

PHYSICS FROM BOSE–EINSTEIN CORRELATIONS IN
HIGH ENERGY MULTIPARTICLE PRODUCTION* **

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Bose–Einstein correlations are being exploited to obtain information about the structure of the sources of hadrons in multiple particle production processes. In this paper the principles of this approach are described and some of the controversies about their implementation are discussed.

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1. Introduction

Bose–Einstein correlations among the momenta of identical particles produced in a high energy multiple particle production process yield information about the structure of the source of hadrons in the process. This has been pointed out in the very first paper on these correlations [1]. In a recent Physics Report issue [2] Wiedemann and Heinz write: *Two-particle correlations provide the only known way to obtain directly information about the space-time structure of the source from the measured particle momenta.* Thus the problem of extracting as well as possible the information about the source from the measured correlations is of great importance. It is, however, not an easy problem. One of the founders of this field of research, Goldhaber wrote in 1990 [3]: *What is clear is that we have been working on this effect for thirty years. What is not as clear is that we have come much closer to a precise understanding of the effect.* It is unlikely that the progress made during the last ten years would make G. Goldhaber give a much more optimistic view.

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There is a great variety of multiple particle production processes. In order to illustrate this point we will describe two well-known cases. In an electron–positron annihilation at high energy, at first usually a quark–antiquark pair is produced. These partons radiate gluons. The gluons radiate further gluons, or go over into quark–antiquark pairs. After some steps of this cascade, in a process known as hadronization and not well understood, the partons combine into colour-neutral hadrons. The hadrons are collimated into two narrow jets pointing in opposite directions along the same straight line. A splitting of the jets into more jets is also possible. There is an alternative way of looking at this process. The first generation quark and antiquark are the ends of a colour string. As they fly away the string stretches. After some time the string breaks. At the breaking point a quark antiquark pair is formed, so that each of the pieces of the string is again a string with a quark at one end and an antiquark at the other. The string pieces break again and finally short strings appear, which go over into hadrons. It is natural to expect that there is a time scale τ for the hadronization process. Since, however, the system is highly relativistic, it is necessary to specify in which frame this time should be measured. We choose the centre-of-mass frame and assume that τ is the longitudinal proper time at the creation of the hadron defined by

$$\tau = \sqrt{t^2 - z^2}, \quad (1)$$

where t is the centre-of-mass time, when the hadron was created, and z is the corresponding coordinate measured along the jet direction (the transverse dimensions are less important). An estimate of the velocity of the hadron is z/t . These formulae have an interesting implication. Hadron production begins at $t = \tau$ and first slow particles close to the interaction point are produced ($|z|$ small). Only later and further from the interaction point (both t and $|z|$ large) do the fast hadrons appear. This production mechanism is known as the inside-outside-cascade [4].

A very different multiple particle production process are the central heavy-nucleus–heavy-nucleus collisions, known also as central heavy ion collisions. Here the usual picture is that of two spheres Lorentz-contracted into coaxial discs — we consider the centre-of-mass system — penetrating through each other. When the discs fly apart, many strings are simultaneously stretched in a tube with a transverse radius of the order of the radii of the colliding nuclei. For heavy nuclei and high energies the strings are so numerous that they merge, *e.g.* into a quark gluon plasma. Then another poorly understood process, known as freeze out, converts the plasma (or whatever is the intermediate state) into hadrons.

There are many obvious questions to ask. What is the transverse radius of the tubular (?) region, where the hadrons are created? One would expect

about one fermi or less for e^+e^- annihilations and several fermi for heavy ion collisions. What is the formation time t , which elapses between the moment of collision and the moment, when the last hadron is produced directly? What is the time Δt between the direct production of the first hadron and of the last? There are also many model dependent questions. In thermodynamical models one asks about the temperature, in hydrodynamic models about the velocity of the collective flow *etc.*

Let us review now the main results of the famous GGLP paper [1]. Even if one does not quite share G. Goldhaber's opinion quoted above, this is certainly a very important paper and some familiarity with its content is necessary for any discussion of the Bose–Einstein correlations in multiparticle production processes.

2. The GGLP contribution

Consider two π^+ -s with momenta \mathbf{p}_1 and \mathbf{p}_2 produced at \mathbf{r}_1 and \mathbf{r}_2 . If the pions were distinguishable, the probability amplitude to observe them both at \mathbf{r} would be a product of the single particle contributions:

$$A_{\text{Dis}}(\mathbf{r}) = \exp[i\phi_1 + i\mathbf{p}_1 \cdot (\mathbf{r}_1 - \mathbf{r})] \exp[i\phi_2 + i\mathbf{p}_2 \cdot (\mathbf{r}_2 - \mathbf{r})]. \quad (2)$$

In each of the square brackets, ϕ_i is the phase acquired by the particle at birth and the other term is the phase accumulated while propagating from \mathbf{r}_i to \mathbf{r} with momentum \mathbf{p}_i . Since, however, the two pions are identical bosons, it is mandatory to symmetrize the amplitude and a more realistic formula is

$$A_{\text{Und}}(\mathbf{r}) = \frac{1}{\sqrt{2}} \exp[i(\phi_1 + \phi_2) + i(\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{r}] \\ \times [\exp[i(\mathbf{p}_1 \cdot \mathbf{r}_1 + \mathbf{p}_2 \cdot \mathbf{r}_2)] + \exp[i(\mathbf{p}_1 \cdot \mathbf{r}_2 + \mathbf{p}_2 \cdot \mathbf{r}_1)]] . \quad (3)$$

Physically this means that we have added coherently the contribution corresponding to the possibility that the two pions have exchanged their birth points. Classically for different momenta and given birth points, if before the exchange the two pions can reach the point \mathbf{r} , then in general after the exchange they must miss it. One should keep in mind, however, that the distance between the points \mathbf{r}_1 and \mathbf{r}_2 is of the order of a fermi, while the distance between either of them and the point \mathbf{r} is of the order of a meter. Classically this would not help, but quantum-mechanically in this situation the probability of reaching \mathbf{r} by both pions is in the two cases the same for all practical purposes. For comparison with experiment the result should be averaged over all the possible pairs of points $\mathbf{r}_1, \mathbf{r}_2$. By averaging the amplitude (3) nothing interesting is obtained. Therefore, GGLP assumed that one should average the square of the absolute value

of the amplitude (3). Physically this means that the contributions from all the pairs of points $\mathbf{r}_1, \mathbf{r}_2$ add incoherently. Then the probability of finding a pair of pions with momenta $\mathbf{p}_1, \mathbf{p}_2$ is

$$\langle |A_{\text{Und}}|^2 \rangle = 1 + \langle \cos[\mathbf{q} \cdot (\mathbf{r}_1 - \mathbf{r}_2)] \rangle, \quad (4)$$

where $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2$ and the Dirac brackets $\langle \dots \rangle$ denote averaging, with a suitable weight, over all the pairs of points $\mathbf{r}_1, \mathbf{r}_2$. Using as an example a Gaussian weight function

$$\rho(\mathbf{r}_1, \mathbf{r}_2) = (2\pi R^2)^{-3} \exp[-(r_1^2 + r_2^2)/(2R^2)], \quad (5)$$

where R is a constant with the dimension of length, GGLP found

$$\langle |A_{\text{Und}}|^2 \rangle = 1 + \exp[-\mathbf{q}^2 R^2]. \quad (6)$$

The parameter R can be interpreted as the radius of the sphere, where the pions are produced. Thus, finding R from a fit to the experimental data yields the size of the hadronization region. Note that formulae qualitatively similar to (6) hold for a broad class of weight functions. For $\mathbf{q}^2 = 0$ the cosine being averaged equals one whatever are the points \mathbf{r}_1 and \mathbf{r}_2 . Thus the right hand side of (6) for $\mathbf{q}^2 = 0$ must be equal two. For large values of \mathbf{q}^2 , the cosine is a rapidly oscillating function of the difference $\mathbf{r}_1 - \mathbf{r}_2$. For smooth weight functions, therefore, its average is very small and the right hand side of (6) equals approximately one. If the weight function contains only one parameter with the dimension of length, let us denote it R , the width of the region in \mathbf{q}^2 over which the right hand side of (6) drops from the value 2 to, say, 1.5 must be proportional to R^{-2} for purely dimensional reasons. Thus it is not difficult to improve over the Gaussian Ansatz. There is a problem, however.

It is certainly not true that the probability of producing two identical pions with momenta \mathbf{p}_1 and \mathbf{p}_2 depends only on the momentum difference \mathbf{q} . GGLP could have tried to save the theory by introducing some more complicated weight functions, which would depend also on the momenta. They found, however, a much simpler and more brilliant solution. Note that in the model the squared modulus of the amplitude for distinguishable particles is a constant. Thus one may claim that the right hand side of (6) is not the two-particle probability distribution, but the ratio

$$C_2(\mathbf{p}_1, \mathbf{p}_2) = \frac{\rho(\mathbf{p}_1, \mathbf{p}_2)}{\rho_{\text{Dis}}(\mathbf{p}_1, \mathbf{p}_2)}. \quad (7)$$

Here $\rho(\mathbf{p}_1, \mathbf{p}_2)$ is the experimentally observed two-particle distribution and $\rho_{\text{Dis}}(\mathbf{p}_1, \mathbf{p}_2)$ is the distribution, which would be observed, if the identical

pions were distinguishable, *i.e.* if the Bose–Einstein symmetrization were switched off. Of course the numerator of this expression cannot be measured, which has led to much discussion as described in the following section. This is the famous normalization problem. GGLP chose again a very simple solution. They substituted for ρ_{Dis} the distribution of π^+, π^- pairs, where of course symmetrization does not occur. The experimental results for the ratio C_2 could be fitted reasonably well with formula (6) and thus both the apparent attraction in momentum space among pions of the same charge got explained and an estimate of the radius R was given.

Let us mention one more fruitful idea from the GGLP paper. The right-hand side of formula (6) depends on the Lorentz frame, where the momenta are measured. Choosing the rest frame of the pair, where the energies of the two pions are equal, we can rewrite the result in a covariant form:

$$C_2(\mathbf{p}_1, \mathbf{p}_2) = 1 + \exp[Q^2 R^2], \quad (8)$$

where Q^2 is the square of the four-vector $p_1 - p_2$.

3. Normalization

The problem of finding the best denominator for the function C_2 has attracted much attention. Theorists often suggest to replace $\rho_{\text{Dis}}(\mathbf{p}_1, \mathbf{p}_2)$ by the product of single particle distributions $\rho_1(\mathbf{p}_1)\rho_1(\mathbf{p}_2)$. With this choice, the function C_2 becomes the familiar, standard two-particle correlation function. Moreover, in many models terms cancel between the numerator and the denominator making the formula simpler. One may also notice that a two-particle density is normalized to $\langle n(n-1) \rangle$, while the single particle distribution is normalized to $\langle n \rangle$. Thus a better choice of the denominator might be to multiply the product of single particle distributions by $\langle n(n-1) \rangle / \langle n \rangle^2$. The advantages and disadvantages of using this factor have been recently discussed in [5]. Since it does not depend on q^2 , usually it does not have much effect on the parameters of the source found from fits. The identification of C_2 with the standard correlation function, in spite of its advantages, is not very popular with experimentalists. In order to explain the reason let us consider the following simple example. Consider a high-energy reaction, where the two initial particles go over into two well collimated jets. By momentum conservation the two jets must be back to back in their centre-of-mass system. Let us assume that the orientation of their common axis can point with equal probability in any direction — is isotropic. Then the single particle distribution of momentum is also isotropic. On the other hand the opening angle between the momenta of two particles is either small, when the two particles are extracted from the same jet, or close to π if the two particles are from different jets. Thus the correlation function exhibits

a very large peak for $\theta \approx 0$. This peak has, of course, nothing to do with Bose–Einstein correlations. Experimentalists would prefer a definition of C_2 , where the forward peak reflects Bose–Einstein correlations and nothing else, because this makes the interpretation much easier. Sometimes a compromise is chosen. For instance in e^+e^- annihilations, where the situation is similar to that from our example, the z -axis is often chosen along the jet axis and not along a direction fixed in the laboratory frame. With this choice the single particle distribution becomes also strongly peaked for small angles θ and in the correlation function the bumps due to the two-jet structure of the events are largely eliminated.

The most popular choices, however, are improvements over the choice of GGLP. For instance one uses “mixed” samples, where ρ_{Dis} is the distribution of pairs of π^+ -ses, but with each π^+ taken from a different event. Sometimes Monte Carlo generated samples are used, with Monte Carlo generators which do not include Bose–Einstein correlations. This procedure is not very safe, because such generators contain a number of free parameters, which are fitted to the data, where the Bose–Einstein correlations are present. One also uses ratios of functions C_2 obtained from the data to functions C_2 obtained according to the same prescription from Monte Carlo. The wide variety of methods of calculating the denominator of the function C_2 is one of the reasons, why the comparison of results from different experimental groups is very difficult. This is, however only part of the story. Some groups correct for final state interactions (mostly coulombic) and/or resonances, others do not. Various cuts defining the data samples are used. Some groups assume that every negative particle is a π^- , while other have particle identification. Because of all that great care is necessary when interpreting the experimental results and their stated errors. This difficulty has been known for a long time *cf. e.g.* [6]. For a more recent (pessimistic) review *cf.* [7].

4. Beyond spherical symmetry

The GGLP weight factor is a function of $r^2 = x^2 + y^2 + z^2$ and, therefore, it is spherically symmetric. It is natural to replace r^2 by an arbitrary quadratic form in x, y, z , provided the eigenvalues are positive so that the weight function can be normalized to unity. Performing the averaging of the cosine one obtains

$$C_2(\mathbf{q}, \mathbf{K}) = 1 + \lambda(\mathbf{K}) \exp \left[- \sum_{i,j=1}^3 R_{ij}^2(\mathbf{K}) q_i q_j \right]. \quad (9)$$

Here besides abandoning spherical symmetry two improvements have been introduced. In agreement with experimental observation (*cf. e.g.* [6])

a factor $0 < \lambda \leq 1$ has been included and a dependence of all the coefficients on $\mathbf{K} = (\mathbf{p}_1 + \mathbf{p}_2)/2$ has been allowed. The coefficients denoted R_{ij}^2 do not have to be all positive. Out of the nine coefficients R_{ij}^2 , three are eliminated by the symmetry condition $R_{ij}^2 = R_{ji}^2$. Moreover, choosing the y axis so that $K_y = 0$ and assuming reflection symmetry with respect to the x, z plane we have $R_{yz}^2 = R_{zy}^2 = 0$. Thus there are four independent coefficients left. The time component q_0 can be easily obtained from the identity $K^\mu q_\mu = (p_1^2 - p_2^2)/2 = 0$.

It is convenient to choose the z -axis in the longitudinal direction *i.e.* for central heavy ion collisions along the beam axis and for e^+e^- annihilations along the jet axis. The y -axis is perpendicular to the z -axis and to the vector \mathbf{K} . This fixes also the direction of the x -axis. With this choice, the x, y, z directions are often referred to as the *out* direction, the *side* direction and the *longitudinal* direction respectively. The R^2 parameters are denoted R_s^2, R_0^2, R_l^2 and R_{0l}^2 [8,9]. There were speculations that the study of R_{out}/R_s could give clues as to whether there is quark-gluon plasma and/or collective flow in the system [8], but the results have not been conclusive and a complete study of all the parameters seems now to be the best strategy. Other choices of parameters are also possible. For instance one can put

$$C_2(\mathbf{q}, \mathbf{K}) = 1 + \lambda \exp[-R_x^2 q_x^2 - R_y^2 q_y^2 - R_z^2 q_z^2 - T^2 q_0^2] \quad (10)$$

[10], or

$$C_2(\mathbf{q}, \mathbf{K}) = 1 + \lambda \exp[-R_T^2 q_T^2 - R_{||}^2 (q_L^2 - q_0^2) - (R_0^2 + R_{||}^2) \gamma^2 (q_0^2 - v q_z^2)], \quad (11)$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2}}, \quad (12)$$

and $u = \gamma(1, \mathbf{0}_T, v)$. This parametrization proposed in [11] and improved in [12] is particularly popular and is often referred to as the YKP parametrization.

5. Time dependence

In the GGLP picture the production of all the hadrons was instantaneous at some time t_0 . Formally this assumption is difficult to disprove. Choosing the time t_0 after all the hadrons have been produced and interacted and before the time when they were observed, one can calculate the distributions at the observation time using the state at time t_0 as the initial condition. Whether the hadrons existed before time t_0 , is irrelevant for this calculation. In particular, one may assume that all the hadrons were created at time t_0 .

Guessing the initial condition at time t_0 is, however, very hard in this approach. Therefore, it is more practical to choose a more realistic conjecture about the origin of the hadrons, because then the initial conditions are more natural and easier to guess.

In particular, several authors (*cf. e.g.* [13–15]) assumed that the production of hadrons is from sources, which fly away from each other and such that in the rest frame of a source the production of hadrons is isotropic. Consider the simplest case of just two sources. If their relative velocity is large and the momenta of the hadrons in the rest frames of the corresponding sources are moderate, then it is very unlikely that two identical pions from different sources have momenta close to each other. On the other hand, all the information about the structure of the source comes from pairs with small momentum differences $|\mathbf{q}|$. For large values of $|\mathbf{q}|$ function C_2 is flat and carries no information. Consequently, the observed function C_2 contains only information about the single sources and no information about the distance between them. One finds that the effective production region is spherical in its rest frame, while the actual production region composed of the two sources is elongated. This is an important piece of information. What we observe is not the total size of the hadronization region, but the average size of the so-called regions of homogeneity [16], *i.e.* regions, where hadrons with similar momenta are produced. For this reason the effective hadronization regions observed in experiment are approximately spherical, while we expect that at high collision energy the actual production regions are strongly elongated, because of their stringy origin.

There is one more interesting result connected with the problem of the time span of the hadronization process [8]. Consider instantaneous hadronization from a spherical shell of thickness δR . The length δR should be reflected in the momentum correlations along the direction x_{out} , because the particles, which come from the inner part of the shell, are out of phase with those, which come from the outer part of the shell. The same effect can be obtained, however, if some particles are produced later than others. From the experimental fact that R_{out} is not particularly large, one concludes that the hadronization process does not last very long. This excludes models, where hadrons are produced from the quark-gluon plasma in a first order phase transition with a large latent heat. This process would be too slow to be made consistent with the observed moderate time interval of the hadron production.

6. More quantum physics

The formulation of the GGLP model is quasiclassical — one talks about a pion with momentum \mathbf{p} created at point \mathbf{r} , which is not quantum mechanics. Much work has been done on formulations consistent with quantum mechanics. When dealing with incoherent superpositions of states one should use density matrices or density operators. The starting point may be the density matrix in coordinate representation $\rho(\mathbf{x}, \mathbf{x}', t)$ or in momentum representation $\rho(\mathbf{p}, \mathbf{p}', t)$. Often it is convenient to replace the vectors \mathbf{a}, \mathbf{a}' by their linear combinations

$$\mathbf{a}_+ = \frac{1}{2}(\mathbf{a} + \mathbf{a}'); \quad \mathbf{a}_- = (\mathbf{a} - \mathbf{a}'). \quad (13)$$

In particular, the GGLP results can be derived from the density matrix

$$\rho(\mathbf{p}, \mathbf{p}', t) = \int d\mathbf{r} \langle \mathbf{p} | \mathbf{r} \rangle \rho(\mathbf{r}) \langle \mathbf{r} | \mathbf{p}' \rangle. \quad (14)$$

This incidentally shows that in spite of its pseudoclassical formulation the GGLP model can be translated into respectable quantum mechanics.

The density matrices, however, do not combine explicitly the information about the space distribution of sources and the momentum distribution of the final pions. Therefore, in order to derive the space distribution of sources from the momentum distributions of the observed final particles other approaches have been proposed.

One can use the Wigner function W defined in term of the density matrix by the formula

$$W(\mathbf{p}_+, \mathbf{x}_+) = \int d\mathbf{p}_- e^{i\mathbf{p}_- \cdot \mathbf{x}_+} \rho(\mathbf{p}, \mathbf{p}', t). \quad (15)$$

The properties of the Wigner functions are described in detail in the famous review article [17]. For a recent application to the description of multiple production of identical particles *cf.* [18]. In a well defined sense [17] Wigner's function is the best quantum mechanical analogue of the classical phase space distribution. In this formulation Heisenberg's uncertainty principle is easily implemented. The density in phase space should not be too large. Quantitatively the condition is

$$|W(\mathbf{p}_+, \mathbf{p}_-, t)|^2 \leq (\pi\hbar)^{-3n}, \quad (16)$$

where $3n$ is the dimension of the vectors \mathbf{p}_\pm , or equivalently n is the number of particles described by the Wigner function W .

An alternative approach is to introduce the classical position and momentum vectors $\boldsymbol{\xi}, \boldsymbol{\pi}$ besides the quantum mechanical position and momentum

vectors \mathbf{x}, \mathbf{p} . Each pair of vectors $\boldsymbol{\xi}, \boldsymbol{\pi}$ defines a quantum mechanical wave packet *e.g.*

$$\langle \mathbf{x} | \boldsymbol{\xi}, \boldsymbol{\pi} \rangle = \left(\frac{\sigma^2}{\pi} \right)^{\frac{3}{4}} \exp \left[\frac{1}{2} (\mathbf{x} - \boldsymbol{\xi})^2 + i \boldsymbol{\pi} \cdot \mathbf{x} \right], \quad (17)$$

or equivalently in momentum representation

$$\langle \mathbf{p} | \boldsymbol{\xi}, \boldsymbol{\pi} \rangle = \left(\frac{1}{\pi} \right)^{\frac{3}{4}} \exp \left[\frac{1}{2\sigma^2} (\mathbf{p} - \boldsymbol{\pi})^2 + i \boldsymbol{\xi} \cdot (\boldsymbol{\pi} - \mathbf{p}) \right]. \quad (18)$$

Coherent, or incoherent superpositions of such wave packets are legal quantum mechanical states. On the other hand, the values of \mathbf{x} and \mathbf{p} are constrained to be close to the values of $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$. If the distribution of the parameters $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$ is calculated from some classical model, the result is directly translated into a distribution of the observables \mathbf{x} and \mathbf{p} , which is consistent with quantum mechanics. This approach was pioneered in Ref. [19]. For a recent application and detailed discussion *cf.* [20].

Still another approach is to introduce a source function S related to the density matrix by the formula

$$\rho(p, p') = \int d^4x \exp[ip_- x_+] S(x_+, p_+). \quad (19)$$

In this formula x, x', p, p' are four-vectors. The similarity of this formula to the formula relating Wigner's function to the density matrix caused that the source function is often referred to as a Wigner function, a kind of Wigner function, a pseudo Wigner function *etc.* (*cf. e.g.* [21,22]). In fact the relation between the source function and Wigner's function is not unique and may be quite complicated. We will not discuss this problem here, but let us note that the density matrices and Wigner's function were calculated at a given time, while here the time dependence must be known. In simple cases the time dependence of a Wigner function can be found and used (*cf. e.g.* [23]), but it is quite complicated and unlikely to be found by a simple guess. What is more, the source function depends on two time arguments.

There is one case, however, when the source function has an easy interpretation. If the pions are produced by classical currents J , then [21,24]

$$S(x_+, p_+) = \int \frac{d^4x_-}{2(2\pi)^3} e^{-ip_+ x_-} \langle J^*(x) J(x') \rangle, \quad (20)$$

where the averaging is over all the incoherent components of the currents $J(x)$.

7. Momentum–position correlations

In the GGLP model and in many others the momentum spectrum of the produced particles does not depend on the position of the source. When momentum–position correlations are introduced in the model, unexpected results may be obtained. Let us consider for example quasiclassical sources with the single particle position-momentum distribution

$$\rho(\mathbf{x}, \mathbf{p}) = \sqrt{2\pi R^2}^3 \delta(\mathbf{p} - \lambda \mathbf{r}) \exp \left[-\frac{r^2}{2R^2} \right]. \quad (21)$$

Correlations of the type $\mathbf{p} \sim \mathbf{r}$ occur in classical versions of various models, *e.g.* in string models and in models, where hadronization is preceded by a rapid flow of hot matter. The average of the cosine from the interference term in the present generalized GGLP model is

$$\langle \cos[\mathbf{q} \cdot (\mathbf{r}_1 - \mathbf{r}_2)] \rangle = \lambda^6 \exp \left[-\frac{\mathbf{K}^2}{\lambda^2 R^2} \right] \exp \left[-\frac{\mathbf{q}^2}{4\lambda^2 R^2} \right] \cos \left[\frac{\mathbf{q}^2}{\lambda} \right]. \quad (22)$$

In comparison with the GGLP result for the Gaussian weight function ρ , two changes are striking. A dependence on the sum of momenta \mathbf{K} has appeared and the dependence on the momentum difference \mathbf{q} is no more Gaussian. For non-Gaussian distributions the effective radius of the distribution of sources in space is usually defined by

$$R_{\text{eff}}^2 = - \left(\frac{d}{d\mathbf{q}^2} \langle \cos[\dots] \rangle \right)_{\mathbf{q}^2=0}. \quad (23)$$

For the GGLP model with a Gaussian weight function this reproduces the usual result $R_{\text{eff}} = R$. In the present case, however, we obtain

$$R_{\text{eff}} = \frac{1}{2\lambda R}. \quad (24)$$

Due to the position-momentum correlations, R_{eff} becomes inversely proportional to R !

A more realistic model with similar correlations can be built as follows [25, 26]. For the source function we choose

$$S(x, p) = \delta(t^2 - z^2 - \tau_0^2) \exp \left[\frac{(p - \lambda x)_+ (p - \lambda x)_-}{2\delta_{\parallel}^2} - \frac{(\mathbf{p} - \lambda \mathbf{x})_{\text{T}}^2}{2\delta_{\text{T}}^2} - \frac{\mathbf{x}_{\text{T}}^2}{2R^2} \right]. \quad (25)$$

This formula has a simple physical interpretation. The subscript T denotes the vector component transverse with respect to the beam axis z . The last

term in the exponent implies that the particles are created not too far from the beam axis, typical distances being of the order of the parameter R . The first two terms in the exponent impose the condition $\mathbf{p} \approx \lambda \mathbf{x}$. The subscripts \pm refer to the four-vector components $a_0 \pm a_z$. The dispersion of the transverse components is of the order of δ_{\perp}^2 and that of the longitudinal component of the order of δ_{\parallel}^2 . The Dirac delta implies that for each drop of the hot matter hadronization takes place after the same longitudinal invariant time τ_0 . It is useful that for this source function the corresponding density matrix in the momentum representation can be calculated in closed form. One finds in particular

$$|\rho(\mathbf{K}, \mathbf{q})|^2 \sim \left(\delta_{\perp}^2 + \frac{R^2 M_{\perp}^2}{\tau_0^2} \right) \exp \left[-\frac{\mathbf{K}_{\perp}^2 + R^2 \delta_{\perp}^2 \mathbf{q}_{\perp}^2}{\delta_{\perp}^2 + R^2 M_{\perp}^2 \tau_0^{-2}} \right] \exp \left[\frac{2M_{\perp}^2}{\delta_{\parallel}^2} \right] |K_0(\alpha)|^2, \quad (26)$$

where K_0 is the modified Bessel function,

$$\alpha^2 = \frac{M_{\perp}^4}{\delta_{\parallel}^4} - \frac{\tau_0^2 (\mathbf{K}_{\perp} \cdot \mathbf{q}_{\perp})^2}{M_{\perp}^2} - \frac{\tau_0^2 m_{1T}^2 m_{2T}^2}{4M_{\perp}^2} \sinh^2(y_1 - y_2) + \frac{2i\tau_0 M_{\perp} \mathbf{K}_{\perp} \mathbf{q}_{\perp}}{\delta_{\parallel}^2}, \quad (27)$$

and $m_{iT}^2 = m_{\pi}^2 + \mathbf{p}_{iT}^2$ and $M_{\perp}^2 = K_{\perp}^2 + \mathbf{K}_{\perp}^2$ are transverse masses of the two particles and of the pair. The single particle distribution can be calculated from the formula $\rho_1(\mathbf{p}) = \rho(\mathbf{p}, 0)$.

This model was found to reproduce reasonably well the data for single particle distributions and for Bose–Einstein correlations in e^+e^- annihilations at LEP energies [27].

8. Final state interactions and (partial) coherence

In the GGLP model and in many later models, after hadronization the pions propagate as free particles. In fact we know, that the majority of pions is generated in resonance decays. Moreover, there are the nonresonant strong and the electromagnetic interactions among the pions. All that has been studied for years, but many problems are still controversial.

Strong nonresonant interactions are usually neglected, but it has been pointed out [22] that absorption of the produced pions can lead to a decrease of the effective radius R_{eff} with increasing momentum of the pair $|\mathbf{K}|$. The effect of resonances seems to be much more important. The long lived resonances, mostly η and η' , usually decay far from the centre of the interaction region. Therefore they simulate a large hadronization region and produce a narrow peak in the plot of C_2 versus \mathbf{q}^2 . This peak is narrower than the experimental resolution and consequently its main effect is to reduce the parameter λ . The resonances with life times neither very long, nor very

short, like the ω resonance, produce for small \mathbf{q}^2 a steep rise of C_2 with decreasing \mathbf{q}^2 . Short lived resonances, like the ρ meson, increase only slightly the measured radius of the interaction region. It has been suggested [7,28] that in e^+e^- annihilation it may be difficult to explain why after correcting for the resonance effects the Bose–Einstein correlations remain as strong as observed in experiment.

Coulomb interactions can be easily included by replacing in the description of the propagation of the pions the plane waves by Coulomb wave functions, which leads to the introduction of the so called Gamow factor [19]. This is now known to grossly overestimate the effect (for a review *cf.* [29]). The physical reason is that the introduction of the Coulomb wave functions describes the evolution of an isolated pair of pions, while in reality there are many other pions around, which partly screen the Coulomb interaction within the pair. A direct experimental argument is that the reasoning leading to the Gamow factor, when applied to $\pi^+\pi^-$ pairs, gives a strong attraction, which is not seen in the data. A simple way of correcting for the screening effect is given in Ref. [30] (see also [29]). A screening radius r_0 is introduced. The interaction potential of the pair is continuous at $r = r_0$, Coulomb for $r > r_0$ and constant for $r < r_0$. The parameter r_0 is chosen so as to reproduce correctly the experimental data for $\pi^+\pi^-$ pairs. Coulomb corrections calculated in this way are rather small.

Another assumption of GGLP, which has been put into doubt, was that the production process is completely incoherent. Complete coherence would kill the effect, but some degree of coherence seems difficult to avoid in realistic models. It is easy to write down general formulae including partial coherence (*cf. e.g.* [19]), but it is not clear how to use them fruitfully. At a time it was suggested that the parameter λ measures the degree of coherence ($\lambda = 1$ no coherence, $\lambda = 0$ complete coherence), but now it is clear that this parameter is strongly affected by the dynamics of the production process, in particular by resonance production, and by experimental conditions (*e.g.* particle misidentification).

9. Two recent models

In order to illustrate how Bose–Einstein correlations are nowadays analysed, we present two recent models. The first (*cf.* [2] and references quoted there) is based on analogies with hydrodynamics and thermodynamics. It is being used to describe central heavy ion collisions. The starting point is the (single particle) source function

$$S(x, p) \sim m_T \cosh(y - \eta) \exp \left[\frac{m_T \cosh y \cosh \eta_T(r_T) - p_T \frac{x}{r} \sinh \eta_T(r_T)}{T} \right] \exp \left[-\frac{r_T^2}{2R^2} - \frac{\eta^2}{2(\Delta\eta)^2} - \frac{(\tau - \tau_0)^2}{2(\Delta\tau)^2} \right]. \quad (28)$$

In this formula \sim means that a normalization constant has been omitted, y is the rapidity along the z axis *i.e.* along the beam direction. The pseudo-rapidity

$$\eta = \frac{1}{2} \ln \frac{t+z}{t-z}. \quad (29)$$

The distance from the z axis $r_T = \sqrt{x^2 + y^2}$. The rapidity of the transverse flow has been assumed in the form

$$\eta_T(r_T) = \eta_f \frac{r_T}{R}, \quad (30)$$

where η_f is a constant. The model contains six free parameters: $R, T, \eta_f, \Delta\eta, \tau_0$ and $\Delta\tau$. These parameters have been estimated by comparison with the data from the NA49 experiment for collisions of lead nuclei at an energy of 158 GeV per nucleon. Some of the results have interesting physical implications. The parameter R , interpreted as the transverse radius of the hadronization region, is about 7 fm. From the known size of the lead nucleus one could have expected a number about twice smaller. This means that there is substantial transverse spreading of the interaction region before hadronization takes place. The parameter T interpreted as the local temperature is about 130 MeV. This is less than the temperatures found in models used to calculate the chemical composition of the produced hadrons, which could mean that the spreading is accompanied by cooling. The parameter η_f is about 0.35. This is a very reasonable value. The velocity of sound in a plasma is about 1/3 (in units of the velocity of light in vacuum). The parameters τ_0 and $\Delta\tau$ are about 9 fm and 1.5 fm respectively. It means that the time span of hadronization is short compared to the time between the original interaction and the onset of hadronization. This would exclude models, where hadronization is a first order phase transition with a large latent heat, because in such models the time span of hadronization is large. One should keep in mind, however, the authors' warning that the determination of the parameter $\Delta\tau$ from the data is poor. This example shows that given a model one can extract from the data much interesting information. Little is known, however, about the model dependence of these results.

A very different picture of the Bose-Einstein correlations [31–34] has been inspired by a string model of the Lund type. This model is tailored

for e^+e^- annihilations. An annihilation is depicted as the formation of a string with a quark at one end and an antiquark at the other. This string stretches with the speed of light. Then somewhere along the string a quark–antiquark pair is produced and the string breaks. The pieces stretch and break again. Finally sufficiently short bits of strings hadronize. In the z, t plane the trajectories of the ends of all these strings form a closed contour. Let us denote the area enclosed by this contour by A . The key assumption is that the probability amplitude for a given final state is

$$M \sim \exp[i\xi A]. \quad (31)$$

We have not written explicitly the factor related to the transverse momenta of the produced particles. Denoting by $b/2$ the imaginary part of ξ one finds

$$|M|^2 \sim e^{-bA}. \quad (32)$$

This result is well known from the Lund model. For the description of the Bose–Einstein correlations, however, it is the real part of ξ which is the important one. It is expected to be of the order of the string tension *i.e.* of the order of 1 GeV/fm. The point is that the production amplitude has to be symmetrized with respect to exchanges of identical particles. Such an exchange, however, changes the area A and because of the nonzero real part of ξ the phase of the amplitude. Numerical calculations show that this model gives a reasonable description of the Bose–Einstein correlations in e^+e^- annihilations. This is very interesting, because this model, contrary to all the previous ones, does not contain incoherent components, random phases *etc.* It is curious what happens, when in a process more than one string is initially produced. In e^+e^- annihilations there are events, where two W bosons are produced, which implies two strings. It has been suggested [34] that in this case only pions from a single string should exhibit Bose–Einstein correlations. To be sure, nobody doubts that pions are bosons and that consequently their production amplitude should be suitably symmetrized, but the attraction in momentum space, known since the GGLP paper as Bose–Einstein correlations, requires in addition certain phase relationships, which may not be realized for pions originating from different strings. In central heavy ion collisions many strings are produced. By extension of the previous argument one could expect very weak Bose–Einstein correlations, which experimentally is not the case. This suggests that the state from which hadronization occurs in heavy ion collisions is not a bunch of strings. The strings must somehow merge and form a very different object, perhaps a volume of quark–gluon plasma.

10. Conclusion

Bose–Einstein correlations attract much interest. Many hundreds of papers have been published on this subject. In principle they offer the only access to some important information about the hadronization process. Typical questions are: what is the size and shape of the hadronization region, how is hadronization distributed in time, what is the coherence of the sources, what is the hadronization mechanism *etc.* In practice it is difficult to find model independent, definitive answers to these questions. The common approach is to study models, which either demonstrates the viability of certain scenarios, like the sting model discussed above, or give tentative answers to the questions concerning hadronization.

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