RENYI ENTROPIES IN MULTIPARTICLE PRODUCTION *

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Renyi entropies are calculated for some multiparticle systems. Arguments are presented that measurements of Renyi entropies as functions of the average number of particles produced in high energy collisions carry important information on the produced system.

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1. Introduction

The process of multiparticle production in high-energy collisions is a complex phenomenon which can be described from different points of view, given the large number of variables involved. Indeed, the full knowledge of the produced system requires information about the probability distribution $P_N(q_1,....,q_N)dq_1...dq_N$, where N is the number of produced particles and q_i are their 4-momenta. Such information is, clearly, never available: neither theoretically (except in some simple, unrealistic models) nor experimentally. It is therefore of great importance to investigate some *averaged* quantities which can be practically measured and — at the same time — provide some well-defined (although admittedly non complete) information about $P_N(q_1,...,q_N)$. Many such quantities (mostly the moments of kinematic variables and of the multiplicity distribution in different phase-space regions [1]) were measured and studied in the past. Practically all our present knowledge about multiparticle production is based on these analyses.

Recently, we have proposed [2–4] to investigate the so-called Renyi entropies H_l, $l=2,3,4...$ which were introduced many years ago [5] as a tool for

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studies of the dynamical systems[1] and are closely related to the thermodynamic entropy of the system (the Shannon entropy [7]). Their novel aspect is that they are not related to the moments of the kinematical variables and/or multiplicity distributions but rather to moments of the probability distributions themselves [3]. In order to understand better the physical significance of this new measurement, it seems useful to consider in detail some simplified systems resembling those which are encountered in the physics of multiparticle production. This is the purpose of the present paper.

After reviewing shortly the method of measurement (Section 2) and the general properties of Renyi entropies (Section 3), we discuss the results for specific examples. The ideal Bose gas (including effects of Bose condensation) is reviewed in Section 4 and 5, and a multiparticle system exhibiting long-range correlations in momentum space in Section 6. Our conclusions are listed in the last section.

2. Measurement of Renyi entropies

As shown in [3], Renyi entropies H_l for integer $l = 2, 3, 4...$ can be determined from the measurement of the "coincidence probabilities" C_l, $l = 2, 3, 4...$, which generalize the notion introduced some time ago by Ma [8]. The measurement of C_l can be performed in the following steps [4]:

(i) One has to discretize the momentum distribution by splitting the investigated region of (momentum) phase-space into a number of cells of a given size[2] Δq. Denoting the number of cell by M, each event is now characterized by the set of integers $(n_1, ..., n_M)$ denoting the number of particles in each of the cells. The whole system is then described by the probability distribution $W(n_1, ..., n_M)$ which is -of course- trivially related to $P_N(q_1, ..., q_N)$ introduced in Section 1.

(ii) The measurement is performed by considering a certain number $N^{\rm tot}$ of events (each characterized by a particular set $n_1, ..., n_M$) and counting how many of these sets are identical (*i.e.* how many of them *coincide*). The coincidence probabilities are defined as

$$C_l = N_l / N_l^{\rm tot} \ , \tag{1}$$

where N_2 is the number of *identical* pairs of events, N_3 is the number of *identical* triples, *etc.* Correspondingly, $N_2^{\rm tot} = N^{\rm tot}(N^{\rm tot} - 1)$ is the total number of pairs recorded, $N_3^{\rm tot} = N^{\rm tot}(N^{\rm tot} - 1)(N^{\rm tot} - 2)$ is the total number of triples, *etc.*

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[1] For a recent review, see *e.g.* [6].

[2] For simplicity we consider here all cells of the same size. This is not necessary, in general.

(iii) The Renyi entropies are defined as

$$H_l = \frac{1}{1-l} \log C_l \ . \tag{2}$$

One sees that this measurement is conceptually very simple. The catch is that, for realistic systems, the coincidence probabilities are usually very small and thus one needs a rather large statistics (large N^{tot}) to obtain a significant result [3, 9].

3. Renyi entropies and multiparticle production

Some features of Renyi entropies (and coincidence probabilities) seem very attractive for description of multiparticle systems. Let us list some of them

(i) It is not difficult to show that C_l's are moments of the distribution $W(N_1, ..., n_M)$. The relation is [3]:

$$C_l = \sum_{n_i} [W(n_1, ..., n_M)]^l = \left\langle [W(n_1, ..., n_M)]^{l-1} \right\rangle \ . \tag{3}$$

It explains directly the physical meaning of the coincidence probabilities.

(ii) The formula (3) gives definition of C_l which can be used for any l and is thus more general than that given by (1) (which can be applied only for integer l). Actually (3) was the original definition of Renyi [5]. Unfortunately, there seems to be no simple way to measure C_l for non-integer l, except by determining first the probabilities $W(n_1, ..., n_M)$ and then using directly the formula (3). This method would, however, require a tremendous statistics and thus does not seem practical - except for very small systems (small number of particles and/or small number of bins [3, 9, 10]).

(iii) When Eq. (3) is inserted into (2) and continued analytically to $l = 1$, one obtains relation to the Shannon entropy S:

$$H_1 = S \equiv \sum_{n_i} W(n_1, \ldots, n_M) \log W(n_1, \ldots, n_M) = \langle \log W(n_1, \ldots, n_M) \rangle . \tag{4}$$

This feature is of course very important: by extrapolation to $l = 1$ of the *measured* Renyi entropies for $l = 2, 3, ...$ one may obtain information on the fundamental quantity S characterizing the statistical system in question. The problem is that the extrapolating procedure is of course not unique and thus brings an additional error to this measurement[3].

(iv) One sees from (1) that the coincidence probabilities are mostly sensitive to event-by-event fluctuations. They may thus be very useful as a quantitative measure of this important feature of the data.

(v) The measurement requires, as a first step, a discretization of the system in question. Consequently, the results depend on the way the system is discretized (on the size of the cell Δq, in particular). A careful investigation of this dependence may be very interesting and fruitful for understanding of the dynamics[4].

4. The ideal Bose gas

We start with the simplest example of an ideal gas of identical bosons at equilibrium.

The probability of having n_1 bosons with energy ε_1, n_2 bosons with energy ε_2,... is given by

$$P(n_1, n_2, ...n_M) = \prod_{m=1}^{M} p_{m,n_m} \ , \tag{5}$$

where

$$p_{m,n_m} = (1 - x_m)[x_m]^{n_m} \ , \tag{6}$$

is the probability to have n_m bosons with energy ε_m and

$$x_m = \mathrm{e}^{-\beta(\varepsilon_m - \mu)}; \quad \beta = \frac{1}{T} \ , \tag{7}$$

where $\mu = \mu(T)$ is the chemical potential.

From (5) and (3) we obtain for coincidence probabilities

$$C_l = \prod_{m=1}^{M} \frac{(1 - x_m)^l}{1 - x_m^l} \ , \tag{8}$$

and thus

$$H_l = -\sum_{m=1}^{M} \log(1 - x_m) + \frac{1}{1-l} \sum_{m=1}^{M} \log\left(\frac{1 - x_m}{1 - x_m^l}\right) \ . \tag{9}$$

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[3] For a more detailed discussion, see [3].

[4] Such studies, proved very fruitful already, in a different context [1, 11].

In the limit of continuous spectrum, the sums in this formula can be replaced by the integrals:

$$H_l = \int \frac{d^3p d^3x}{(2\pi)^3} \log\left(1 - \mathrm{e}^{-\beta(\varepsilon-\mu)}\right) + \frac{1}{1-l} \int \frac{d^3p d^3x}{(2\pi)^3} \log\left(\frac{1-\mathrm{e}^{-\beta(\varepsilon-\mu)}}{1-\mathrm{e}^{-l\beta(\varepsilon-\mu)}}\right) . \tag{10}$$

In case of the photon gas ($\varepsilon = |p|, \mu = 0$), the integrals can be worked out with the result [3]

$$H_l = \frac{1}{4}\left(1 + \frac{1}{l} + \frac{1}{l^2} + \frac{1}{l^3}\right) S . \tag{11}$$

Thus we conclude that in the case of photon gas the Renyi entropies are simply proportional to the standard (Shannon) entropy of the system. One important consequence is that H_l are all proportional to the average number of particles (photons in this case) in the system:

$$H_l = \frac{1}{4}\left(1 + \frac{1}{l} + \frac{1}{l^2} + \frac{1}{l^3}\right) \frac{2\pi^4}{45\zeta(3)} N , \tag{12}$$

where

$$N \equiv \sum_{m=1}^{M} \sum_{n_m=0}^{\infty} n_m p_{m,n_m} = \sum_{m=1}^{M} \frac{x_m}{1-x_m} . \tag{13}$$

Note that for photons $H_l \sim S \sim N \sim (\beta)^{-3}$, hence for low temperatures they all vanish with the same temperature dependence.

For massive particles and/or for non-vanishing chemical potential it is not possible, in general, to obtain simple analytic results. The exception is the limit of very low temperature $T \to 0$ which we are going to consider.

First, we observe that in the limit $T \to 0$ we have $\beta \to \infty$ and thus $\mathrm{e}^{\beta\varepsilon_m} \to \infty$. Thus — since in this limit all particles should reside on the lowest level — we can determine $\mu(T)$ by demanding that the lowest energy level contains — on the average — N particles. This means

$$N = \frac{1}{\mathrm{e}^{\beta(\varepsilon_1-\mu)} - 1} \quad \to \quad x_1 = \mathrm{e}^{-\beta(\varepsilon_1-\mu)} = \frac{N}{N+1} . \tag{14}$$

Incidentally, from (6), we get

$$p_{1,n_1} = \left(1 - \frac{N}{N+1}\right)\left(\frac{N}{N+1}\right)^{n_1} , \tag{15}$$

for the multiplicity distribution in the lowest level. Using (14) and taking into account that no other state is occupied, the Renyi entropies can be trivially obtained from (9) with the result

$$H_l = -\log(1-x_1)+\frac{1}{1-l}\log\left(\frac{1-x_1}{1-x_1^l}\right) = \frac{1}{l-1}\log\left((1+N)^l - N^l\right) . \quad (16)$$

In the limit $N \to \infty$ this gives

$$H_l \sim \log(1+N) + \frac{\log l}{l-1} . \quad (17)$$

One can easily verify that the formula (17) remains valid also in the limit $l \to 1$ (*i.e.* for Shannon entropy).

We thus conclude that in the limit of very low temperatures and for a *discrete energy spectra* the entropies of the (condensed) Bose gas reveal the logarithmic dependence on the average multiplicity instead of the proportionality to N shown in Eq. (12).

However, when we are dealing with a *continuous energy spectrum* a critical temperature, T_c, appears, and we loose the logarithmic dependence of H_l on N. To see this we employ the following relation (proven in Appendix A) valid for an ideal Bose gas

$$\begin{aligned} H_l(T) \; = \; & \frac{1}{l-1}\left(\left[lS(T) - S\left(\frac{T}{l}\right)\right] + \left[\frac{\mu N(T)}{\frac{T}{l}} - \frac{\mu N\left(\frac{T}{l}\right)}{\frac{T}{l}}\right]\right. \\ & \left. - \left[\frac{E(T)}{\frac{T}{l}} - \frac{E\left(\frac{T}{l}\right)}{\frac{T}{l}}\right]\right) . \end{aligned} \quad (18)$$

(Note that this relation gives, correctly, $H_1 = S$. Note also that the temperature dependence of μ is nor scaled down with l.)

This relation is convenient because we can simply copy the well known textbook expressions for $S(T)$ and $E(T)$. For $T < T_c$, we have $\mu = 0$ and (18) simplifies. Because and N is finite and given, the second bracket on the r.h.s. of (18) disappears. Copying $S(T)$ and $E(T)$ from *e.g.* Fetter + Walecka [12] we obtain

$$lS(T) - S\left(\frac{T}{l}\right) = \frac{gV}{4\pi^2}k_B\left(\frac{2mk_B}{\hbar^2}\right)^{\frac{3}{2}}\zeta\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)\frac{5}{2}T^{\frac{3}{2}}\left(l - \frac{1}{l^{\frac{3}{2}}}\right) , \quad (19)$$

$$\frac{E(T)}{\frac{T}{l}} - \frac{E\left(\frac{T}{l}\right)}{\frac{T}{l}} = \frac{gV}{4\pi^2}k_B\left(\frac{2mk_B}{\hbar^2}\right)^{\frac{3}{2}}\zeta\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)T^{\frac{3}{2}}\left(l - \frac{1}{l^{\frac{3}{2}}}\right) , \quad (20)$$

and finally

$$H_l(T) = \left(\frac{l^{\frac{5}{2}} - 1}{l^{\frac{5}{2}} - l^{\frac{3}{2}}}\right)\frac{2}{5}S(T) . \quad (21)$$

Since $S(T) \sim (N - N_{\rm c})$, where $N_{\rm c}$ is the average number of Bosons in the ground state, we have also $H_l(T) \sim (N - N_{\rm c})$. Since (at fixed temperature) $N_{\rm c} \sim N$, we obtain again proportionality of all Renyi entropies (including the Shannon entropy) to the total number of particles. Note, however, that this need not to be true if the number of particles is changed by changing the temperature of the system.

Let us see what happens when we have a spatially finite system of Bosons, *i.e.* when we put it in a trap.

5. Bose gas in a harmonic trap

Bose gas at low temperatures has recently been experimentally investigated in the so called harmonic traps (compare *e.g.* [13], and review article [14]). To discuss such a system we can use our formulae (5)–(9) replacing, for a spherically symmetric trap,

$$\varepsilon_m \to \varepsilon_{n_x,n_y,n_z} = \left(n_x + n_y + n_z + \tfrac{3}{2}\right)\hbar\omega \ , \tag{22}$$

where ω determines the finite size, a, of the trap

$$a = \sqrt{\frac{\hbar}{m\omega}} \qquad m - \text{mass of one Boson} \ . \tag{23}$$

As it turns out [14] the critical temperatures , $T_{\rm c}$, of the observed condensates are $k_{\rm B}T \approx$(20–200)$\hbar\omega$ ($\hbar\omega$ being of the order of a few nano-Kelvins). Thus $k_{\rm B}T \gg \hbar\omega$ and it makes good sense to take the continuous limit of the oscillator spectrum.

All relevant formulae for $N(T)$, $E(T)$, $S(T)$ have been worked out in [14] for $T < T_{\rm c}$ (note that we have now $\mu = \frac{3}{2}\hbar\omega$):

$$S(T) = 4k_{\rm B}\left(\frac{k_{\rm B}T}{\hbar\omega}\right)^3\zeta(4) \ , \tag{24}$$

and

$$E(T) = N\frac{3}{2}\hbar\omega + \frac{3}{(\hbar\omega)^3}(k_{\rm B}T)^4\zeta(4) \ . \tag{25}$$

Inserting them into (18) we obtain

$$H_l(T) = \frac{1}{4}\left(1 + \frac{1}{l} + \frac{1}{l^2} + \frac{1}{l^3}\right)4k_{\rm B}\left(\frac{k_{\rm B}T}{\hbar\omega}\right)^3\zeta(4) \ . \tag{26}$$

Thus

$$H_l(T) = \frac{1}{4}\left(1 + \frac{1}{l} + \frac{1}{l^2} + \frac{1}{l^3}\right)S(T) \ . \tag{27}$$

So, we obtain again the relation (11) proven for photons. We also have

$$S(T) \sim N - N_{\rm c} \tag{28}$$

and thus the multiplicity dependence is the same as in the case of the unconstrained Bose gas (with continuous energies) discussed in the previous section.

It would be interesting to investigate the system of Bosons at such low temperatures (or in such small traps) that the system would become discrete (and loose $T_{\rm c}$). Then we would have to satisfy: $k_{\rm B}T \ll \hbar\omega$. This would mean that we would have to maintain temperatures $< 10^{-1}$ nano-Kelvins in the presently available traps. The other possibility would be to make the traps more than 10 times smaller. Is this possible? We do not know.

6. Long range correlations

The second example we want to consider is the system characterized by multiplicity distribution which reveals important long-range correlations. Let us thus consider a distribution of particles in M bins given by

$$P(n_1,...,n_M) = \int dtF(t)\mathrm{e}^{-Nt}\prod_{i=1}^{M}\left(\frac{(t\omega_i)^{n_i}}{n_i!}\right)\,, \tag{29}$$

where N is the total (average) multiplicity and ω_i is the average multiplicity in the bin i. $F(t)$ is the KNO function [15]. As one sees, this is just a superposition of convolutions of Poisson distributions describing the physics where the total multiplicity (summed over all bins) is distributed according to

$$P(n) = \int dtF(t)\mathrm{e}^{-Nt}\frac{(tN)^n}{n!} \tag{30}$$

and there are no other correlations of shorter range.

Our problem is to calculate C_l, *i.e.*

$$C_l = \sum_{n_1,...,n_M}[P(n_1,...,n_M)]^l. \tag{31}$$

To this end we put multiplicities in all bins equal to each other $\omega_i \equiv \omega = N/M$ and obtain

$$C_l = \int dt_1F(t_1)...dt_lF(t_l)\mathrm{e}^{-N(t_1+...+t_l)}\left[G_l\left(l\omega(t_1...t_l)^{1/l}\right)\right]^M\,, \tag{32}$$

where

$$G_l(z) = \sum_n\left(\frac{z}{l}\right)^{nl}(n!)^l\,. \tag{33}$$

One sees from this definition that at small z

$$[G_l(z)]^M \approx 1 + M\left(\frac{z}{l}\right)^l . \tag{34}$$

This property allows us to determine the limit N fixed, $M \to \infty$. In this case $z \to 0$ and the result is

$$C_l = \left(\int dt F(t) \mathrm{e}^{-Nt}\right)^l \tag{35}$$

and thus

$$H_l = \frac{l}{1-l} \log\left(\int dt F(t) \mathrm{e}^{-Nt}\right) . \tag{36}$$

For negative binomial distribution

$$F(t) = \frac{k^k}{\Gamma(k)} t^{k-1} \mathrm{e}^{-kt} \tag{37}$$

the integral can be evaluated and the result is

$$H_l = \frac{kl}{l-1} \log\left(1 + \frac{N}{k}\right) . \tag{38}$$

This result is interesting for two reasons:

(a) It shows that the limit $M \to \infty$ does exist (except for the Shannon entropy, $l = 1$);

(b) At large N it gives the entropies proportional to $\log N$ and not to N. This is certainly the consequence of long-range correlations: For $k \to \infty$ we recover proportionality to N (again no surprise, because this limit corresponds to $F(t) = \delta(t-1)$, *i.e.* pure Poisson distribution with no long-range correlations [5].

For finite number of bins M, we were not able to find analytic results in closed form. Instead, we have carried out the numerical analysis for $l = 2$ and for negative binomial distribution (37). In this case, after some algebra, we obtain

$$C_2 = \left(\frac{k^k}{\Gamma(k)}\right)^2 \int dt_1 dt_2 (t_1 t_2)^{k-1} \mathrm{e}^{-k(t_1+t_2)} \mathrm{e}^{-N(\sqrt{t_1}-\sqrt{t_2})^2} \left[\hat{G}(2\omega\sqrt{t_1 t_2})\right]^M , \tag{39}$$

where

[5] The limit $M \to \infty$ at fixed N implies that we are dealing here with a "linearized" Poisson distribution. For a regular Poisson distribution in one bin one gets, for large N,$H_l = \frac{1}{2}\log(2\pi N) + \frac{1}{2}\frac{\log l}{l-1}$ (note a misprint in an analogous formula in [3]).

$$\hat{G}(z) = \mathrm{e}^{-z} I_0(z) \tag{40}$$

and I_0 is the Bessel function.

The double integral in (39) can be reduced to a single one by an appropriate change of variables (see Appendix B). The result is

$$C_2 = \left(\frac{k^k}{\Gamma(k)}\right)^2 \frac{4}{(2\omega)^{2k}} \int_0^\infty dx x^{2k-1} \mathrm{e}^{-wx} [\hat{G}(x)]^M \mathrm{e}^{\Omega x} K_0(\Omega x). \tag{41}$$

$$w = \frac{k}{\omega} = \frac{kM}{N}; \qquad \Omega = M + w\,. \tag{42}$$

This formula was investigated numerically for $k = 1, 2, 4$ and $\omega = 0.5, 4.0$: the second Renyi entropy, H_2, behaves for large N like $(\log N)$. More precisely, it reaches rather soon ($N \sim 30$–50) its asymptotic form (38)).

We thus conclude that in the presence of long-range correlations in momentum space one expects the Renyi entropies to follow the asymptotic behaviour $H_l \sim \log N$ and that this limit is obtained rather soon, *i.e.* for moderately large multiplicities.

7. Summary and conclusions

As shown recently [3, 4], the Renyi entropies provide a novel tool for investigation of multiparticle phenomena, particularly well suited for studying of the event-by-event fluctuations. Therefore, we found it interesting to investigate them for some idealized multiparticle systems in order to obtain an insight into the qualitative behaviour one may expect when actual measurements shall be performed.

In the present paper we present an evaluation of the Renyi entropies for

(a) ideal Bose gas in the limit of both very high and very low temperatures, and

(b) for a multiparticle system characterized by strong long-range correlations. We have studied the dependence of Renyi entropies H_l on particle multiplicity N and on the rank l.

The results of this exercise can be summarized as follows.

(i) If long-range correlations are absent, the Renyi entropy is proportional to N.

$$H_l \sim N \ , \tag{43}$$

If, however, the system reveals positive long-range correlations, (43) is no longer valid and one obtains instead

$$H_l \sim \log N \ , \tag{44}$$

at least for large enough N.

In case of Bose Gas in the limit of very high temperature (so that the masses and chemical potentials can be neglected, and N is determined by T) one obtains Renyi entropies simply proportional to the Shannon entropy. This implies also proportionality to the number of particles:

$$H_l = \frac{1}{4}\left(1 + \frac{1}{l} + \frac{1}{l^2} + \frac{1}{l^3}\right) S; \qquad S \sim N \, . \tag{45}$$

At low temperatures, the behaviour of the Bose gas depends crucially on the nature of the energy spectrum. If the spectrum is discrete, the (positive) long-range correlations appear when all particles fall to the lowest energy state (the multiplicity distribution corresponds to the negative binomial one with $k = 1$). This is just the special case of the system discussed in Section 6 ($M = 1$, as all momenta are the same) and thus the entropy is proportional to the logarithm of the number of particles. The same is true for the Renyi entropies. Moreover, in the limit of large N, the dependence of the Renyi entropies on the rank is only marginal.

For the continuous spectrum the behaviour is different, because at the critical temperature the number of bosons in the ground state vanishes. Thus no long-range correlations are present. For the gas in an infinite volume one obtains for $T < T_{\rm c}$

$$H_l(T) = \left(\frac{l^{\frac{5}{2}} - 1}{l^{\frac{5}{2}} - l^{\frac{3}{2}}}\right)\frac{2}{5} S(T) \tag{46}$$

with

$$S \sim N - N_{\rm c} \, , \tag{47}$$

where $N_{\rm c}$ is the number of particles in the ground state.

Similar situation is obtained for bosons in a harmonic trap, except that in this case the dependence on rank is the same as that for very high temperature *i.e.* given by (45).

Since for fixed temperature $N_c \sim N$, in this case all Renyi entropies are proportional to the average number of particles.

We also considered a distribution of particles partly motivated by what is qualitatively known from the existing measurements and analysis of multiparticle production processes [1]. It exhibits positive long-range correlations and no other correlations of shorter range[6]. In this case one can study the dependence of Renyi entropies on both the bin size (*i.e.* discretization procedure) and on the total number of particles. Our analysis shows that in this case Renyi entropies are proportional to $\log N$. For negative binomial distribution, in the limit of vanishing bin size one obtains a simple result

$$H_l = \frac{kl}{l-1} \log\left(1 + \frac{N}{k}\right). \tag{48}$$

Numerical studies have shown that this result holds also for finite bin size in a large interval of multiplicities.

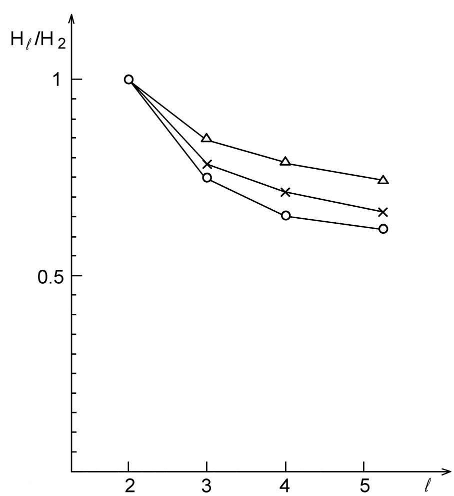

Fig. 1. Dependence of the Renyi entropies H_l on the rank l. Crosses: Photon gas, Eq. (11). Triangles: Nonrelativistic Bose Gas at low temperature (continuous spectrum), Eq. (21). Circles: Asymptotic formula (38).

[6] In realistic systems the short-range correlations are also present, so our analysis can at best be only a crude approximation.

(ii) The analytic form of the dependence of H_l on the rank l differs markedly from one system to another, as exhibited in Eqs. (45), (46) and (48). Numerical evaluation, however, shows that these differences are actually not so great, as seen in the Fig. 1. Thus rather precise measurements of Renyi entropies may be necessary to determine to which class the investigated system belongs.

A practical conclusion which follows from this discussion is that the dependence of Renyi entropies on particle multiplicity represents an important feature of a multiparticle system. Proportionality to N indicates an equilibrated system with no strong long-range correlations. On the other hand, for non-equilibrated systems and/or superposition of system with different properties, proportionality to $\log N$ seems to be a more natural behaviour. It would be very interesting to observe these differences in nature. One may speculate, *e.g.*, that they should show up in comparison between nucleon-nucleon and the central nucleus-nucleus collisions (if equilibration of the system does indeed occur in the latter case).

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Appendix A

To obtain (18) we observe that with the help of (5)–(8) we can write

$$H_l = \frac{1}{1-l}\left[l\sum_{m=1}^{M}\log(1-x_m(T)) - \sum_{m=1}^{M}\log\left(1-x_m\left(\frac{T}{l}\right)\right)\right] . \qquad (49)$$

On the other hand the same formulae lead to

$$\begin{aligned} S &= -\sum_{m=1}^{M}\left[\log(1-x_m)+\frac{x_m}{1-x_m}\log x_m\right] \\ &= -\sum^{M}\log(1-x_m)-\frac{\mu N}{T}+\frac{E}{T} . \end{aligned} \qquad (50)$$

Thus

$$-\sum_{m=1}^{M}\log(1-x_m(t)) = S(T)+\frac{\mu N(T)}{T}-\frac{E(T)}{T} . \qquad (51)$$

Employing (51) in (49) we obtain (18).

Appendix B

To calculate C given by (39) we perform the change of variables

$$x = 2\omega\sqrt{t_1 t_2}; \quad y = \omega(\sqrt{t_1} - \sqrt{t_2})^2 \,. \tag{52}$$

The Jacobian is

$$dt_1 dt_2 = \frac{x dx}{2\omega^2}\frac{dy}{\sqrt{y(y+2x)}} \tag{53}$$

so that the integral (39) becomes

$$C_2 = \left(\frac{k^k}{\Gamma(k)}\right)^2 \frac{4}{(2\omega)^{2k}} \int\limits_0^\infty dx x^{2k-1} \mathrm{e}^{-wx} [\hat{G}(x)]^M \int\limits_0^\infty \frac{dy}{\sqrt{y(y+2x)}} \mathrm{e}^{-\Omega y} \,, \tag{54}$$

where w and Ω are given by (42).

The next point is to employ the formula (Gradstein, 3.364.3)

$$\int\limits_0^\infty \frac{dy}{\sqrt{y(y+2x)}} \mathrm{e}^{-\Omega y} = \mathrm{e}^{\Omega x} K_0(\Omega x) \,. \tag{55}$$

Using this we obtain (41)

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