psi}(\lambda+i)\,, \\ J_+\tilde{\Psi}(\lambda) &= h(\lambda)\tilde{\Psi}(\lambda-i)\,. \end{aligned} \tag{41}$$

Using Eq. (39) one can rewrite the action of J's in terms of r-dependent wave functions:

$$\begin{aligned} &(J_-\Psi)(r) = -r^2\Psi(r)\,, \\ &(J_+\Psi)(r) = \left(-\frac{1}{4}\frac{d^2}{dr^2} - \frac{c}{4}\frac{1}{r}\frac{d}{dr} - \frac{j(j+1)}{r^2} - \frac{(c+1)(c-3)}{16r^2}\right)\Psi(r)\,. \end{aligned} \tag{42}$$

In our case

$$c = (N-2) + aN(N-1)\,, \tag{43}$$

the first term on the right hand side comes from the Jacobian of transformation to spherical coordinates in the space of relative coordinates while the second one is the result of gauging out the $(\prod\limits_{i<j}(x_i-x_j))^a$ factor.

In order to compare Eqs. (29) and (42) we insert Eq. (20) into (29). The result coincides with Eq. (42) provided

$$j = \frac{k}{2} + \frac{N-5+aN(N-1)}{4}\,. \tag{44}$$

We conclude that, given the Calogero polynomial $P_k(x)$, the wave functions (37) span the $D^{(+)}$ representation of our $\mathrm{sL}(2,\mathbb{R})$ algebra. It is easy to show that diagonalizing the $J_3(=\frac{1}{2}(J_+ - J_-))$ generator we reveal the $\mathrm{sL}(2,\mathbb{R})$ structure of the wave functions of Calogero model [15].

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