Vol. 30 (1999) ACTA PHYSICA POLONICA B No 9

STAR PRODUCTS AND DEFORMED YANGIANS

M. Mansour and K. Akhoumach

Laboratoire de Physique Théorique
Département de Physique, Université de Mohamed V
B. P. 1014 Rabat, Maroc

(Received May 13, 1999; Revised version received July 26, 1999)

A pseudotriangular Hopf algebra structure on a deformed Yangian $Y_F(g)$ associated to a simple Lie algebra g is given by using a star-product on the corresponding simple Poisson Lie group.

PACS numbers: 11.10.–z

1. Introduction

Quantum groups, (or quantum algebras) introduced by Drinfeld (1986), are a subject much discussed both by physicists and mathematicians. Its essence crystallised from the intensive developements of the quantum inverse problem method (Faddeev, 1982) and from the investigation related to Yang–Baxter equation (Jimbo, 1986). Other quantum type algebras called Yangians were introduced by Drinfeld (1985, 1986). The Yangian symmetries appear in many physical models, such as long-rang interaction model (Haldane *et al.*, 1992; Kato and Kuramoto, 1995; Hikami, 1995), the one dimensional Hubbard model (Uglov and Korepin, 1984; Gohmann and Inozemtsev, 1996) and the two-dimensional chiral model with or without topological terms (Bardeen, 1991).

Quantum groups are defined as non-abelian Hopf algebras (Takhtajan, 1989; Abe, 1980). A way to generate them consists of deforming the abelian product of the Hopf algebra of functions into a non-abelian one (star-product), using the so-called quantization-deformation or star-quantization (Bayen *et al.*, 1978a, 1978b; Flato *et al.*, 1975).

The existence of a star product has been studied by Vey (1975), Neroslavsky and Vlassov (1981), who proved the existence of a star product on a symplectic manifold with a vanishing third De Rham cohomology group and by De Wilde and Lecopmte (1983) in the general case. From a geometrical point of view, Omori *et al.* (1994) and Fedesov (1994) constructed star products for arbitrary symplectic manifold. Recently Kontsevich (1997), by using different methods has also constructed and classified differential star products on a Poisson manifold.

The star-product on a Poisson–Lie group is used to develop a theory of h-deformed Lie algebras in (Mansour, 1997), a theory of quantum algebras in (Mansour, 1999), a theory of quantum superalgebras (Mansour, 1998a) and a theory of quasi-quantum groups (Mansour, 1998b).

The present paper shows explicitly that the star product on an exact simple Poisson Lie group leads to the structure of pseudotriangular Hopf algebra on the deformed Yangian of the corresponding simple Lie algebra, and that equivalents star-products generate isomorphic Yangians.

2. Preliminaries

In this section we review some general aspects of the theory of Yangians, following mainly the presentation of Drinfeld (1985) and we recall the theory of star-products introducted by Bayen *et al.* (1978a).

2.1. Yangians

Definition 1 *Let g be a simple Lie algebra over C, given by the generartors $\{X_i\}$ and the relations $[X_i, X_j] = C_{ij}^k X_k$, where $\{X_i\}$ is an orthonormal basis with respect to the Killing form. Then the Yangian $Y(g)$ is an associative algebra with unity, generated by elements $\{X_i\}$ and $\{T_j\}$ and the following relations*

$$[X_i, X_j] = C_{ij}^k X_k, \qquad [X_i, T_j] = C_{ij}^k T_k\,, \tag{1}$$

$$[T_i, [T_j, X_k]] - [X_i, [T_j, T_k]] = a_{ijk}^{lmn}\{X_l, X_m, X_n\}\,. \tag{2}$$

Here $a_{ijk}^{lmn} = \frac{1}{24} C_{il}^p C_{jm}^q C_{kn}^r C_{pq}^r$ *and* $\{X_i, X_j, X_k\} = \sum_{i\neq j\neq k} X_i X_j X_k\,.$

Definition 2 *The Hopf algebra structure on the Yangian is given by the following costructures*

— *Coproduct*

$$\Delta(X_i) \;=\; X_i \otimes 1 + 1 \otimes X_i\,, \tag{3}$$

$$\Delta(T_k) \;=\; T_k \otimes 1 + 1 \otimes T_k + \frac{1}{2} C_{ij}^k X_i \otimes X_j\,; \tag{4}$$

— *Counit:*

$$\varepsilon(X_i) = \varepsilon(T_i) = 0\,; \tag{5}$$

— *Antipode*

$$S(X_i) = -X_i\,, \qquad S(T_i) = -T_i - \tfrac{1}{2} C_{ij}^k X_j X_k\,. \tag{6}$$

For any $u \in C$ we define an automorphism J_u of $Y(g)$ by the formulas

$$J_u(X_i) = X_i\,, \qquad J_u(T_i) = T_i + uX_i\,. \tag{7}$$

As usual, we denote by $\Delta^{\rm op}$ the opposite comultiplication.

The pseudo-triangular Hopf algebra structure on the Yangian is given by the following theorem

Theorem 1 *(Drinfeld, 1985) There exists a unique*

$$R(u) = 1 \otimes 1 + \sum_{k=1}^{\infty} R_k u^{-k}\,, \qquad R_k \in Y(g)^{\otimes 2}\,; \qquad R_1 = \sum_i X_i \otimes X_i$$

such that

$$(\Delta \otimes id)R(u) = R_{13}(u)R_{23}(u)$$

$$(J_u \otimes id)\Delta^{\rm op}(a) = R(u)((J_u \otimes id)\Delta(a))R^{-1}(u), \quad \forall a \in Y(g)$$

$$(J_u \otimes J_v)R(w) = R(w+u-v)$$

$$R_{12}(u)R_{21}(-u) = 1 \otimes 1$$

$$R_{12}(u_1-u_2)R_{13}(u_1-u_3)R_{23}(u_2-u_3) = R_{23}(u_2-u_3)R_{13}(u_1-u_3)R_{12}(u_1-u_2)\,.$$

2.2. Star products

Let G be a complex simple Lie group , g its finite dimensional complex simple Lie algebra and $\{X_i\}$ is as basis of g. Let $r = r^{ij}X_i \otimes X_j \ \in (g \otimes g)$ be a solution of the classical Yang–Baxter equation

$$[\![r,r]\!] = [r_{12},r_{13}] + [r_{12},r_{23}] + [r_{13},r_{23}] = 0\,. \tag{8}$$

Then the Lie bialgebra structure $(g, \delta(r))$ on g is given by the algebra 1-cocycle

$$\begin{aligned} \delta(r) : g \ &\longrightarrow \ g \otimes g\,, \\ y \ &\longmapsto \ (ad_y \otimes 1 + 1 \otimes ad_y)r\,, \end{aligned} \tag{9}$$

where ad_y stands for the adjoint representation and the Poisson–Lie structure on Lie group G is given by

$$\{\phi,\psi\} = \sum_{i,j} \ r^{ij}(X_i^r(\phi)X_j^r(\psi) - X_i^l(\phi)X_j^l(\psi))\,, \tag{10}$$

where $X_i^r = (R_g)_* X_i$ and $X_i^l = (L_g)_* X_i$ are the right and left vectors fields on the group G, $(R_g)_*, (L_g)_*$ are the derivatives mapping corresponding to the right and left translation .
A particular solution of the rational classical Yang–Baxter equation

$$\begin{aligned}&[r_{12}(u_1-u_2), r_{23}(u_2-u_3)] + [r_{12}(u_1-u_2), r_{13}(u_1-u_3)]\\&+[r_{13}(u_1-u_3), r_{23}(u_2-u_3)] = 0\end{aligned} \tag{11}$$

is given by

$$r(u) = u^{-1} X_i \otimes X_i$$

which leads to the Lie bialgebra structure $(g[u], \phi)$, where

$$\phi(a(u)) = [a(u)\otimes 1 + 1 \otimes a(v), r(u-v)] .$$

Here we have identified $g[u]\otimes g[u]$ with $(g\otimes g)[u]$.

Recall also that the action of $U(g)$ on $\boldsymbol{F}(G)$ (the space of smooth functions on the Poisson–Lie group G) is given by

$$< X, Y^l(\phi) >=< XY, \phi > \tag{12}$$

and

$$< X, Y^r(\phi) >=< YX, \phi > . \tag{13}$$

Now we give the following definition (Moreno and Valero, 1992).

Definition 3 *A star product on the Poisson Lie group is defined as a bilinear map*

$$\begin{aligned}F(G)\times F(G) &\longrightarrow F(G)[[h]] ,\\(\phi,\psi) &\longmapsto \phi * \psi = \sum_j h^j C_j(\phi,\psi) ,\end{aligned} \tag{14}$$

such that
(i) when the above map is extended to $F(G)[[h]]$, it is formally associative

$$(\phi * \psi) * \chi = \phi * (\psi * \chi) ; \tag{15}$$

(ii) $C_0(\phi,\psi) = \phi.\psi = \psi.\phi$; (iii) $C_1(\phi,\psi) = \{\phi,\psi\}$; (iv) the two cochains $C_k(\phi,\psi)$ are bidifferential operators on $F(G)$.

In this definition, the Hopf algebra $F(G)[[h]]$, with a new product $*$ and unchanged coproduct is considered to be a topological Hopf algebra. We recall that the deformations with unchanged coproduct are called preferred deformations (Bonneau et al, 1995). This condition is imposed on quantization because of the invariance property of the Poisson Lie group bracket

$$\Delta(\{\phi,\psi\}) = \{\Delta(\phi), \Delta(\psi)\} .$$

It is therefore natural to impose the same compatibility condition of the star-product with respect to the coproduct of $F(G)$, *i.e.* :

$$\Delta(\phi * \psi) = (\Delta(\phi) * \Delta(\psi)) \tag{16}$$

is satisfied. The star-product on the right side is canonically defined on $F(G) \otimes F(G)$ by

$$(\phi \otimes \psi) * (\phi' \otimes \psi') = (\phi * \phi') \otimes (\psi * \psi') \,. \tag{17}$$

If we introduce the element of $U(g) \otimes U(g)[[h]]$

$$F = 1 + \sum_{i \geq 1} F_i h^i$$

which satisfies the cocycle condition

$$(\Delta_0 \otimes id) F (F \otimes 1) = (id \otimes \Delta_0) F (1 \otimes F) \,, \tag{18}$$

(where Δ_0 is the usual coproduct of the enveloping algebra $U(g)$), we obtain the star product on the Poisson Lie group by the following expression (Takhtajan, 1990)

$$\phi * \psi = \mu((F^{-1})^r . F^l (\phi \otimes \psi)) \,. \tag{19}$$

In fact, the product defined in this way is associative:

$$(\phi * \psi) * \chi = \phi * (\psi * \chi).$$

Following Mansour (1998b) the quantized enveloping algebra $U(g)[[h]]$ is endowed with a structure of Hopf algebra with the coproduct Δ_F given by:

$$< \Delta_F(X), \phi \otimes \psi > = < X, \phi * \psi > \tag{20}$$

for all $\phi, \psi \;\in\; \boldsymbol{F}(G)$, and $X \in U(g)$.

In fact, using the equations (12) and (13) we obtain:

$$\Delta_F(X) = F^{-1} . \Delta_0(X) . F \,. \tag{21}$$

We can easily show that the twisted coproduct Δ_F is coassociative :

$$(\Delta_F \otimes id) \Delta_F(X) = (id \otimes \Delta_F) \Delta_F(X) \,. \tag{22}$$

For the equivalence of star-products, we give the following definition

Definition 4 *(Gutt and Rawnsley, 1998)*
Two star products $*_1$ and $*_2$ on a Poisson Lie group are said to be equivalent if there is a series

$$T = id + \sum_{r=1}^{\infty} h^r T_r \,,$$

where the T_r are linear operators on $C^\infty(G)$ such that

$$T(f *_1 g) = Tf *_2 Tg \,.$$

3. Star products and deformed Yangians

Consider $Y_h(g) = Y(g)[[h]]$. Clearly $Y_h(g)$ contains $U(g)[[h]]$ as a Hopf subalgebra. The element F definnig the star-product can be viewed as an element of $Y_h(g)^{\otimes 2}$. Obviously, one can extend the Hopf algebra structure to $Y_h(g)$. Now using the star-product F we can define a new Hopf algebra $Y_h^F(g)$, which has the same multiplication as $Y_h(g)$ but the comultiplication is given by

$$\Delta_F(X) = F^{-1}.\Delta_0(X).F \,,$$

which is obviously coassociative on $Y_h^F(g)$. The pseudotriangular Hopf algebra structure on $Y_h^F(g)$ is given by the element

$$R_F(u) = F_{21}^{-1}.R(u)F \,, \tag{23}$$

where R(u) is given in theorem 1. In fact

(i) If we denote by $\Delta_F^{\rm op}$ the opposite coproduct of Δ_F then we can easily show that $\forall a \in Y_h^F(g)$

$$\begin{aligned}(J_u\otimes id)\Delta_F^{\rm op}(a) &= (F^{21})^{-1}R(u)((J_u\otimes id)\Delta_F(a))R^{-1}(u)F^{21}\\ &= (F^{21})^{-1}R(u)((J_u\otimes id)\Delta_0(a))R^{-1}(u)F^{21}\\ &= ((F^{21})^{-1}R(u)F)((J_u\otimes id)\Delta_F(a))((F^{21})^{-1}R(u)F)^{-1}\\ &= R_F(u)((J_u\otimes id)\Delta_F(a))R_F^{-1}(u)\,.\end{aligned} \tag{24}$$

(ii) Secondly we have

$$\begin{aligned}(\Delta_F\otimes id)R_F(u) &= (F^{12})^{-1}(\Delta_0\otimes id)((F^{21})^{-1}R(u)F)F^{12}\\ &= (F^{13})^{-1}R_{13}(u)F^{13}(F^{32})^{-1}R_{23}(u)F^{23}\\ &= (R_F)_{13}(R_F)_{23}\,.\end{aligned} \tag{25}$$

(iii) Similarly we have

$$(id\otimes\Delta_F)R_F(u) = (R_F)_{13}(R_F)_{12}\,. \tag{26}$$

(iv) After:

$$\begin{aligned}(J_v \otimes J_w)R_F(u) &= (J_v \otimes J_w)((F^{21})^{-1}R(u)F)\,,\\ &= (J_v \otimes J_w)(F^{21})^{-1}(J_v \otimes J_w)R(u)(J_v \otimes J_w)F\,,\\ &= (F^{21})^{-1}R(u+v-w)F = R_F(u+v-w)\,, \end{aligned} \quad (27)$$

where we have used the fact that

$$(J_v \otimes J_w)F = F\,.$$

(v) Finally, we have

$$\begin{aligned}(R_F)_{12}(u)(R_F)_{21}(-u) &= (F^{21})^{-1}R(u)FF^{-1}R_{21}(-u)F\,,\\ &= (F^{21})^{-1}R(u)R_{21}(-u)(F^{21})^{-1}\,,\\ &= (F^{21})^{-1}F^{21} = 1 \otimes 1\,. \end{aligned} \quad (28)$$

Using the fact that

$$\phi * 1 = 1 * \phi = \phi\,, \quad (29)$$

which implies that

$$(id \otimes \varepsilon)F = (\varepsilon \otimes id)F = 1\,, \quad (30)$$

we obtain:

$$(\varepsilon \otimes id)(R_F) = (id \otimes \varepsilon)(R_F) = 1\,. \quad (31)$$

Equations (22),(24)–(28),(31) imply that $(Y^F(g), \Delta_F, R_F(u))$ is a pseudo-triangular Hopf algebra.

Let us consider now two star-products $*_1$ and $*_2$ and let F and $\bar{F}$ be the two corresponding invertible elements of the Hopf algebra $Y(g)^{\otimes 2}[[h]]$ such that

$$(\Delta_0 \otimes id)F.F_{12} = (id \otimes \Delta_0)F.F_{23}\,,$$

$$(\Delta_0 \otimes id)\bar{F}.\bar{F}_{12} = (id \otimes \Delta_0)\bar{F}.\bar{F}_{23}\,.$$

Let $A = (Y^F(g)[[h]], \Delta_F, R_F, S_F)$ and $\bar{A} = (Y^{\bar{F}}(g)[[h]], \Delta_{\bar{F}}, R_{\bar{F}}, S_{\bar{F}})$ be the resulting deformed Yangians, where

$$\Delta_F = F.\Delta_0.F^{-1}, \qquad R_F(u) = F_{21}^{-1}.R(u)F\,, \quad (32)$$

$$\Delta_{\bar{F}} = \bar{F}.\Delta_0.\bar{F}^{-1}, \qquad R_{\bar{F}}(u) = \bar{F}_{21}^{-1}.R(u)\bar{F}\,. \quad (33)$$

Then it is easily seen that $\bar{A}$ can be obtained from A by applying the twist $\hat{F} = F^{-1}.\bar{F}$. In fact

$$\Delta_{\bar{F}}(a) = \hat{F}.\Delta_F(a).\hat{F}^{-1}, \qquad \forall a \in Y^F((g)[[h]] \quad (34)$$

and

$$R_{\bar{F}}(u) = \hat{F}_{21}^{-1}.R_F(u).\hat{F}. \quad (35)$$

Proposition 1 *The element $\hat{F} = F^{-1}\bar{F}$ satisfies the following relation*

$$(\Delta_F \otimes id)\hat{F}.\hat{F}_{12} = (id \otimes \Delta_F)\hat{F}.\hat{F}_{23} \tag{36}$$

i.e., the element $\hat{F}$ is a star-product relatively to the coproduct Δ_F, and the element $\hat{F}^{-1}$ is a star-product relatively to the twisted coproduct $\Delta_{\bar{F}}$.

Proof
For the first part of this proposition, we have:

$$\begin{aligned}(\Delta_F \otimes id)\hat{F}.\hat{F}_{12} &= (F^{-1}\Delta_0 F \otimes id)F^{-1}\bar{F}.F_{12}^{-1}.\bar{F}_{12}\,,\\ &= F_{12}^{-1}(\Delta_0 \otimes id)F^{-1}\bar{F}.F_{12}\,,\\ &= F_{23}^{-1}(id \otimes \Delta_0)F^{-1}.(\Delta_0 \otimes id)\bar{F}.\bar{F}_{12}\,,\\ &= F_{23}^{-1}(id \otimes \Delta_0)F^{-1}.(id \otimes \Delta_0)\bar{F}.\bar{F}_{23}\,,\\ &= F_{23}^{-1}(id \otimes \Delta_0)F^{-1}\bar{F}.F_{23}.F_{23}^{-1}.\bar{F}_{23}\,,\\ &= (id \otimes \Delta_F)F^{-1}\bar{F}.F_{23}^{-1}.\bar{F}_{23}\,,\\ &= (id \otimes \Delta_F)\hat{F}.\hat{F}_{23}\end{aligned} \tag{37}$$

for the second part, taking the inverse of the equation (36) we obtain

$$\hat{F}_{12}^{-1}.(\Delta_F \otimes id)\hat{F}^{-1} = \hat{F}_{23}^{-1}.(id \otimes \Delta_F)\hat{F}^{-1}$$

so,

$$\bar{F}_{12}^{-1}F_{12}(F^{-1}\Delta_0 F \otimes id)(\bar{F}^{-1}F) = \bar{F}_{23}^{-1}F_{23}(id \otimes F^{-1}\Delta_0 F)(\bar{F}^{-1}F)$$

and

$$\bar{F}_{12}^{-1}(\Delta_0 \otimes id)\bar{F}^{-1}F)F_{12} = \bar{F}_{23}^{-1}(id \otimes \Delta_0)(\bar{F}^{-1}F).F_{23}\,.$$

Therefore

$$(\Delta_{\bar{F}} \otimes id)(\bar{F}^{-1}F)\bar{F}_{12}^{-1}F_{12} = (id \otimes \Delta_{\bar{F}})(\bar{F}^{-1}F).\bar{F}_{23}^{-1}F_{23}\,.$$

Finally, we have:

$$(\Delta_{\bar{F}} \otimes id)\hat{F}^{-1}.\hat{F}_{12}^{-1} = (id \otimes \Delta_{\bar{F}})\hat{F}^{-1}.\hat{F}_{23}^{-1}\,. \tag{38}$$

If the two star products are equivalent *i.e.* the corresponding elements F and $\bar{F}$ are related by the following expression

$$\bar{F} = \Delta_0(E^{-1}).F.(E \otimes E) \tag{39}$$

for some invertible element E of $U(g)[[h]]$, then the coproduct $\Delta_{\bar{F}}$ can be rewritten as

$$\Delta_{\bar{F}}(X) = (E^{-1} \otimes E^{-1})\Delta_F(E.X.E^{-1}).(E \otimes E)\,. \tag{40}$$

Similarly, the pseudotriangular structures are related by

$$R_{\bar{F}}(u) = (E^{-1} \otimes E^{-1}).R_F(u).(E \otimes E)\,. \tag{41}$$

So, the induced isomorphism maps the pseudotriangular structures as well.

REFERENCES

Abe, E., *Cambridge Tracts in Math.* No. 74, Cambridge University Press, 1980.
Bardeen, D., *Prog. Theor. Phys. Suppl.* **123**, 1 (1991).
Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D., *Ann. Phys.* **111**, 61 (1978a).
Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D., *Ann. Phys.* **110**, 111 (1978b).
Bonneau, P., Flato., M., Gerstenhaber, M., Pincezon, G., *Commun. Math. Phys.* **161**, 125 (1994).
De Wilde, M., Lecomte, P.B., *Lett. Math. Phys.* **1**, 521 (1983).
Drinfeld, V.G., *Sov. Math. Dokl.* **32**, 254 (1985).
Drinfeld, V.G., Proc. Int. Congress of Mathematicians, Berkely, 1986, Vol. 1. (1986) p.798.
Faddeev, L.D., Inegrable Models in (1+1)-Dimensional Quantum Field Theory, in Les Houches lectures XXXIX, Elsiever, Amsterdam 1982, p.563.
Fedesov, P.V., *J. Diff. Geom.* **40**, 213 (1994).
Flato, M., Lichnerowicz, A., Sternheimer, D., *Compostio Mathematic* **31**, 41 (1975).
Gohmann, F., Inozemstev, V., *Phys. Lett.* **A214**, 161 (1996).
Gutt, S., Rawnsley, J., Equivalence of star products on a symplectic manifold, Warwick preprint 31 /1998.
Haldane, F.D.M., Ha, Z.N C., Talstro, J.C., Bernard, D., Pasquier, V., *Phys. Rev. Lett.* **69**, 2021 (1992).
Hikami, K., *J. Phys.* **A28**, L131 (1995).
Jimbo, M., *Lett. Math. Phys.* 10247 (1986).
Kato, Y., Kuramoto, Y., *Phys. Rev. Lett.* **74**, 1222 (1995).
Kontsevish, M., Deformation quantization of poisson manifolds, q-alg/9709040.
Mansour, M., *Int. J. Theor. Phys.* **Vol.36**, 3007 (1997).
Mansour, M., Star-products and quantum superalgebras, Trieste, ICTP, 1998a, pre-print IC/98/124.
Mansour, M., *Int. J. Theor. Phys.* **vol.38**, 1467 (1999).
Mansour, M., *Int. J. Theor. Phys.* **Vol.37**, 2995 (1998-b).
Moreno, C., Valero, L., *J. Geom. Phys.* **9**, 369 (1992).
Neroslavsky, O.M., Vlassov, A.T., *C. R. Acad. Sci. Paris Ser. I Math. 2* **181**, 71 (1981).
Omori, H., Maeda, Y., Yoshioda, A., *Contemp. Math.* AMS **179**, 213 (1994).
Takhtajan, L. A., *Lectures on Quantum Groups* M. Ge and B. Zhao eds., World Scientific, Singapore 1989.
Uglov, D.B., Korepin, V.E., *Phys. Lett.* **A190**, 238 (1984).
Vey, J., *Com. Math. Helvet.* **50**, 421 (1975).