

ON THE CALOGERO MODEL WITH NEGATIVE HARMONIC TERM*

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The Calogero model with negative harmonic term is shown to be equivalent to the set of negative harmonic oscillators. Two time-independent canonical transformations relating both models are constructed: one based on the recent results concerning quantum Calogero model and one obtained from dynamical $\mathfrak{sl}(2, \mathbb{R})$ algebra. The two-particle case is discussed in some detail.

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1. Introduction

It has been shown recently [1] that the quantum Calogero model [2] can be transformed by a similarity transformation into the set of noninteracting harmonic oscillators. It is also known [3, 4] that the Calogero–Moser model (*i.e.* the model with purely inverse square interaction) is equivalent to the set of free particles, both on classical and quantum levels.

In the present paper we show that the “inverted” classical Calogero model, *i.e.* the model obtained from the Calogero model by changing the sign of the oscillator term is equivalent to the set of inverted harmonic oscillators. The equivalence is proven by constructing, more or less explicitly, the relevant time-independent canonical transformation.

Two canonical transformations doing the job are presented. First, we construct the classical counterpart of the similarity transformation proposed in Ref. [1]. It is easy to check that for the inverted case this transformation is unitary which, in classical case, corresponds to real canonical transformation. In Section 3 an alternative mapping is presented which is based on $\mathfrak{sl}(2, \mathbb{R})$ dynamical algebra of rational Calogero model. In order to show that they are really different transformation we consider in some detail the

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$N = 2$ case. Both mappings are here constructed explicitly which makes their inequivalence transparent.

2. Equivalence of the models

Let us start with “inverted” Calogero model described by the Hamiltonian

$$H_C = \sum_i \left(\frac{p_i^2}{2} - \frac{\omega^2 q_i^2}{2} \right) + \frac{g}{2} \sum_{i \neq j} \frac{1}{(q_i - q_j)^2}. \quad (1)$$

In order to get rid of the last term on the right hand side we perform three successive canonical transformations.

(i) First we put

$$\begin{aligned} q_i &= q'_i, \\ p_i &= p'_i + \omega q'_i. \end{aligned} \quad (2)$$

This is a canonical transformation and transforms H_C into

$$H_C = \sum_i \frac{p_i'^2}{2} + \frac{g}{2} \sum_{i \neq j} \frac{1}{(q'_i - q'_j)^2} + \omega \sum_i q'_i p'_i. \quad (3)$$

(ii) Next, consider the one-parameter family of canonical transformations of the form

$$\begin{aligned} q'_i(\lambda) &= e^{-\lambda H_{\text{CM}}(q'', p'')} * q''_i \equiv q''_i + \lambda \{q''_i, H_{\text{CM}}(q'', p'')\} + \dots, \\ p'_i(\lambda) &= e^{-\lambda H_{\text{CM}}(q'', p'')} * p''_i \equiv p''_i + \lambda \{p''_i, H_{\text{CM}}(q'', p'')\} + \dots, \end{aligned} \quad (4)$$

where H_{CM} is Calogero–Moser Hamiltonian

$$H_{\text{CM}}(q, p,) = \sum_i \frac{p_i^2}{2} + \frac{g}{2} \sum_{i \neq j} \frac{1}{(q_i - q_j)^2}. \quad (5)$$

Actually, Eq. (4) describes the dynamics of Calogero–Moser model, λ playing the role of time variable. It is easy to check that the choice $\lambda = -\frac{1}{2\omega}$ gives

$$e^{\frac{1}{2\omega} H_{\text{CM}}(q'', p'')} * \left(H_{\text{CM}}(q'', p'') + \omega \sum_i q''_i p''_i \right) = \omega \sum_i q''_i p''_i. \quad (6)$$

Therefore, as a second canonical transformation we take

$$\begin{aligned} q'_i &= e^{\frac{1}{2\omega}H_{CM}(q'',p'')} * q''_i, \\ p'_i &= e^{\frac{1}{2\omega}H_{CM}(q'',p'')} * p''_i \end{aligned} \quad (7)$$

implying

$$H_C = \omega \sum_i q''_i p''_i. \quad (8)$$

(iii) Finally, let

$$\begin{aligned} q''_i &= \frac{1}{\sqrt{2}}(Q_i + \frac{1}{\omega}P_i), \\ p''_i &= \frac{1}{\sqrt{2}}(P_i - \omega Q_i), \end{aligned} \quad (9)$$

so that

$$\omega \sum_i q''_i p''_i = \sum_i \left(\frac{P_i^2}{2} - \frac{\omega^2 Q_i^2}{2} \right). \quad (10)$$

The transformation $(q, p) \rightarrow (Q, P)$ given by the composition of (i)÷(iii) reduces “inverted” Calogero model to the inverted harmonic oscillator.

3. An alternative formulation

It is well known that the rational Calogero model posses dynamical $\mathfrak{sl}(2, \mathbb{R})$ symmetry. In fact, define

$$\begin{aligned} T_+ &\equiv \frac{1}{\omega} \left(\sum_i \frac{p_i^2}{2} + \frac{g}{2} \sum_{i \neq j} \frac{1}{(q_i - q_j)^2} \right), \\ T_- &\equiv \omega \sum_i \frac{q_i^2}{2}, \\ T_0 &\equiv \frac{1}{2} \sum_i q_i p_i, \end{aligned} \quad (11)$$

where ω is a fixed, nonzero but otherwise arbitrary, frequency. The relevant Poisson bracket read

$$\begin{aligned} \{T_0, T_{\pm}\} &= \pm T_{\pm}, \\ \{T_+, T_-\} &= -2T_0. \end{aligned} \quad (12)$$

Define

$$\begin{aligned} T_1 &= \frac{1}{2}(T_+ + T_-), \\ T_2 &= \frac{1}{2}(T_+ - T_-), \\ T_3 &= iT_0; \end{aligned} \tag{13}$$

the commutation rules (12) take the form

$$\{T_i, T_j\} = i\varepsilon_{ijk}T_k. \tag{14}$$

Consider now the canonical transformation

$$\begin{aligned} q_i(\lambda) &= e^{\lambda T_1(q', p')} * q'_i, \\ p_i(\lambda) &= e^{\lambda T_1(q', p')} * p'_i; \end{aligned} \tag{15}$$

then

$$\begin{aligned} T_2(q(\lambda), p(\lambda)) &= e^{\lambda T_1(q', p')} * T_2(q', p'), \\ &= T_2(q', p') \cos \lambda + T_3(q', p') \sin \lambda. \end{aligned} \tag{16}$$

Therefore, putting

$$\begin{aligned} q_i &= e^{\frac{\pi}{2}T_1(q'', p'')} * q''_i, \\ p_i &= e^{\frac{\pi}{2}T_1(q'', p'')} * p''_i, \end{aligned} \tag{17}$$

one obtains

$$T_2(q, p) = T_3(q'', p''). \tag{18}$$

Let us define

$$\tilde{T}_i = T_i(g = 0). \tag{19}$$

Then $T_3 = \tilde{T}_3$ and the transformation

$$\begin{aligned} q''_i &= e^{-\frac{\pi}{2}\tilde{T}_1(Q, P)} * Q_i, \\ p''_i &= e^{-\frac{\pi}{2}\tilde{T}_1(Q, P)} * P_i \end{aligned} \tag{20}$$

converts $\tilde{T}_3 = T_3$ into $\tilde{T}_2$,

$$\tilde{T}_2(P, Q) = \tilde{T}_3(q'', p''). \tag{21}$$

Composing the mappings (17) and (20) one gets finally

$$H_C = \sum_i \left(\frac{P_i^2}{2} - \frac{\omega^2 Q_i^2}{2} \right). \quad (22)$$

The canonical transformation (20), written out explicitly, reads

$$\begin{aligned} q_i'' &= \frac{1}{\sqrt{2}} \left(Q_i - \frac{1}{\omega} P_i \right), \\ p_i'' &= \frac{1}{\sqrt{2}} \left(P_i + \omega Q_i \right), \end{aligned} \quad (23)$$

and coincides with Eq. (9).

4. The $N = 2$ case

In Sections 2 and 3 we defined two canonical transformations reducing the inverted Calogero Hamiltonian to dilatation. However, we still do not know whether these are different transformations or two forms of the same transformation. To show that they are really different it is sufficient to consider the $N = 2$ case. Separating the center of mass and relative coordinates

$$\begin{aligned} X &= \frac{1}{2}(q_1 + q_2), \pi = p_1 + p_2, \\ q &= q_1 - q_2, p = \frac{1}{2}(p_1 - p_2), \end{aligned} \quad (24)$$

and ignoring the former we get

$$H_C = p^2 + g/q^2 - \frac{\omega^2}{4} q^2. \quad (25)$$

The transformations described in Sections 2 and 3 can be given explicitly. In particular, from Eqs. (2) and (7) one gets

$$\begin{aligned} q &= \operatorname{sgn}(q'') \sqrt{\frac{g + (q'' p'' - E/\omega)^2}{E}}, \\ p &= \operatorname{sgn}(q'') \frac{(q'' p'' - E/\omega)}{\sqrt{\frac{g + (q'' p'' - E/\omega)^2}{E}}} + \operatorname{sgn}(q'') \frac{\omega}{2} \sqrt{\frac{g + (q'' p'' - E/\omega)^2}{E}}, \\ E &= p''^2 + g/q''^2. \end{aligned} \quad (26)$$

We easily check that

$$p^2 + g/q^2 - \frac{\omega^2}{4}q^2 = \omega q''p'' \quad (27)$$

as it should be.

On the other hand Eq. (17) takes the form

$$\begin{aligned} q &= \text{sgn}(q'') \sqrt{\frac{2E}{\omega^2} - \frac{2q''p''}{\omega}}, \\ p &= \text{sgn}(q'') \frac{(\omega q''^2/2 - E/\omega)}{\sqrt{\frac{2E}{\omega^2} - \frac{2q''p''}{\omega}}}, \\ E &= p''^2 + g/q''^2 + \frac{\omega^2}{4}q''^2. \end{aligned} \quad (28)$$

Again one easily verifies that the Eq. (27) holds. By comparying Eqs. (26) and (28) we conclude that the transformations described in Sections 2 and 3 are really different.

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