CASIMIR ENERGY OF THE NAMBU–GOTO STRING WITH GAUSS–BONNET TERM AND POINT-LIKE MASSES AT THE ENDS*

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We calculate the Casimir energy of the rotating Nambu-Goto string with the Gauss-Bonnet term in the action and point-like masses at the ends. This energy turns out to be negative for every values of the parameters of the model.

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It seems to exist a common belief that the construction of (even approximate) string representation of QCD could be crucial for understanding non-perturbative properties of quantum chromodynamics, such as the nature of the ground state or mechanism of confinement. The conjecture of existence of such a description is supported by a number of facts [1,2], to mention only the nature of the $1/N_c$ expansion [3], success of the dual models in description of Regge phenomenology, area confinement law found in the strong coupling lattice expansion [4] or the existence of flux-line solutions in confining gauge theories [5,6] and the analytical results concerning two-dimensional QCD [7]. The results obtained recently in the framework of M theory are also very promising (see, for instance, [8]).

It is well known that the simplest, Nambu–Goto string model [9], when treated as a quantum system, has many drawbacks [10,11] which include the non-physical dimension of the space-time ($D = 26$) or tachion and unwanted massless states in the spectrum. It is therefore reasonable to study modifications of the Nambu–Goto model, and among them the simplest ones, which preserve the equations of motion for the interior of the string while changing the boundary conditions imposed at the string ends.

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The model we investigate in this letter is defined through the action functional

$$S = -\int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma \, \sqrt{-g} \left(\gamma + \frac{\alpha}{2} R\right) - \sum_{i=1}^{2} m_i \int_{\tau_1}^{\tau_2} d\tau \sqrt{(\partial_\tau X)^2} \,. \tag{1}$$

Here γ, with the dimension $(\text{mass})^2$, is the string tension, α is a dimensionless parameter and $g = \det g_{ab}$ is the determinant of the induced metric tensor $g_{ab} = \partial_a X^\mu \partial_b X_\mu$ $(a, b = \tau, \sigma)$. R, the inner curvature scalar, can be written in the form

$$R = \left(g^{ab} g^{cd} - g^{ad} g^{bc}\right) \nabla_a \nabla_b X_\mu \nabla_c \nabla_c X^\mu \,,$$

where ∇_a is a covariant (with respect to the induced metric g_{ab}) derivative. This model belongs to the family of the string models studied for instance in [12]. Partial analysis of classical solutions of the model specified by (1) was performed in [13].

The inclusion of the Gauss–Bonnet term

$$S_{\text{GB}} = \int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma \, \sqrt{-g} \, R$$

into the action (1) is a rather natural construction in the context of the effective QCD string. The QCD string action should contain — apart from the X^μ fields — also infinitely many fields describing for instance the transverse shape of the chromoelectric flux joining the color sources. In constructing the effective string action, one integrates over such a fields and this procedure inevitably leads to emergence of the intrinsic curvature term in the action functional. Of course, it is then only the first one out of the infinitely many terms with the growing number of derivatives.

The worldsheet parametrization can be completely fixed by imposing the manifestly Lorentz invariant conditions [10]:

$$(\dot{X} \pm X')^2 = 0 \,, \tag{2}$$

$$(\ddot{X} \pm \dot{X}')^2 = -\frac{1}{4} q^2 \,, \tag{3}$$

where the dot and the prime mean differentiation with respect to τ and σ and q is a parameter with the dimension of mass. The appearance of this parameter can be traced back to the assumption, that σ takes values in the fixed interval $[0, \pi]$.

It can be shown (for details see [10,14,15]), that in this parametrization every solution of the string equations of motion and boundary conditions,

following from the action (1), corresponds to the solution of the complex Liouville equation [16]:

$$\ddot{\Phi} - \Phi'' = 2q^2 \mathrm{e}^{\Phi} \,, \tag{4}$$

supplemented with the boundary conditions:

$$\begin{cases} \gamma - \alpha q^2 \mathrm{e}^{2\mathrm{Re}\,\Phi} &=& (-1)^i m_i \frac{\partial}{\partial\sigma} \left(\mathrm{e}^{\mathrm{Re}\,\Phi/2}\right) \,, \\ \alpha \frac{\partial}{\partial\tau} \mathrm{Re}\,\Phi &=& 0 \,, \\ \alpha \cos\left(\mathrm{Im}\,\Phi/2\right) &=& 0 \,, \\ \frac{\partial}{\partial\sigma} \mathrm{Im}\,\Phi &=& 0 \,, \qquad\qquad \text{for} \quad \sigma = 0, \pi \,. \end{cases} \tag{5}$$

The correspondence is explicitly established through the relations:

$$\mathrm{e}^{\Phi} = -\frac{1}{q^2} \frac{F_{\mathrm{L}}'(\tau+\sigma) F_{\mathrm{R}}'(\tau-\sigma)}{\sin^2\left[\frac{F_{\mathrm{L}}(\tau+\sigma) - F_{\mathrm{R}}(\tau-\sigma)}{2}\right]} \,, \tag{6}$$

$$X^{\mu}(\tau,\sigma) = X^{\mu}_{\mathrm{L}}(\tau+\sigma) + X^{\mu}_{\mathrm{R}}(\tau-\sigma) \,, \tag{7}$$

$$\frac{\partial}{\partial\tau} X^{\mu}_{\mathrm{L,R}} = \frac{q}{2|F_{\mathrm{L,R}}'|} \left(\cosh \mathrm{Im}\, F_{\mathrm{L,R}},\ \cos \mathrm{Re}\, F_{\mathrm{L,R}},\ \sin \mathrm{Re}\, F_{\mathrm{L,R}},\ \sinh \mathrm{Im}\, F_{\mathrm{L,R}}\right) \,, \tag{8}$$

where $F_{\mathrm{L,R}}$ are arbitrary complex functions which give single valued Φ satisfying the boundary conditions (5).

A distinguished class of solutions of the Liouville equation (4) is composed of static, *i.e.* τ-independent fields. They are of the form

$$\mathrm{e}^{\Phi_0} = -\frac{\lambda^2}{q^2} \frac{1}{\cos^2\left(\lambda\sigma - d\right)} \,, \tag{9}$$

where λ and d satisfy the set of algebraic equations,

$$\begin{aligned} \frac{\gamma q}{\lambda^2} \cos^4 d - m_1 \sin d \cos^2 d - \frac{\alpha\lambda^2}{q} &= 0 \,, \\ \frac{\gamma q}{\lambda^2} \cos^4(\pi\lambda - d) - m_1 \sin(\pi\lambda - d) \cos^2(\pi\lambda - d) - \frac{\alpha\lambda^2}{q} &= 0 \,, \end{aligned} \tag{10}$$

following from the boundary conditions (5) for the Liouville field of the form (9).

The Liouville field Φ_0 describes a straight string which rotates with a constant angular velocity in some plane and by choosing a convenient reference frame we can write the string coordinates in a form

$$X^{\mu} = \frac{q}{\lambda^2} \Big(\lambda\tau, \cos\lambda\tau\ \sin(\lambda\sigma - d), \sin\lambda\tau\ \sin(\lambda\sigma - d), 0\Big) \,. \tag{11}$$

Let us note, that in the presence of the inner curvature term in the action (1) the velocities of the string ends,

$$\begin{aligned} v_1 &= \left|\frac{dX^i}{dX^0}\right|_{\sigma=0} = |\sin d|, \\ v_2 &= \left|\frac{dX^i}{dX^0}\right|_{\sigma=\pi} = |\sin(\pi\lambda - d)|\,, \end{aligned} \tag{12}$$

remain smaller than the the velocity of light even in the limit of vanishing masses $m_i = 0$.

For fixed values of the external parameters γ, α and $m_i - s$ this is in fact a family of solutions, parameterized by the value of q. By increasing q we increase the string length,

$$L = \frac{2q}{\lambda^2}\left[\sin^2\frac{d}{2} + \sin^2\left(\frac{\pi\lambda - d}{2}\right)\right], \tag{13}$$

as well as its classical energy,

$$E_0 = \frac{\pi q\gamma}{\lambda}\left[1 + \frac{\sin\pi\lambda\,\cos(\pi\lambda - 2d)}{\pi\lambda}\right] + m_1\cos d + m_2\cos(\pi\lambda - d)\,. \tag{14}$$

In order to calculate the Casimir energy of the rotating string we have to find the frequencies of small oscillations around this configuration. If we write

$$\Phi(\tau,\sigma) = \Phi_0(\tau,\sigma) + \Phi_1(\tau,\sigma)\,, \tag{15}$$

where Φ_0 is given by (9) and Φ_1 is assumed to be small, then from (4) we get the equation

$$\partial_\tau^2\Phi_1 - \partial_\sigma^2\Phi_1 + \frac{2\lambda^2}{\cos^2(\lambda\sigma - d)}\Phi_1 = 0\,, \tag{16}$$

and (5) leads to the boundary conditions for the Φ_1 field of the form

$$\Phi_1 = 0, \qquad \mathrm{Im}\;\partial_\sigma\Phi_1 = 0 \qquad \text{for} \qquad \sigma = 0, \pi. \tag{17}$$

General solution of the equation (16) satisfying the conditions (17) is

$$\Phi_1(\tau,\sigma) = \sum_{n=1}^{\infty} a_n\cos\left(\omega_n\tau + \phi_n\right)\left[\frac{\partial}{\partial\sigma} + \lambda\tan(\lambda\sigma - d)\right]\cos(\omega_n\sigma - \delta_n)\,, \tag{18}$$

where

$$\tan\delta_n = \frac{\lambda}{\omega_n}\tan d\,,$$

ω_n are positive roots of the equation

$$D(\omega) \equiv \omega^2 \sin \pi\omega - \lambda\omega \left[\tan d + \tan(\pi\lambda - d)\right] \cos \pi\omega \\ -\lambda^2 \tan d \tan(\pi\lambda - d) \sin \pi\omega = 0\,, \tag{19}$$

excluding $\omega_0 = \lambda$ and a_n, ϕ_n are arbitrary, real constants.

It is convenient to introduce the abbreviations

$$\eta = \lambda \tan(\pi\lambda - d), \qquad \rho = \lambda \tan d\,,$$

what allows to rewrite

$$D(\omega) = \left(\omega^2 - \rho\eta\right) \sin \pi\omega - (\rho + \eta)\omega \cos \pi\omega\,.$$

Using Eqs. (6)–(8) one checks that the Liouville field Φ_1 describes a set of decoupled string oscillations with frequencies

$$\nu_n = \frac{\lambda}{q}\, \omega_n\,. \tag{20}$$

The Casimir energy is defined as a (appropriately regularised and renormalized) sum

$$E_{\mathrm{Cas}} = \left(\sum_{n=1}^{\infty} \frac{1}{2} \nu_n \right)_{\mathrm{ren}}\,. \tag{21}$$

We choose to work with the ζ function regularization (let us stress, however, that the final result is independent of the chosen regularization method — for instance, the cut-off regularization gives the same ultimate formulae) and define after [17]

$$\tilde{E}_{\mathrm{Cas}} \stackrel{\mathrm{def}}{=} \frac{1}{4} \lim_{\varepsilon \to 0} \left[\mu^{\varepsilon} \zeta(-1 + \varepsilon) + \mu^{-\varepsilon} \zeta(-1 + -\varepsilon)\right]\,, \tag{22}$$

where, for $\mathrm{Re}\, s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \nu_n^{-s} \tag{23}$$

and the parameter μ with dimension of mass is introduced to ensure that the r.h.s. of the expression (22) has the dimension of energy for arbitrary complex s. The physically interesting value $s = -1$ is obtained from (23) through the analytic continuation.

Using the standard methods of contour integration in the complex plane one writes

$$\sum_n \nu_n^{-s} = \frac{1}{2\pi i} \left(\frac{\lambda}{q}\right)^{-s} \int\limits_{\mathcal{C}_1} dz z^{-s} \frac{d}{dz} \log D(z)\,, \tag{24}$$

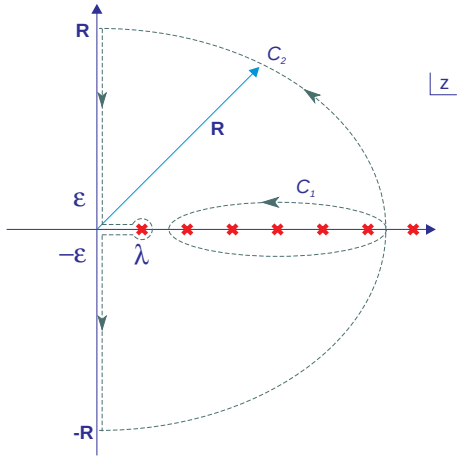

Fig. 1. The integration contours in the complex plane.

where the integration contour $\mathcal{C}_1$ (Fig. 1) surrounds zeroes of the function D excluding $\nu_0 = \frac{\lambda^2}{q}$.

The analicity of the function $D(z)$ allows to deform the integration contour $\mathcal{C}_1$ into $\mathcal{C}_2$ and, after a straightforward calculation, one arrives at the formula

$$\begin{aligned}\tilde{E}_{\rm Cas} &= \frac{\lambda}{2\pi q}\left[\eta^2\log\frac{\eta^2}{\tilde{\mu}^2}+\rho^2\log\frac{\rho^2}{\tilde{\mu}^2}\right]\\ &+\frac{\lambda}{2\pi q}\left\{\int_0^\infty dy\ \log\left[1-\frac{(y-\rho)(y-\eta)}{(y+\rho)(y+\eta)}\mathrm{e}^{-2\pi y}\right]-\lambda\right\},\end{aligned}\tag{25}$$

where $\tilde{\mu}$ is also an arbitrary, but now dimensionless constant.

Following [18,19] we interpret terms in the first square bracket in Eq. (25) as renormalising the classical string mass. This is also supported by the expectation, that the Casimir energy should vanish for infinitely long strings, while the discussed terms fail to satisfy this condition.

Our final expression for the Casimir energy thus reads

$$E_{\rm Cas} = \frac{\lambda}{2\pi q}\left\{\int_0^\infty dy\ \log\left[1-\frac{(y-\rho)(y-\eta)}{(y+\rho)(y+\eta)}\mathrm{e}^{-2\pi y}\right]-\lambda\right\}.\tag{26}$$

For every values of masses m_1, m_2 and the parameters γ, α the Casimir energy (26) is negative.

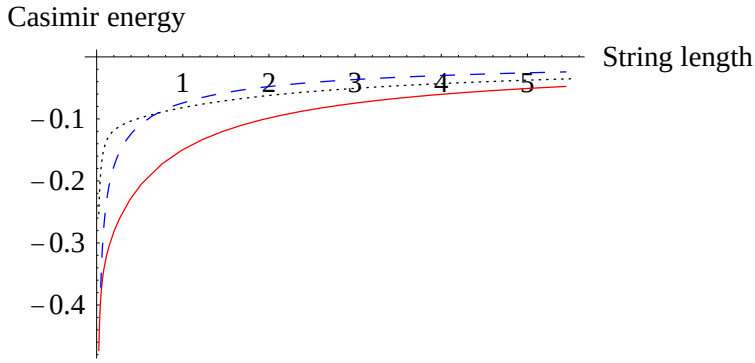

Fig. 2. The Casimir energy versus string length for various values of masses and the parameter α : $\alpha = 0.2, m_1 = 0.1, m_2 = 0.2$ (solid line), $\alpha = 0.2, m_1 = 0.1, m_2 = 30$ (dashed line) and $\alpha = 2, m_1 = 0.1, m_2 = 0.2$ (dotted line). All dimensionful quantities in the system of units $\gamma = 1$.

For long strings ($\sqrt{\gamma}L \to \infty$) formula (26) gives

$$E_{\mathrm{Cas}} = -\frac{1}{12}\frac{1}{L} + o\left(L^{-1}\right) .$$

This is different from the celebrated Lüsher term [20],

$$E_C^{\mathrm{L}} = -\frac{\pi}{12}\frac{1}{L},$$

but the reasons are obvious. First, Lüsher term is derived for the string with fixed ends and the oscillation frequencies equal

$$\nu_n^{\mathrm{L}} = \frac{\pi n}{L},$$

while in our, rotating string case we have

$$\nu_n(\sqrt{\gamma}L \to \infty) = \frac{2n}{L}.$$

Second, in considered model we have only planar oscillations and this gives additional factor $1/2$.

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