# SYMMETRY-BREAKING IN THE LIPKIN MODEL*

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Generalized version of the standard Lipkin model is presented. Parity-breaking is studied in both standard and generalized Lipkin models. The generalized Lipkin model Hamiltonian is derived from an octupole-octupole Hamiltonian. It is shown that only the generalized Lipkin model gives asymptotic zero energy splitting between the first positive and the first negative parity states for both even and odd numbers of particles as one expects in the case of octupole deformation.

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## 1. Introduction

Simple models, especially if they can be exactly solved, serve as meaningful tools for the testing of various many-body approximations. An outstanding example is the Lipkin model (LM) presented by Lipkin, Meshkov and Glick in 1965 [1]. They considered a system of $N$ fermions occupying a symmetric two-level space and interacting via a monopole-monopole interaction. The two levels each having an $N$-fold degeneracy are separated by an energy $E$. Individual states are labeled by two quantum numbers, $\sigma$, which has the value $+1$ in the upper level and $-1$ in the lower level, and a quantum number $m$ specifying the particular degenerate state within the level.

## 2. Lipkin model Hamiltonian

The model Hamiltonian can be written as [1]

$$\hat{H} = \frac{1}{2}E\sum_{m\sigma}\sigma a^{\dagger}_{m\sigma}a_{m\sigma} + \frac{1}{2}V\sum_{mm'\sigma} a^{\dagger}_{m\sigma}a^{\dagger}_{m'\sigma}a_{m'-\sigma}a_{m-\sigma} + \frac{1}{2}W\sum_{mm'\sigma} a^{\dagger}_{m\sigma}a^{\dagger}_{m'-\sigma}a_{m'\sigma}a_{m-\sigma}\,, \tag{1}$$

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where $a^\dagger_{m\sigma}$ and $a_{m\sigma}$ are respectively creation and annihilation operators acting on a particle in the $m$, $\sigma$ state. $V$ and $W$ stand for the strengths of the interactions. Eq. (2) can be rewritten in terms of quasi-spin operators as [1]

$$\hat{H} = E\hat{K}_0 + \frac{1}{2}V \left(\hat{K}_+\hat{K}_+ + \hat{K}_-\hat{K}_-\right) + \frac{1}{2}W \left(\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+\right) , \qquad (2)$$

where

$$\begin{aligned} \hat{K}_0 &= \frac{1}{2}\sum_{m=1}^{N}\left(a^\dagger_{m+}a_{m+} - a^\dagger_{m-}a_{m-}\right) , \\ \hat{K}_+ &= \sum_{m=1}^{N} a^\dagger_{m+}a_{m-} , \hat{K}_- = \left(\hat{K}_+\right)^\dagger \end{aligned} \qquad (3)$$

are generators of quasi-spin algebra satisfying angular commutation rules:

$$\left[\hat{K}_+, \hat{K}_-\right] = 2\hat{K}_0 , \qquad \left[\hat{K}_0, \hat{K}_\pm\right] = \pm\hat{K}_\pm . \qquad (4)$$

In (2) we neglected exchange terms in the last interaction which cause only an energy shift of the model Hamiltonian. The Hamiltonian (2) commutes with the total quasi-spin operator squared:

$$\hat{K}^2 = \frac{1}{2}\left(\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+\right) + \hat{K}_0^2 . \qquad (5)$$

The interaction term of the Hamiltonian (2) proportional to $V$ scatters a pair of particles in one level into the other, keeping the same quantum number $m$. The term proportional to $W$ scatters one particle up while another is scattered down.

## 3. Standard Lipkin model

The exact solution of the Hamiltonian (2) can be obtained by diagonalization in the quasi-spin basis containing $N + 1$ basis states labeled by quantum numbers $K = N/2$ and $K_0 = (N_+ - N_-)/2$, where $N_+$ and $N_-$ are numbers of particles in the upper and lower level, respectively.

The interaction term proportional to $W$ does not mix configurations and is diagonal in the quasi-spin basis. Since the main purpose of Lipkin *et al.* [1] was to test the treatment of ground-state correlations in the RPA, they set $W = 0$. Although they pointed out that another possible choice for the interaction parameters, namely $V = W$, might be useful for the study of the instability of the HF state against collective oscillations where the RPA breaks down, the Hamiltonian with $W = 0$ became known as the standard LM (SLM) Hamiltonian and is used in testing of various approximations.

Recently, the SLM has been used to test the transition to the parity-breaking solution [2] in the HF treatment (the quantum number $-\sigma$ is interpreted as parity). The parity-breaking HF solution is searched for using trial wave function chosen as

$$|\Phi_0\rangle = \prod_{m=1}^{N} \alpha_{m-}^{\dagger} |\rangle \,, \tag{6}$$

where

$$\begin{pmatrix} \alpha_{m-} \\ \alpha_{m+} \\ \alpha_{m-}^{\dagger} \\ \alpha_{m+}^{\dagger} \end{pmatrix} = \begin{pmatrix} D_{--}^{*} & D_{-+}^{*} & 0 & 0 \\ D_{+-}^{*} & D_{++}^{*} & 0 & 0 \\ 0 & 0 & D_{--} & D_{-+} \\ 0 & 0 & D_{+-} & D_{++} \end{pmatrix} \begin{pmatrix} a_{m-} \\ a_{m+} \\ a_{m-}^{\dagger} \\ a_{m+}^{\dagger} \end{pmatrix} . \tag{7}$$

Since the matrix $\hat{D}$ is unitary, we can choose

$$D_{--} = \cos\alpha \,, \qquad D_{-+} = \sin\alpha \exp\left(-i\phi\right) . \tag{8}$$

Then the mean value of the SLM Hamiltonian in the state $|\Phi\rangle$ can be written as [3]

$$E_{\mathrm{g.s.}} = \langle\Phi_0|\hat{H}(W=0)|\Phi_0\rangle = -\frac{EN}{2}\left[\cos 2\alpha + \frac{\chi}{2}(\sin 2\alpha)^2 \cos 2\phi\right] , \tag{9}$$

where $\chi = -V(N-1)/E$. $E_{\mathrm{g.s.}}$ is minimal for $\phi = 0$ and for

1. $\chi \leq 1$: $\alpha = 0$ (no parity-breaking solution),

2. $\chi > 1$: parity-breaking solution, $\alpha$ can be determined from $\chi \cos 2\alpha = 1$. Since now parity is not a good number for the ground-state wave function, we have to project it onto good parity states (variation before projection – VBP). For comparison of the exact and the VBP results on the energy splitting between the first positive and the first negative parity states (ES), see Fig. 1.

In the degenerate SLM ($E = 0$), the asymptotic behaviour (the interaction strength $-V$ large enough) of the SLM ($E \neq 0$) can be studied. In this case, VBP gives zero ES's (the HF treatment). On the contrary, the exact solution gives zero ES's only for odd numbers $N$ of particles (the numbers of positive and negative parity states are the same for odd $N$). In Fig. 2, minimum exact ES's for even numbers $N$ of particles are depicted. They are non-zero for all $N$ values, but asymptotically approaching zero for high

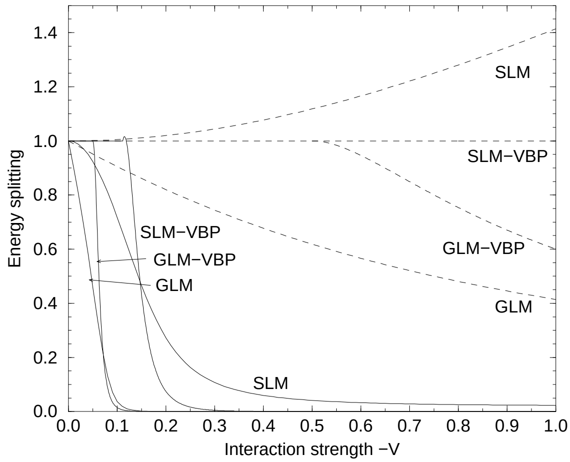


Fig. 1. Energy splitting between the first positive and the first negative parity states as a function of the interaction strength $-V$ for different number of particles $N$ ($N = 2$ – dashed lines, $N = 10$ – solid lines) for exact and VBP solutions of SLM and GLM. Energy splitting and $-V$ in $E = 1$ units.

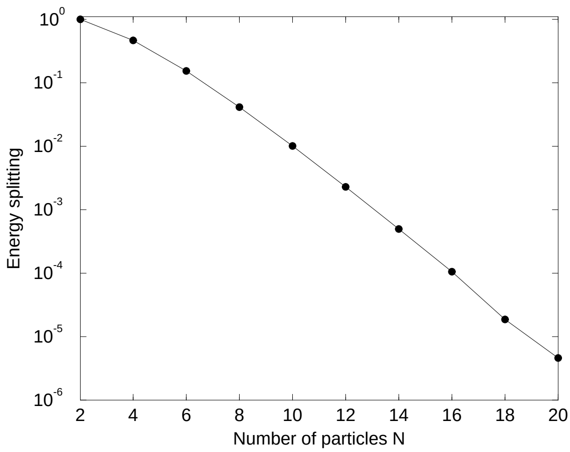


Fig. 2. Minimum exact energy splitting (in $E = 1$ units) between the first positive and the first negative parity states as a function of the number of particles $N$ (even) for the SLM.

$N$ (for fixed interaction strength $V$). The non-zero value of ES results from the different numbers of positive and negative parity states ($1 + N/2$ positive and $N/2$ negative parity states). Since ES's are proportional to the interaction strength $V$, one can always choose $-V$ large enough to get any energy splitting (larger than a minimum positive value for $N$ fixed).

## 4. Generalized Lipkin model

A simple octupole-octupole Hamiltonian can be written as (the same notation as for the LM is used, exchange terms are neglected):

$$\hat{H}_{\mathrm{OO}} = \frac{1}{2}V\left[\sum_m \langle j_+ m|r^3 Y_{30}|j_- m\rangle (a^\dagger_{m+}a_{m-} + a^\dagger_{m-}a_{m+})\right]^2 . \tag{10}$$

$\hat{H}_{\mathrm{OO}}$ simplifies under the assumption that all the $\langle j_+ m|r^3 Y_{30}|j_- m\rangle$ matrix elements are of the same magnitude, *i.e.* we assume $\langle j_+ m|r^3 Y_{30}|j_- m\rangle = 1$. Then

$$\hat{H}_{\mathrm{OO}} \approx \frac{1}{2}V\left(\hat{K}_+ + \hat{K}_-\right)^2 \tag{11}$$

and the generalized LM (GLM) Hamiltonian can be written as

$$\hat{H}_{\mathrm{GLM}} = E\hat{K}_0 + \frac{1}{2}V\left(\hat{K}_+\hat{K}_+ + \hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+ + \hat{K}_-\hat{K}_-\right) . \tag{12}$$

The GLM Hamiltonian corresponds to a special case ($V = W$) of the LM Hamiltonian (2).

The HF treatment of the GLM Hamiltonian using the trial wave functions (6) gives

$$\begin{aligned} E_{\mathrm{g.s.}} &= \langle \Phi_0|\hat{H}(W=V)|\Phi_0\rangle \\ &= -\frac{EN}{2}\left\{\cos 2\alpha + \chi\left[(\sin 2\alpha)^2\cos^2\phi + \frac{1}{N-1}\right]\right\} , \end{aligned} \tag{13}$$

where $\chi = -V(N-1)/E$. $E_{\mathrm{g.s.}}$ is minimal for $\phi = 0$ and for

1. $\chi \leq 1/2$: $\alpha = 0$ (no parity-breaking solution),

2. $\chi > 1/2$: parity-breaking solution, $\alpha$ can be determined from $2\chi\cos 2\alpha = 1$. Similarly as in the case of the SLM, we have to project the parity-breaking solution onto good parity states (variation before projection — VBP). For comparison of the exact and the VBP results on ES's, see again Fig. 1.

In the degenerate GLM ($E = 0$), the asymptotic behaviour (the interaction strength $-V$ large enough) of the GLM ($E \neq 0$) can be studied. The GLM gives zero both exact and VBP ES's for all numbers of particles occupying the model space.

## 5. Conclusion

The GLM Hamiltonian has been derived from a simple octupole-octupole Hamiltonian (octupole-octupole interaction in a symmetric two-level system). Parity-breaking HF solutions have been studied in both SLM and GLM. In the SLM, VBP gives asymptotic (for infinitely strong interaction strength) zero ES's and the exact solution gives nonzero ES's for even numbers of particles. On the other hand, in the GLM, both VBP and exact asymptotic ES's are zero. A model pretending to describe octupole deformation phenomena should give zero ES's for infinitely strong interaction. This is not fulfiled in the SLM for even numbers of particles. In this case the GLM is more convenient than the SLM for the description of octupole phenomena and testing of various many-body approximations leading to octupole-deformed (parity-breaking) solutions.

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