

CRITICAL PROPERTIES OF TOOM CELLULAR AUTOMATA*

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The following paper is the continuation of our earlier considerations on cellular automata with Toom local rule (TCA) as the alternative to kinetic Ising systems. The arguments for TCA stationary states not being the equilibrium states are found in simulations.

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1. Motivation

The Monte Carlo method applied to statistical physics denotes simulation of some stochastic system for which time averages restore equilibrium ensemble expectations [1–3]. Although the dynamical system obtained in this way is often completely artificial to the original equilibrium system, but much freedom in designing dynamics offers possibility to verify different hypothesis about micro scale interactions in the system. In this way, the Monte Carlo simulations allows to examine links between the micro dynamics and resulting equilibrium system. In case of the widely known Lenz model of interacting spins, the Monte Carlo method provides the so-called kinetic Ising models.

Why not search among other dynamical systems, such systems which also can mimic properties of some equilibrium system?

The proposition originated from cellular automata is qualitatively distinct from the mentioned kinetic models. The cellular automata are complex dynamical systems. It means that we are given the set of local rules instead of the notion of energy and changes in the system, means evolution, are synchronised. The steps of synchronised update are interpreted as time steps.

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One can notice two “temperatures” playing in the Domany model. The lower temperature ε_3^D is assigned to the homogeneous with respect to the spin state areas what protects them, while the higher temperature acts “kicking” more frequently the mixed neighbourhoods.

3. Critical properties of TCA

The renormalization procedure applied to equilibrium statistical mechanics systems provides the scaling laws, *i.e.* gives the relations satisfied by the critical exponents $\alpha, \beta, \gamma, \delta, \nu, \eta$ [3]. Each critical exponent characterises the power law dependence of some observable when the system is undergoing a continuous phase transition. Applying the standard methods for estimate singularities [11, 12] we have found the following four of the critical exponents:

β — magnetisation exponent: $m(\varepsilon) \sim (\varepsilon - \varepsilon_{\text{cr}})^\beta$ at $\varepsilon > \varepsilon_{\text{cr}}$

γ — magnetic susceptibility exponent: $\chi(\varepsilon) \sim |\varepsilon - \varepsilon_{\text{cr}}|^{-\gamma}$ at $\varepsilon \rightarrow \varepsilon_{\text{cr}}$

ν — correlation length exponent: $\xi(\varepsilon) \sim |\varepsilon - \varepsilon_{\text{cr}}|^{-\nu}$ at $\varepsilon \rightarrow \varepsilon_{\text{cr}}$

η — correlation decay at the critical point: $C(i, j) \sim |i - j|^{-\eta}$ at $\varepsilon = \varepsilon_{\text{cr}}$

Our results and the methods used to obtain them are presented in Figs. 1–4.

It is easy to check that with our estimations, namely:

- $\varepsilon_{\text{cr}} = 0.822 \pm 0.002$
- $\beta = 0.12 \pm 0.03$
- $\gamma = 1.75 \pm 0.03$
- $\nu = 0.88 \pm 0.02$
- $\eta = 0.56 \pm 0.02$

the scaling laws:

- $\gamma = \nu(2 - \eta)$
- $\beta = \frac{1}{2}\nu\eta$

are not satisfied.

One can notice that our values obtained for β and γ are in the good agreement with those characterising the two-dimensional Ising system while the rest critical parameters ν and η are much different from the corresponding ones in the Ising system.

Hence, *TCA system seems to neither belong to the Ising class of universality nor be an equilibrium system.*

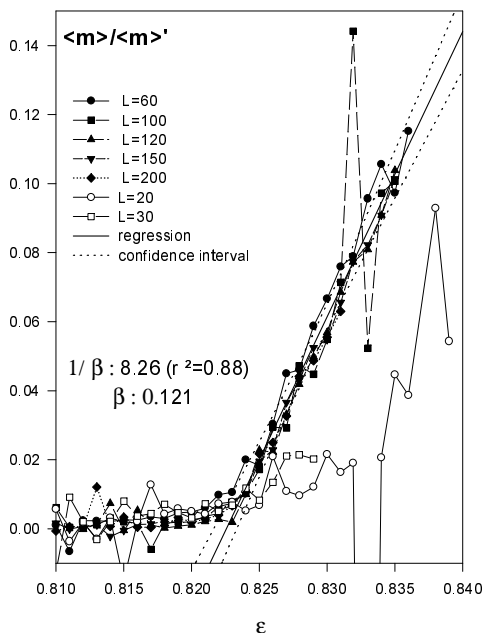


Fig. 1. Estimates for β in Toom CA. Data collected in 430, 280, 120 experiments with lattices: $L = 100, 150, 200$, respectively.

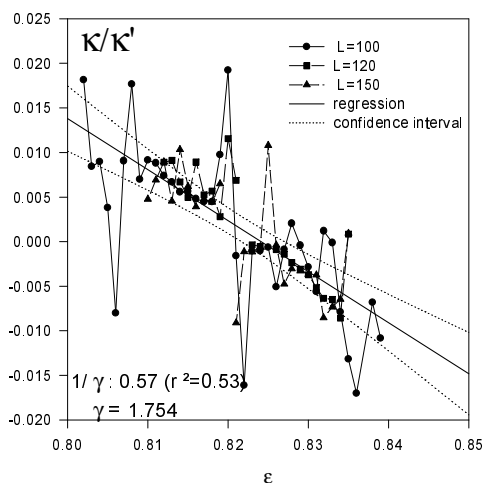


Fig. 2. Estimates for γ in TCA from the relation $\kappa = \langle m^2 \rangle - \langle m \rangle^2$. Data collected in 430, 180, 120 experiments with lattices: $L = 100, 120, 150$, respectively.

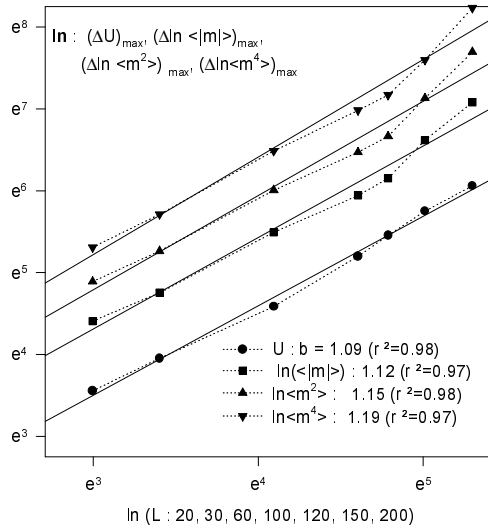


Fig. 3. Estimates for $1/\nu$ in TCA on the base of finite size theory from the maximal slope of derivative of the 4th order magnetisation cumulant as well as from the maximum slope of the logarithm derivatives of: absolute value of magnetisation, square magnetisation, fourth power of magnetisation. Data collected in 1400, 860, 660, 430, 280, 120 experiments with lattices: $L = 20, 30, 60, 100, 150, 200$, respectively.

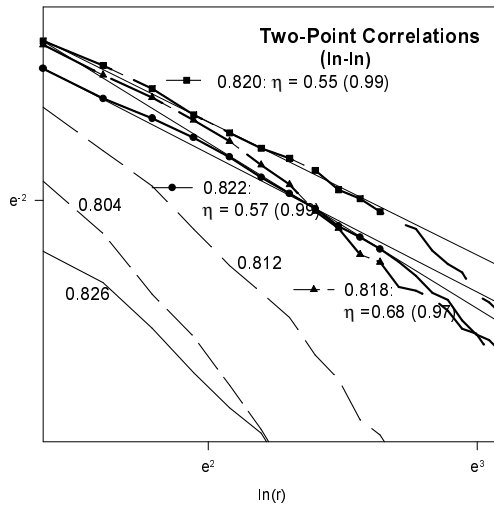


Fig. 4. The decay rate η for the two-point correlation function of magnetisation obtained on the lattice with $L = 100$ at different ε .

4. Features of TCA measures

Statistical mechanics offers tools to investigate if a given stationary state can be represented by some Gibbs measure [7, 13, 14]. For a probability measure μ to be a Gibbs measure denotes that for any finite configuration $\{\sigma_\Lambda\}$, $\ln \mu(\sigma_\Lambda)$ exists and means the energy carried by the configuration $\{\sigma_\Lambda\}$. The two basic features of Gibbs measures are the, so-called, quasilocality of interactions and the proper properties of large deviations.

4.1. Quasilocality

The idea of the quasilocality is shown in Fig. 5. Using the notation used in Fig. 5 one can say that nothing can arrive to some finite area Λ from infinity without changing the σ_Γ configuration. In particular, if a measure μ is quasilocal then for the average magnetisation one have

$$| \langle m(\{\sigma_\Lambda\} |_{\sigma_\Gamma}) \rangle_\mu - \langle m(\{\sigma'_\Lambda\} |_{\sigma_\Gamma}) \rangle_\mu | \longrightarrow_{\Gamma \rightarrow \infty} 0. \quad (3)$$

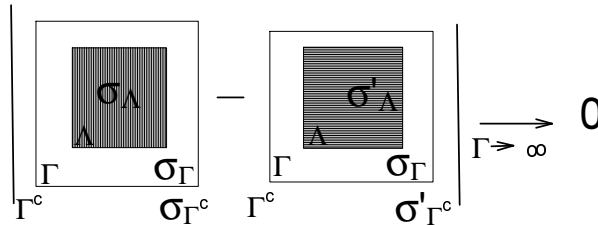


Fig. 5. The illustrative definition of quasilocality of interaction in the lattice system. Λ and Γ are any finite subsets of a lattice $\mathcal{L}$. The idea is that in case of large Γ there is not observable influence on the configuration in Λ coming from any configuration σ_{Γ^c} which is outside to the fixed configuration of σ_Γ .

In the following experiments we test the presence of the above property in TCA.

(i) Let σ_Γ be the fixed configuration evolving according to the Toom dynamics at some ε . Let σ_{Γ^c} — the boundary configuration, be :

a) the homogeneous configuration of all spins +1

$$\begin{array}{cccccc}
 + & + & + & + & + & + \\
 + & . & m_-^\varepsilon & . & . & + \\
 + & . & . & \mathbf{O} & . & + \\
 + & . & m_+^\varepsilon & . & . & + \\
 + & + & + & + & + & +
 \end{array}
 \xrightarrow{\text{TCA}}
 \begin{array}{cccccc}
 + & + & + & + & + & + \\
 + & . & . & . & . & + \\
 + & . & m_+^{\varepsilon+\varepsilon'} & \mathbf{O} & . & + \\
 + & . & . & . & . & + \\
 + & + & + & + & + & +
 \end{array}$$

Then the (+) boundary built from the rectangle of pluses *rises* the probability to find a spin in +1 state at the origin $\Lambda = \mathbf{O}$ and ε' depends on the distance from the right and top sides of the rectangle (see Fig. 6 for details).

Remark: If Γ is not of the rectangle shape then the Toom interactions enlarge Γ to have the right angle between the bottom and left sides.

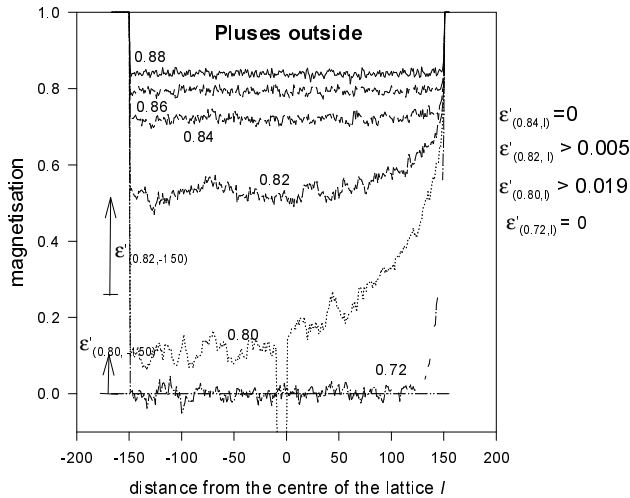


Fig. 6. TCA with (+) boundary added after reaching stabilisation. Distinct curves correspond to different values of stochastic perturbation ε . With ε' we measure the extra magnetisation observed at the origin $\mathbf{O}$ in stationary TCA.

- b) the flat-interface configuration: +1 on the right and -1 on the left of the flat-interface:

$$\begin{array}{cccccc}
 - & - & - & + & + & + \\
 - & . & m_-^\varepsilon & . & . & + \\
 - & . & . & \mathbf{O} & . & + \\
 - & . & m_+^\varepsilon & . & . & + \\
 - & - & - & + & + & +
 \end{array}
 \xrightarrow{\text{TCA}}
 \begin{array}{cccccc}
 - & + & + & + & + & + \\
 - & . & . & . & . & + \\
 - & . & m_+^{\varepsilon+\varepsilon'} & \mathbf{O} & . & + \\
 - & . & . & . & . & + \\
 - & + & + & + & + & +
 \end{array}$$

Then the position of the flat-interface moves to the left side of the rectangle and stays there thanks to the periodic boundary conditions. The dependence ε' on lattice site is the same as in the previous experiment.

- (ii) Let σ_{Γ^c} evolves along TCA rule at some $\varepsilon_{\text{out}} > 0$. Let the initial state for both *in*- and *out*- configurations is (+). Let ε_{out} be chosen such that it generates the stationary state dominated by (+) phase, $\varepsilon_{\text{out}} \gg \varepsilon_{\text{cr}}$. Then we observe:

<i>in</i> -state	<i>out</i> -state	resulting state
$\varepsilon_{\text{in}} \gg \varepsilon_{\text{cr}}$		$m_{\Gamma}(l) = m(\sigma' \text{ typical for } \mu_+^{\varepsilon_{\text{in}}})$, $m_{\Gamma^c}(l) = m(\sigma' \text{ typical for } \mu_+^{\varepsilon_{\text{out}}})$
$\varepsilon_{\text{in}} \approx \varepsilon_{\text{cr}}$		$m_{\Gamma}(l) = m(\sigma' \text{ typical for } \mu_+^{\varepsilon_{\text{cr}} + \varepsilon'})$, $m_{\Gamma^c}(l) = m(\sigma' \text{ typical for } \mu_+^{\varepsilon_{\text{out}}})$
$\varepsilon_{\text{in}} \approx \varepsilon_{\text{cr}} + \varepsilon'$	$\varepsilon_{\text{out}} \gg \varepsilon_{\text{cr}}$	$m_{\Gamma}(l) = m(\sigma' \text{ typical for } \mu_+^{\varepsilon_{\text{cr}}}) = 0$, $m_{\Gamma^c}(l) \neq 0$ RANDOM
$\varepsilon_{\text{in}} \ll \varepsilon_{\text{cr}}$	$\Rightarrow$	$m_{\Gamma}(l) = m_{\Gamma^c}(l) = 0$

These results together with the case when the initial *out*-phase is (−) are presented in Fig. 7. Let us comment our observation as follow:

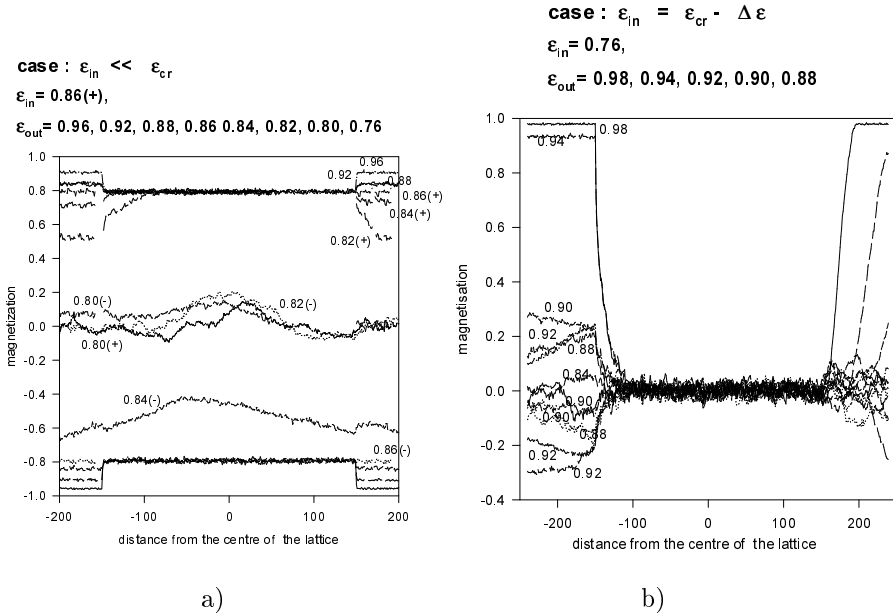


Fig. 7. Interaction between two TCA systems: *in* and *out* distinct from each other because of different stochastic parameters: ε_{in} and ε_{out} , suitable. a) *in* initial state is (+)- phase, $\varepsilon_{\text{in}} = 0.86$, *out* initial states are changed. Labels in the figure correspond to different values of ε_{out} . b) Propagation of self-created clusters of one phase from *in* area into the *out* configuration observed as random changes in magnetisation of *out* state.

- After adjusting the phase of the *in* configuration, the regions of distinct temperatures evolves independently. The two-point correlation function dies in one lattice unit.
- The *out*-phase picks up the (+) phase from the *in*-system when the *in*-system is in the phase transition regime.
- The large homogeneous structures created in the *in*-system undergoing the phase transition, propagates freely to the *out* configuration. Since the phase of these structures is random, then the phase of the *out* configuration changes randomly.
- The random *in*-system propagates random errors outside destroying the *out*-phase magnetisation.

Remark: The case when $\varepsilon_{\text{out}} \geq \varepsilon_{\text{cr}}$ provides always $m_{\Gamma}(l) = m_{\Gamma^c}(l) = 0$.

Concluding:

- If $\varepsilon_{\text{in}} < 0.76$ then the conditional distribution for the magnetisation in Γ is independent of Γ -configuration.
Instead, there is observed dependence in σ_{Γ^c} on random clusters created in σ_{Λ} if ε_{in} is about the limit value and $\varepsilon_{\text{out}} \gg \varepsilon_{\text{cr}}$.
- If $\varepsilon_{\text{in}} > 0.76$ then the conditional distribution for the magnetisation depends on the system outside, in the sense that the probability for the spin at the origin o to take +1 state satisfies the inequality:

$$\left| \text{Prob}\{\sigma_O = +1 \mid \sigma_{\Gamma}(\varepsilon_{\text{in}}) \text{ and } \sigma_{\Gamma^c}(\varepsilon'_{\text{out}})\} - \text{Prob}\{\sigma_O = +1 \mid \sigma_{\Gamma}(\varepsilon_{\text{in}}) \text{ and } \sigma_{\Gamma^c}(\varepsilon''_{\text{out}})\} \right| > 0.$$

So that the conditional probabilities are discontinuous, what in the lattice system topology denotes that the property (3) is not satisfied. The stationary measure might be strongly non-Gibbsian [14]. However, if $\varepsilon_{\text{in}} > 0.84$ then after thermalization time, what means time allowing system for adjusting the phase of the *in*-configuration, both areas evolve independently of each other.

4.2. Large deviations

For stationary measures arisen from any Markov process to be or not to be the Gibbsian ones is determined by the zero or non-zero value of the relative entropy density $i(\mu|\nu)$ between different stationary measures μ and ν of the system considered. If $i(\mu|\nu) > 0$ then both measures are non-Gibbsian [7, 13]. Thanks to the large deviation theorems [13] we have the powerful way to estimate $i(\mu|\nu)$.

In case of μ_- and μ_+ : two stationary TCA measures corresponding to $(-)$ and $(+)$ phases, respectively, $i(\mu_-|\mu_+)$ can be extracted from the probability of the large fluctuation event, namely, from the probability that the large area of spins with negative magnetisation occurs in the stationary state described by the μ_+ measure.

From computer experiments we collect data on the magnetisation of square blocks of the size $l \times l$. Then on the base of the formula

$$i_l(\mu_-|\mu_+) = \lim_{l \rightarrow \infty} \frac{1}{l^2} \ln \text{Prob}_{\mu_+} \{m(\sigma_{l \times l}) < 0\} \quad (4)$$

we try to estimate the limit

$$i(\mu_-|\mu_+) = \lim_{l \rightarrow \infty} i_l(\mu_-|\mu_+).$$

Fig. 8 is to present the density of the relative entropy $i_l(\mu_-|\mu_+)$ for different lattice sizes: $L = 60, 100, 200$. Although averaging time was very large (10 000 time steps), the strong time dependence is noticeable. This dependence moves to the calculations of $i_l(\mu_-|\mu_+)$ causing difficulties in estimating the limit $l \rightarrow \infty$. Anyway, we tried to find out the block size where the relative entropy density arrives at the zero. According to the presented results for the lattice size $L = 200$ this limit should occur if the blocks are of the size $l > 50$. In the next figure, Fig. 9, we present the failure of this suggestion. What we observe in simulations is that with the increase of the block size the relative entropy density decay slows down.

Moreover, we test the density of relative entropy between the stationary measure arising in TCA evolving little apart from the critical point, namely at $\varepsilon = 0.800$ and the $(-)$ minus phase. (see Fig. 10 for these results). It appears that at $l = 105$ the relative entropy density would reach the zero. But this block size is greater than the lattice size.

Concluding, we can say that *the relative entropy density between stationary measures of Toom cellular automata evolving in the critical regime on the periodic lattice is positive.*

5. Conclusions

Although the critical regime in TCA manifests itself in the way characteristic to any thermodynamic system, *i.e.* by the rapid increase in the two-point-correlation function of spin states, but the phenomena driving the system seems to be different from any equilibrium system.

The TCA system undergoing the phase transition stays in the extremely dynamically fragile state. There is kept the sensitive balance between two processes: the processes of self-organising spins state to enlarge pure phase

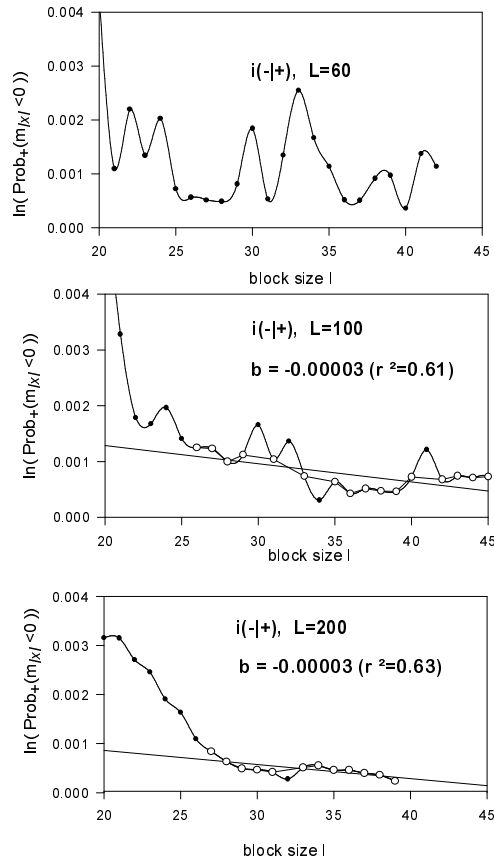


Fig. 8. The relative entropy density between μ_+ and μ_- stationary measures in the critical regime of TCA versus block size for lattices $L = 60, 100, 200$.

clusters and the stochastic process which destroys this clusters. Since the homogeneous clusters are not specially protected as it happens in the Domany CA, the area of cluster of one phase varies. In consequence, the correlations between spins are damped much stronger than in Domany CA and Ising kinetic models. However, it is astonishing that the relations between spin sites do not influence the critical behaviour of the order parameter. Therefore, one can say that the phenomenological conjecture that all two-dimensional ferromagnetic systems belong to the same universality class — the class of Ising system [15], is “partially” satisfied (if the participation into the universality class could be partial).

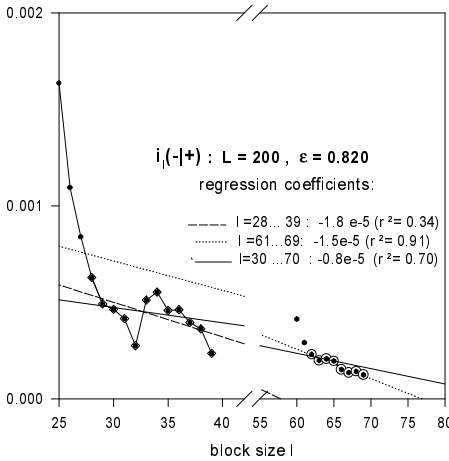


Fig. 9

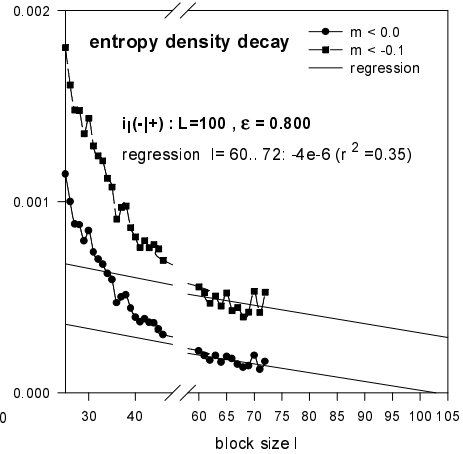


Fig. 10

Fig. 9. The relative entropy density between μ_+ and μ_- stationary measures for large blocks.

Fig. 10. Density of the relative entropy between stationary measures μ_- and ν — the measure for the system moved a little from the critical point.

Testing locality of interactions we have found that at about the critical point there exists a boundary for the interaction to be finite. If the stochastic perturbation is sufficiently strong than the information from distant spins is lost. Otherwise, no matter how far away spins are, their influence on each other is evident.

Our examinations on large deviation properties as well as quasilocality have hit the boundary of the finite lattice size. Therefore one can say that the system we studied does not restore properties of any infinite system. Such a conclusion often accompanies complex system considerations. Complex systems, existing on the, so-called, *edge of order and chaos* [16], are known for that nobody is able to predict what kind of phenomena will arise in a system if one moves a little the system parameters. Studying critical properties of Toom cellular automata we obtain arguments for the suspicion that self-organised criticality coming from the dynamic equilibrium provides systems qualitatively distinct from the thermodynamic systems [17].

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- [17] see B. Boghosian invitation for Statistical Physics of Complex Systems — the satellite meeting to the International Conference on Complex Systems which will be held in Boston on 25–30 October, this year.