

SUPERCONDUCTOR-INSULATOR QUANTUM PHASE TRANSITIONS*

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(Received October 24, 2000)

In these lectures, superconductivity in impure thin films close to the absolute zero of temperature is discussed. The behavior as function of the applied magnetic field and the amount of impurities suggests the presence of a superconductor-insulator transition at zero temperature. The theory of superconductivity in the limit where all the electrons become tightly bound in pairs is used to explain the main characteristics of the transition. In that limit, where the theory becomes equivalent to a phase-only theory, electron pairs exist on either side of the transition. The presentation is pedagogical in nature and includes exercises as a learning aid for those new to the field.

PACS numbers: 74.40.+k, 71.30.+h, 64.60.Fr

1. Introduction

The topic of these lectures is superconductivity in impure thin films close to the absolute zero of temperature. Such superconducting films may by increasing either the amount of impurities, or the applied magnetic field go over to an insulating state. The transition is believed to signal the presence of a quantum critical point at zero temperature [1,2]. Quantum critical phenomena differ from conventional critical phenomena taking place at finite temperature, in that quantum rather than thermal fluctuations are important in the critical region. Since quantum phase transitions occur at zero temperature, they cannot be induced by changing the temperature as can

* Presented at the XL Cracow School of Theoretical Physics, Zakopane, Poland, June 3-11, 2000.

thermal phase transitions, and another parameter is to be varied to trigger the transition. It also means that one can never tune right through the transition as experiments are necessarily done at finite temperature. For low enough temperature though, the presence of a quantum critical point can nevertheless be detected by using finite-size scaling.

A superconducting and an insulating state seems an unlikely combination to be present in the same system. Whereas superconductivity needs an *attractive* interaction for the pairing between electrons, the insulating state needs a *repulsive* interaction. It is a priori not clear how these two requirements can be fulfilled in a single system. Adding to the bewilderment is the striking similarity in the current–voltage characteristics in both phases close to the transition. By interchanging the current and voltage axes in one phase, a characteristic obtained at some value of the applied magnetic field say, can be mapped onto a characteristic of the other phase obtained at a different value of the field. This means that, although the physical properties of the superconducting and the insulating state are completely different, there is a close connection between the conduction mechanisms in the two phases. A last defining aspect of the superconductor–insulator transition is the presence of a $1/r$ Coulomb potential, which at low charge-carrier densities becomes very strong.

In these lectures, we will argue that the main aspects of the superconductor–insulator transition just mentioned can be accounted for by a single theory. Namely, the theory of superconductivity in the limit where all the electrons become tightly bound in pairs — the so-called *composite boson limit* [3, 4]. This is the opposite limit of the conventional weak-coupling BCS limit, where only electrons in a thin shell around the Fermi surface become loosely bound in Cooper pairs. In the composite boson limit, the superconducting state displays a quantum phase transition to an insulating state characterized not by an unbinding of electron pairs, but rather by a quenching of the condensate of composite bosons. In other words, electron pairs exist on both sides of the transition. In the superconducting state they are Bose–Einstein condensed, while in the insulating state they are localized. The attractive interaction responsible for the binding of the electrons in pairs translates into a repulsive interaction between the composite bosons. The theory describing these bosons is the Bogoliubov theory of superfluidity, which is equivalent to a phase-only theory. This implies that the superconductor–insulator transition can be described by a phase-only effective theory without ignoring any degrees of freedom [5, 6].

To be fair, we should mention at this point that no consensus exists to what extent the phase-only theory can be applied to the superconductor–insulator transition. Further experimental and theoretical studies are required to settle this point.

1.1. General scaling

Since quantum rather than thermal fluctuations are relevant in a quantum phase transition, one has to work in spacetime rather than just in space as is appropriate for thermal phase transitions in equilibrium. As a result, in addition to a diverging correlation length ξ , quantum phase transitions have also a diverging correlation time ξ_t . They indicate, respectively, the distance and time period over which the system fluctuates coherently. The two are related, with the diverging correlation time scaling with the diverging correlation length as

$$\xi_t \sim \xi^z, \quad (1)$$

where z is the so-called dynamic exponent. It is a measure for the asymmetry between the time and space directions close to the critical point. The dynamic exponent is to be added to the set of critical exponents used to characterize a thermal phase transition. Since such transitions have only two independent exponents, a quantum phase transition is specified by three independent exponents.

The traditional scaling theory of thermal, continuous phase transitions in equilibrium, first put forward by Widom [7], is easily extended to include the time dimension [8] because the relation (1) implies the presence of only one independent diverging scale.

Let $\delta = K - K_c$, with K the parameter triggering the phase transition, measure the distance from the critical value K_c . A physical observable O at the absolute zero of temperature depends on K as well as on other variable, such as an external field, energy, or momentum. Let us denote these other variables collectively by Γ . In the critical region close to the critical point, O can be written as

$$O(\Gamma, K) = \xi^{d_O} \mathcal{O}(\hat{\Gamma}), \quad (T = 0), \quad (2)$$

where d_O is the scaling dimension of the observable O , $\xi \sim |\delta|^{-\nu}$, with ν the correlation length exponent, and $\hat{\Gamma}$ is obtained from Γ by rescaling it with factors of the correlation length, so that $\hat{\Gamma}$ is independent of that scale. To be specific, if an external field scales as $\Gamma \sim \xi^{d_r}$, then the rescaled field is defined as $\hat{\Gamma} = \xi^{-d_r} \Gamma$. The right side of Eq. (2) does not depend explicitly on K , but only implicitly through ξ .

The data of the observable O as function of an external field Γ obtained at different values of the parameter K triggering the transition can be collapsed on a single curve when instead of $O(\Gamma, K)$, the rescaled quantity $|\delta|^{\nu d_O} O(\Gamma, K)$ is plotted as function not of Γ , but of $\hat{\Gamma}$. Indeed, because of Eq. (2), the combination $|\delta|^{\nu d_O} O(\Gamma, K)$ depends only on $\hat{\Gamma}$ and is thus independent of the distance from the critical point. This is used to determine

critical exponents experimentally. By rescaling the vertical and horizontal axis of the plot with $|\delta|^x$ and $|\delta|^y$, respectively, the best collapse obtained at some value x_0 and y_0 give the combination of critical exponents $\nu d_O = y_0$ and $\nu d_T = x_0$.

Since a physical system is always at some finite temperature, we have to investigate how the scaling law (2) changes when the temperature becomes nonzero. The easiest way to include the temperature in a quantum theory is to go over to imaginary time $\tau = it$, with τ restricted to the interval $0 \leq \tau \leq 1/T$. The temporal dimension thus becomes of finite extent. The behavior at a finite temperature is still controlled by the quantum critical point, provided the correlation time satisfies $\xi_t < 1/T$. If this condition is fulfilled, the system will not see the finite extent of the time dimension. This is what makes quantum phase transitions experimentally accessible. Instead of the zero-temperature scaling (2), we now have the finite-size scaling

$$O(\Gamma, K, T) = T^{-d_O/z} \mathcal{O}(\hat{\Gamma}_T, \xi_t T), \quad (T \neq 0), \quad (3)$$

where instead of using the correlation length to convert dimensionfull quantities in dimensionless ones, the temperature is used: $\hat{\Gamma}_T = T^{d_T/z} \Gamma$.

Notation

We adopt Feynman's notation and denote a spacetime point by $x = x_\mu = (t, \mathbf{x})$, $\mu = 0, 1, \dots, d$, with d the number of space dimensions, while the energy k_0 and momentum $\mathbf{k}$ of a particle will be denoted by $k = k_\mu = (k_0, \mathbf{k})$. The time derivative $\partial_0 = \partial/\partial t$ and the gradient ∇ are sometimes combined in a single vector $\tilde{\partial}_\mu = (\partial_0, -\nabla)$. The tilde on $\tilde{\partial}_\mu$ is to alert the reader for the minus sign appearing in the spatial components of this vector. We define the scalar product $kx = k_\mu x_\mu = k_0 t - \mathbf{k} \cdot \mathbf{x}$ and use Einstein's summation convention. Because of the minus sign in the definition of the vector $\tilde{\partial}_\mu$ it follows that $\tilde{\partial}_\mu a_\mu = \partial_0 a_0 + \nabla \cdot \mathbf{a}$, with a_μ an arbitrary vector.

Integrals over spacetime are denoted by

$$\int_x = \int_{t, \mathbf{x}} = \int dt d^d x,$$

while those over energy and momentum by

$$\int_k = \int_{k_0, \mathbf{k}} = \int \frac{dk_0}{2\pi} \frac{d^d \mathbf{k}}{(2\pi)^d}.$$

When no integration limits are indicated, the integrals are assumed to run over all possible values of the integration variables. Similarly, for functional

integrals we use the notation

$$\int D\phi = \int_{\phi}.$$

We will work in natural units with the speed of light, Boltzmann's constant k_B , and Planck's constant $\hbar$ set to unity.

These lectures include exercises, which are clearly marked. Most of the solutions can be found in Ref. [9].

2. Superconductivity

In this section we study the theory of superconductivity in the limit where all the electrons become tightly bound in pairs [3,4]. The composite boson limit is to be distinguished from the usual weak-coupling BCS limit, where only electrons (of opposite momentum) in a thin shell around the Fermi surface become loosely bound in Cooper pairs.

2.1. BCS theory

As starting point we take the microscopic BCS model specified by the Lagrangian [10]

$$\begin{aligned} \mathcal{L} &= \psi_{\uparrow}^*[i\partial_0 - \xi(-i\nabla)]\psi_{\uparrow} + \psi_{\downarrow}^*[i\partial_0 - \xi(-i\nabla)]\psi_{\downarrow} - \lambda\psi_{\uparrow}^*\psi_{\downarrow}^*\psi_{\downarrow}\psi_{\uparrow} \\ &:= \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \end{aligned} \quad (4)$$

where $\mathcal{L}_0$ is the free theory, and $\mathcal{L}_{\text{int}} = -\lambda\psi_{\uparrow}^*\psi_{\downarrow}^*\psi_{\downarrow}\psi_{\uparrow}$ is a local electron–electron interaction term, representing the effective, phonon mediated, attraction between electrons with coupling constant $\lambda < 0$. The field $\psi_{\uparrow(\downarrow)}$ is an anticommuting field describing the electrons with mass m and spin up (down), while $\xi(-i\nabla) = \epsilon(-i\nabla) - \mu$, with $\epsilon(-i\nabla) = -\nabla^2/2m$, is the kinetic energy operator with the chemical potential μ characterizing the ensemble of fermions subtracted. The theory is invariant under global U(1) transformations under which

$$\psi_{\sigma} \rightarrow e^{i\alpha}\psi_{\sigma} \quad (5)$$

with $\sigma = \uparrow, \downarrow$ and α a constant transformation parameter. The superconducting state is characterized by a spontaneous breakdown of this symmetry.

To investigate the superconducting state, we integrate out the fermionic degrees of freedom at the expense of a newly introduced auxiliary field, which

is better equipped to describe this condensed matter state. It is convenient to first introduce the two-component field

$$\psi = \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow}^* \end{pmatrix} \quad \psi^{\dagger} = (\psi_{\uparrow}^*, \psi_{\downarrow}). \quad (6)$$

In this so-called Nambu representation, $\mathcal{L}_0$ becomes

$$\mathcal{L}_0 = \psi^{\dagger} \begin{pmatrix} i\partial_0 - \xi(-i\nabla) & 0 \\ 0 & i\partial_0 + \xi(-i\nabla) \end{pmatrix} \psi, \quad (7)$$

where the anticommuting character of the electron fields is used and total derivatives are neglected. We next replace the electron–electron interaction with an expression involving the auxiliary field Δ

$$\lambda \psi_{\uparrow}^* \psi_{\downarrow}^* \psi_{\downarrow} \psi_{\uparrow} \rightarrow \Delta^* \psi_{\downarrow} \psi_{\uparrow} + \psi_{\uparrow}^* \psi_{\downarrow}^* \Delta - \frac{1}{\lambda} |\Delta|^2. \quad (8)$$

The original interaction is regained when the auxiliary field is integrated out. Physically, Δ , representing a product of two electron fields, describes electron pairs. We shall therefore refer to it as pair field. With this replacement, the partition function

$$Z = \int_{\psi^{\dagger}, \psi} \exp \left(i \int_x \mathcal{L} \right), \quad (9)$$

becomes

$$\begin{aligned} Z = & \int_{\psi^{\dagger}, \psi, \Delta^*, \Delta} \exp \left(\frac{i}{\lambda} \int_x |\Delta|^2 \right) \\ & \times \exp \left[i \int_x \psi^{\dagger} \begin{pmatrix} i\partial_0 - \xi(-i\nabla) & -\Delta \\ -\Delta^* & i\partial_0 + \xi(-i\nabla) \end{pmatrix} \psi \right]. \end{aligned} \quad (10)$$

Since the fermion fields appear quadratically now, they may be integrated out to yield an effective action S_{eff} for Δ and Δ^*

$$S_{\text{eff}}[\Delta^*, \Delta] = -i \text{Tr} \ln \begin{pmatrix} p_0 - \xi(\mathbf{p}) & -\Delta \\ -\Delta^* & p_0 + \xi(\mathbf{p}) \end{pmatrix}, \quad (11)$$

where $p_0 = i\partial_0$ and $\xi(\mathbf{p}) = \epsilon(\mathbf{p}) - \mu$, with $\epsilon(\mathbf{p}) = \mathbf{p}^2/2m$ and $\mathbf{p} = -i\nabla$, so that the partition function becomes

$$Z = \int_{\Delta^*, \Delta} \exp \left(i S_{\text{eff}}[\Delta^*, \Delta] + \frac{i}{\lambda} \int_x |\Delta|^2 \right). \quad (12)$$

The trace Tr appearing in Eq. (11) denotes the trace over discrete indices as well as the integration over spacetime and over energy k_0 and momentum $\mathbf{k}$.

In the mean-field approximation, the functional integral over the pair field in Eq. (12) is approximated by the saddle point, where only the extremal value, satisfying the equation

$$\frac{\delta S_{\text{eff}}}{\delta \bar{\Delta}_0^*(x)} = -\frac{1}{\lambda} \bar{\Delta}_0(x), \quad (13)$$

is included. For a system homogeneous in spacetime, the pair field is a constant $\bar{\Delta}$, and Eq. (13) reduces, after passing to the Fourier representation, to the BCS gap equation [10]:

$$\begin{aligned} \frac{1}{\lambda} &= -i \int_{\mathbf{k}} \frac{1}{k_0^2 - E^2(\mathbf{k}) + i\eta} \\ &= -\frac{1}{2} \int_{\mathbf{k}} \frac{1}{E(\mathbf{k})}. \end{aligned} \quad (14)$$

Here, η is an infinitesimal positive constant which is to be set to zero at the end of the calculation, and

$$E(\mathbf{k}) = \sqrt{\xi^2(\mathbf{k}) + |\bar{\Delta}_0|^2} \quad (15)$$

is the spectrum¹ of the fermionic excitations. A nontrivial solution to the gap equation signals the spontaneous symmetry breakdown of the global $U(1)$ symmetry (5).

2.2. Composite boson limit

For studying the composite boson limit, it proves prudent to swap the coupling constant λ in the gap equation (14) for a more convenient parameter, namely the binding energy ϵ_a of an electron pair in vacuum [11]. Both parameters characterize the strength of the electron–electron interaction. To establish the connection between the two, let us consider the Schrödinger equation for the problem at hand.

In reduced coordinates, it reads

$$\left[-\frac{\nabla^2}{m} + \lambda \delta(\mathbf{x}) \right] \psi(\mathbf{x}) = -\epsilon_a, \quad (16)$$

¹ To avoid confusion, let us repeat that the bar in $\bar{\Delta}_0$ indicates that the pair field is a constant, while the subscript 0 indicates that it satisfies the extremal condition (13).

where the reduced mass is $m/2$ and the delta-function potential, with $\lambda < 0$, represents the attractive local electron–electron interaction $\mathcal{L}_{\text{int}}$ in (4). We stress that this is a two-particle problem in vacuum and not the famous Cooper problem of two interacting fermions on top of a filled Fermi sea. The equation is most easily solved in the Fourier representation, yielding

$$\psi(\mathbf{k}) = -\frac{\lambda}{\frac{\mathbf{k}^2}{m} + \epsilon_a} \psi(0), \quad (17)$$

or

$$-\frac{1}{\lambda} = \int_{\mathbf{k}} \frac{1}{\frac{\mathbf{k}^2}{m} + \epsilon_a}. \quad (18)$$

This bound-state equation allows us to replace the coupling constant λ with the binding energy ϵ_a . When substituted in the gap equation (14), the latter becomes

$$\int_{\mathbf{k}} \frac{1}{\frac{\mathbf{k}^2}{m} + \epsilon_a} = \frac{1}{2} \int_{\mathbf{k}} \frac{1}{E(\mathbf{k})}. \quad (19)$$

By inspection, it follows that this equation has a solution given by [3, 4]

$$\bar{\Delta}_0 \rightarrow 0, \quad \mu \rightarrow -\frac{1}{2}\epsilon_a, \quad (20)$$

where it should be noted that, in contrast to the weak-coupling limit, the chemical potential characterizing the ensemble of fermions is *negative* here. This is the composite boson limit.

To appreciate the physical significance of the specific value found for the chemical potential in this limit and also its name, observe that the spectrum $E_{\text{B}}(\mathbf{q})$ of the two-fermion bound state measured relative to the pair chemical potential 2μ reads

$$E_{\text{B}}(\mathbf{q}) = \frac{\mathbf{q}^2}{4m} - \mu_{\text{B}}, \quad (21)$$

where μ_{B} is defined as $\mu_{\text{B}} = \epsilon_a + 2\mu$ and may be understood as the chemical potential characterizing the ensemble of composite bosons. The negative value for μ found in Eq. (20) is precisely the condition for a Bose–Einstein condensation of an *ideal* gas of composite bosons in the $\mathbf{q} = 0$ state.

Including quadratic terms in $\bar{\Delta}_0$, we obtain as solution to Eq. (19)

$$\mu = -\frac{1}{2}\epsilon_a + \left(1 - \frac{d}{4}\right) \frac{|\bar{\Delta}_0|^2}{\epsilon_a}. \quad (22)$$

This leads to the chemical potential

$$\mu_{\text{B}} = \left(2 - \frac{d}{2}\right) \frac{|\bar{\Delta}_0|^2}{\epsilon_a}, \quad (23)$$

characterizing the now interacting Bose gas.

2.3. Renormalization

For a system homogeneous in spacetime, so that the field $\Delta_0(x)$ is constant, the effective action (11) is readily evaluated. Disassembling the argument of the logarithm as

$$\begin{pmatrix} p_0 - \xi(\mathbf{p}) & -\bar{\Delta}_0 \\ -\bar{\Delta}_0^* & p_0 + \xi(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} p_0 - \xi(\mathbf{p}) & 0 \\ 0 & p_0 + \xi(\mathbf{p}) \end{pmatrix} - \begin{pmatrix} 0 & \bar{\Delta}_0 \\ \bar{\Delta}_0^* & 0 \end{pmatrix}, \quad (24)$$

and expanding the second logarithm in a Taylor series, we recognize the form

$$\begin{aligned} S_{\text{eff}}[\bar{\Delta}_0^*, \bar{\Delta}_0] = & -i \text{Tr} \ln \begin{pmatrix} p_0 - \xi(\mathbf{p}) & 0 \\ 0 & p_0 + \xi(\mathbf{p}) \end{pmatrix} \\ & -i \text{Tr} \ln \left(1 - \frac{|\bar{\Delta}_0|^2}{p_0^2 - \xi^2(\mathbf{p})} \right), \end{aligned} \quad (25)$$

apart from an irrelevant constant. Again passing to the Fourier representation, and carrying out the integral over the loop energy k_0 , we obtain the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \int_{\mathbf{k}} [E(\mathbf{k}) - \xi(\mathbf{k})]. \quad (26)$$

Exercise: Derive this result from Eq. (25), using contour integration. Rather than integrating the logarithms in that equation, one better first differentiate it with respect to the chemical potential as the integral over the loop energy k_0 becomes easier that way. In the end one integrates the resulting expression again over the chemical potential to obtain the desired result.

To the one-loop result (26), the tree term $|\bar{\Delta}_0|^2/\lambda$ is to be added. Expanding $E(\mathbf{k})$ in Eq. (26) in a Taylor series, we see that the effective Lagrangian also contains a term quadratic in $\bar{\Delta}_0$. This term amounts to a renormalization of the coupling constant. Specifically, the renormalized coupling constant λ_r is to this order given by

$$\begin{aligned} \frac{1}{\lambda_r} &= \frac{1}{\lambda} + \frac{1}{2} \int_{\mathbf{k}} \frac{1}{\xi(\mathbf{k})} \\ &= \frac{1}{\lambda} + \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{d/2}} \frac{m^{d/2}}{\epsilon_a^{1-d/2}}, \end{aligned} \quad (27)$$

with, as is appropriate in the composite boson limit,

$$\xi(\mathbf{k}) = \epsilon(\mathbf{k}) + \frac{1}{2}\epsilon_a, \quad (28)$$

and dimensional regularization is used in evaluating the momentum integral. Because of Eq. (18), which may be viewed as the defining equation of the parameter ϵ_a , the right side of Eq. (27) is zero. This implies that in the composite boson limit, $\lambda_r \rightarrow -\infty$ so that we indeed have tightly bound pairs here.

3. Bogoliubov theory

In this section we show that in the composite boson limit, the effective theory obtained after integrating out the fermionic degrees of freedom, is the Bogoliubov theory of superfluidity in an interacting Bose gas. The gas consists of composite bosons with a mass twice the electron mass. The system is known to undergo a quantum phase transition from the superfluid to a (Mott) insulating state [12]. When translated back to the fermionic theory, this transition corresponds to one where the condensate is drained of composite bosons, without breaking them up. We include impurities in the Bogoliubov theory to show that it leads to localization without destroying the superfluid state completely. The insulating state, which is now a result not only because of repulsive interactions, as in a Mott insulator, but also of (Anderson) localization, is called an Anderson–Mott insulator.

3.1. Derivative expansion

We next wish to relax the assumption of homogeneity in spacetime and consider a spacetime-dependent pair field. To this end, we study the effective action (11) and expand $\Delta(x)$ around the constant value $\bar{\Delta}_0$ satisfying the gap equation (14),

$$\Delta(x) = \bar{\Delta}_0 + \tilde{\Delta}(x). \quad (29)$$

We obtain in this way,

$$S_{\text{eff}} = i \text{Tr} \sum_{\ell=1}^{\infty} \frac{1}{\ell} \left[G_0(p) \begin{pmatrix} 0 & \tilde{\Delta} \\ \tilde{\Delta}^* & 0 \end{pmatrix} \right]^{\ell}, \quad (30)$$

where G_0 is the correlation function,

$$\begin{aligned} G_0(k) &= \begin{pmatrix} k_0 - \xi(\mathbf{k}) & -\bar{\Delta}_0 \\ -\bar{\Delta}_0^* & k_0 + \xi(\mathbf{k}) \end{pmatrix}^{-1} \\ &= \frac{1}{k_0^2 - E^2(\mathbf{k}) + i\eta} \begin{pmatrix} k_0 e^{ik_0\eta} + \xi(\mathbf{k}) & \bar{\Delta}_0 \\ \bar{\Delta}_0^* & k_0 e^{-ik_0\eta} - \xi(\mathbf{k}) \end{pmatrix}. \end{aligned} \quad (31)$$

The exponential functions in the diagonal elements of the correlation function are additional convergence factors needed in nonrelativistic theories [13].

When evaluating the effective action (30), the precise meaning of the trace Tr appearing there should be kept in mind. Explicitly, it is defined as

$$S_{\text{eff}} = -i\text{Tr} \ln [K(p, x)] = -i\text{tr} \ln \left[K(p, x) \delta(x - y)|_{y=x} \right], \quad (32)$$

where the trace tr is the usual one over discrete indices. We abbreviated the matrix appearing in (11) by $K(p, x)$ so as to cover the entire class of actions of the form

$$S = \int_x \psi^\dagger(x) K(p, x) \psi(x). \quad (33)$$

The delta function in (32) arises because $K(p, x)$ is obtained as a second functional derivative of the action

$$\frac{\delta^2 S}{\delta \psi^\dagger(x) \delta \psi(x)} = K(p, x) \delta(x - y)|_{y=x}, \quad (34)$$

each of which gives a delta function. Since the action contains only one integral over spacetime, one delta function remains. Because it is diagonal, the delta function may be taken out of the logarithm and (32) can be written as

$$\begin{aligned} S_{\text{eff}} &= -i\text{tr} \int_x \ln [K(p, x)] \delta(x - y)|_{y=x} \\ &= -i\text{tr} \int_x \int_k e^{ikx} \ln [K(p, x)] e^{-ikx}. \end{aligned} \quad (35)$$

In the last step, we used the integral representation of the delta function:

$$\delta(x) = \int_k e^{-ikx}, \quad (36)$$

shifted the exponential function $\exp(iky)$ to the left, which is justified because the derivative p_μ does not operate on it, and, finally, set y_μ equal to x_μ . We thus see that the trace Tr in Eq. (32) stands for the trace over discrete indices as well as the integration over spacetime and over energy and momentum. The integral $\int_k$ arises because the effective action calculated here is a one-loop result with k_μ the loop energy and momentum.

The integrals in (35) cannot in general be evaluated in closed form because the logarithm contains energy–momentum operators and spacetime-dependent functions in a mixed order. To disentangle the integrals resort has

to be taken to a derivative expansion [14] in which the logarithm is expanded in a Taylor series. Each term contains powers of the energy-momentum operator p_μ which acts on every spacetime-dependent function to its right. All these operators are shifted to the left by repeatedly applying the identity

$$f(x)p_\mu g(x) = (p_\mu - i\tilde{\partial}_\mu)f(x)g(x), \quad (37)$$

where $f(x)$ and $g(x)$ are arbitrary functions of spacetime and the derivative $\tilde{\partial}_\mu = (\partial_0, -\nabla)$ acts *only* on the next object to the right. One then integrates by parts, so that all the p_μ 's act to the left where only a factor $\exp(ikx)$ stands. Ignoring total derivatives and taking into account the minus signs that arise when integrating by parts, one sees that all occurrences of p_μ (an operator) are replaced with k_μ (an integration variable). The exponential function $\exp(-ikx)$ can at this stage be moved to the left where it is annihilated by the function $\exp(ikx)$. The energy-momentum integration can now in principle be carried out and the effective action be cast in the form of an integral over a local density $\mathcal{L}_{\text{eff}}$:

$$S_{\text{eff}} = \int_x \mathcal{L}_{\text{eff}}. \quad (38)$$

This is in a nutshell how the derivative expansion works.

Exercise: Apply the derivative expansion to the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\tilde{\partial}_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2 - \frac{1}{4}\lambda \phi^4, \quad (39)$$

with ϕ a real scalar field. The theory has a Z_2 symmetry under which the scalar field flips sign: $\phi \rightarrow \phi' = -\phi$. Show that the effective theory in two space dimensions is given by

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}Z(\phi)(\tilde{\partial}_\mu \phi)^2 - \mathcal{V}_{\text{eff}}(\phi), \quad (40)$$

with

$$\begin{aligned} \mathcal{V}_{\text{eff}}(\phi) &= -\frac{1}{12\pi}(m^2 + \frac{1}{2}\lambda\phi^2)^{3/2}, \\ \mathcal{Z}(\phi) &= \frac{1}{192\pi} \frac{\lambda^2\phi^2}{(m^2 + \frac{1}{2}\lambda\phi^2)^{3/2}}. \end{aligned} \quad (41)$$

3.2. Map onto Bogoliubov theory

We are interested in terms in the effective action (30) quadratic in the field $\tilde{\Delta}$. Using the derivative expansion, we find

$$S_{\text{eff}}^{(2)}(q) = \frac{i}{2} \text{Tr} \frac{1}{p_0^2 - E^2(\mathbf{p})} \frac{1}{(p_0 + q_0)^2 - E^2(\mathbf{p} - \mathbf{q})} \\ \times \left\{ \bar{\Delta}_0^2 \tilde{\Delta}^* \tilde{\Delta} + [p_0 + \xi(\mathbf{p})][p_0 + q_0 - \xi(\mathbf{p} - \mathbf{q})] \tilde{\Delta} \tilde{\Delta}^* \right. \\ \left. + \bar{\Delta}_0^{*2} \tilde{\Delta} \tilde{\Delta} + [p_0 - \xi(\mathbf{p})][p_0 + q_0 + \xi(\mathbf{p} - \mathbf{q})] \tilde{\Delta}^* \tilde{\Delta} \right\}, \quad (42)$$

where $q_\mu = i\tilde{\partial}_\mu$. Let us first ignore the derivatives in this expression. After carrying out the integral over the loop energy k_0 and using the gap equation (14), we obtain

$$\mathcal{L}^{(2)}(0) = -\frac{1}{8} \int_{\mathbf{k}} \frac{1}{E^3(\mathbf{k})} \left(\bar{\Delta}_0^2 \tilde{\Delta}^{*2} + \bar{\Delta}_0^{*2} \tilde{\Delta}^2 + 2|\bar{\Delta}_0|^2 |\tilde{\Delta}|^2 \right). \quad (43)$$

In the composite boson limit, where the spectrum of the fermionic excitations is given by Eq. (28), the integral over the loop momentum becomes elementary, yielding

$$\int_{\mathbf{k}} \frac{1}{E^3(\mathbf{k})} = \frac{4\Gamma(3 - \frac{d}{2})}{(4\pi)^{d/2}} m^{d/2} \epsilon_a^{d/2-3}. \quad (44)$$

We next consider the terms in Eq. (42) involving derivatives. Following Ref. [15] we set $\bar{\Delta}_0$ to zero here. The integral over the loop energy is easily carried out, with the result

$$\mathcal{L}^{(2)}(q) = -\frac{1}{2} \int_{\mathbf{k}} \frac{1}{q_0 - \frac{\mathbf{k}^2}{m} + 2\mu - \frac{q^2}{4m}} \tilde{\Delta} \tilde{\Delta}^* \\ - \frac{1}{2} \int_{\mathbf{k}} \frac{1}{-q_0 - \frac{\mathbf{k}^2}{m} + 2\mu - \frac{q^2}{4m}} \tilde{\Delta}^* \tilde{\Delta}. \quad (45)$$

In the composite boson limit, the remaining momentum integrals become elementary again and after expanding in derivatives we find

$$\int_{\mathbf{k}} \frac{1}{q_0 - \frac{\mathbf{k}^2}{m} - \epsilon_a - \frac{q^2}{4m}} \\ = -\frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{d/2}} m^{d/2} \epsilon_a^{d/2-1} - \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} m^{d/2} \epsilon_a^{d/2-2} \left(q_0 - \frac{q^2}{4m} \right). \quad (46)$$

When combined with the tree term $|\tilde{\Delta}|^2/\lambda$, the first term at the right side of this equation yields the renormalization (27) of the coupling constant. The second term at the right side of Eq. (46) gives, when combined with the contribution (43), the result [15],

$$\mathcal{L}^{(2)} = \frac{1}{2} \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} m^{d/2} \epsilon_a^{d/2-2} \tilde{\Psi}^\dagger M_0(q) \tilde{\Psi}, \quad \tilde{\Psi} = \begin{pmatrix} \tilde{\Delta} \\ \tilde{\Delta}^* \end{pmatrix}, \quad (47)$$

where $M_0(q)$ stands for the 2×2 matrix,

$$M_0(q) = \begin{pmatrix} q_0 - \frac{q^2}{4m} - (2 - \frac{d}{2}) \frac{|\tilde{\Delta}_0|^2}{\epsilon_a} & -(2 - \frac{d}{2}) \frac{\tilde{\Delta}_0^2}{\epsilon_a} \\ -(2 - \frac{d}{2}) \frac{\tilde{\Delta}_0^{*2}}{\epsilon_a} & -q_0 - \frac{q^2}{4m} - (2 - \frac{d}{2}) \frac{|\tilde{\Delta}_0|^2}{\epsilon_a} \end{pmatrix}. \quad (48)$$

As we shall show now, this is the Bogoliubov theory of superfluidity in an interacting Bose gas. That is to say, after integrating out the fermionic degrees of freedom from the theory of superconductivity, we obtain in the composite boson limit a theory describing a gas of repulsively interacting (composite) bosons.

3.3. Quantum phase transition

The Bogoliubov theory is specified by the Lagrangian [16]

$$\mathcal{L} = \phi^* [i\partial_0 - \epsilon(-i\nabla) + \mu_B] \phi - \lambda_B |\phi|^4, \quad (49)$$

where μ_B is the chemical potential characterizing the Bose gas. The self-coupling is taken to be positive, $\lambda_B > 0$, so that the local interaction is repulsive.

At the mean-field, or classical level, where quantum fluctuations are ignored, the theory (49) undergoes a phase transition when the chemical potential changes sign. For $\mu_B > 0$, the global U(1) symmetry is spontaneously broken by a nontrivial ground state, while for $\mu_B < 0$ the symmetry is unbroken. The change in μ_B can be induced by varying the temperature, as in a thermal phase transition, but it can also be induced at zero temperature by varying, for example, the number of charge carriers, or the amount of impurities. The zero-temperature quantum phase transition describes a transition between a superfluid and an insulating state [12].

The ground state of a system homogeneous in spacetime is obtained by considering the shape of the potential energy

$$\mathcal{V} = -\mu_B |\bar{\phi}|^2 + \lambda_B |\bar{\phi}|^4. \quad (50)$$

For $\mu_B > 0$ it indeed has a minimum away from the origin $\phi = 0$ given by

$$|\bar{\phi}_0|^2 = \frac{1}{2} \frac{\mu_B}{\lambda_B}, \quad (51)$$

and the potential becomes

$$\mathcal{V}_0 = -\frac{\mu_B^2}{4\lambda_B}. \quad (52)$$

Since the total particle number density n_B is represented by

$$n_B(x) = |\phi(x)|^2, \quad (53)$$

the quantity $n_0 := |\phi_0|^2$ physically denotes the number density of particles residing in the ground state. A nonzero value for n_0 thus signals Bose–Einstein condensation. For a homogeneous system in its ground state, we see that at the mean-field level $\bar{n}_0 = \bar{n}$ so that all the particles reside in the condensate. This will change when quantum fluctuations are included as a result of which particles are knocked out of the condensate (see below).

To account for the nontrivial ground state, we introduce the shifted field² $\tilde{\phi}(x)$:

$$\phi(x) = \bar{\phi}_0 + \tilde{\phi}(x). \quad (54)$$

The terms in the Lagrangian (49) quadratic in this shifted field may be cast in the matrix form

$$\mathcal{L}_0 = \frac{1}{2} \tilde{\Phi}^\dagger M_0(p) \tilde{\Phi}, \quad \tilde{\Phi} = \begin{pmatrix} \tilde{\phi} \\ \tilde{\phi}^* \end{pmatrix}, \quad (55)$$

with

$$M_0(p) = \begin{pmatrix} p_0 - \epsilon(\mathbf{p}) + \mu_B - 4\lambda_B |\bar{\phi}_0|^2 & -2\lambda_B \bar{\phi}_0^2 \\ -2\lambda_B \bar{\phi}_0^{*2} & -p_0 - \epsilon(\mathbf{p}) + \mu_B - 4\lambda_B |\bar{\phi}_0|^2 \end{pmatrix}. \quad (56)$$

Taking into account only the quadratic terms in the field and neglecting higher-order terms, as we just did, is known as the Bogoliubov approximation.

Comparing this expression with Eq. (48) obtained in the composite boson limit after integrating out the fermionic degrees of freedom from the theory of superconductivity, we conclude that the composite bosons have — as expected — a mass $m_B = 2m$ twice the fermion mass m , and a small chemical potential given by Eq. (23), which we there derived from the gap equation.

² Similar as before, the bar in $\bar{\phi}_0$ denotes a constant value of the field, while the subscript 0 indicates that it satisfies the mean-field equation (51).

It also follows that the number density of composite bosons condensed in the ground state reads

$$\bar{n}_0 = \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} m^{d/2} \epsilon_a^{d/2-2} |\bar{\Delta}_0|^2, \quad (57)$$

and that the interaction λ_B between the composite bosons is

$$\lambda_B = (4\pi)^{d/2} \frac{1 - \frac{d}{4}}{\Gamma(2 - \frac{d}{2})} \frac{\epsilon_a^{1-d/2}}{m^{d/2}}, \quad (58)$$

or, using Eq. (18),

$$\lambda_B = -\frac{1 - \frac{d}{4}}{1 - \frac{d}{2}} \lambda. \quad (59)$$

Note that the parameter $\lambda (< 0)$ characterizing the attractive electron-electron interaction appears below $d = 2$ with a minus sign here, leading to a repulsive interaction between the composite bosons. (In the next subsection, we will see that $d = 2$ is the upper critical dimension of the $T = 0$ Bogoliubov theory.) This brings us to the important conclusion that for $d < 2$ the same interaction responsible for the formation of electron pairs, is also responsible for the stability of the superfluid state, and when this state ceases to exist, for that of the insulating state, which both need a repulsive interaction.

The quantum phase transition encoded in the Bogoliubov theory corresponds, when translated back to the fermionic theory, to one where the condensate is drained of composite bosons, without breaking them up. In other words, composite bosons exist on both sides of the transition, either condensed (superfluid state) or localized (insulating state) [5,6].

3.4. Beyond mean-field theory

We can continue now and improve on the usual mean-field approximation of the theory of superconductivity, where the functional integral over the pair field in the partition function (12) is approximated by the saddle point, by integrating out the field $\tilde{\Psi}$ in Eq. (47), or to simplify notation, the field $\tilde{\Phi}$ in Eq. (55). This leads to the effective potential

$$\mathcal{V}_{\text{eff}} = -\frac{i}{2} \text{tr} \int_k \ln[M_0(k)] = \frac{1}{2} \int_k E(\mathbf{k}). \quad (60)$$

Here, $E(\mathbf{k})$ is the famous single-particle Bogoliubov spectrum [16],

$$\begin{aligned} E(\mathbf{k}) &= \sqrt{\epsilon^2(\mathbf{k}) + 2\mu_B \epsilon(\mathbf{k})} \\ &= \sqrt{\epsilon^2(\mathbf{k}) + 4\lambda_B |\bar{\phi}_0|^2 \epsilon(\mathbf{k})}. \end{aligned} \quad (61)$$

In the limit of large momentum, the spectrum behaves in a way

$$E(\mathbf{k}) \sim \epsilon(\mathbf{k}) + 2\lambda_B |\bar{\phi}_0|^2 \quad (62)$$

typical for a nonrelativistic particle of mass m moving in a background medium, provided by the condensate in this case. The most notable feature of the Bogoliubov spectrum is that it is gapless, behaving for small momentum as

$$E(\mathbf{k}) \sim c |\mathbf{k}|, \quad (63)$$

with $c = \sqrt{\mu_B/m}$.

Exercise: Carry out the integral over the loop energy k_0 in Eq. (60) using contour integration and show that this leads to the right side of that equation. This is best done by first differentiating the expression with respect to the chemical potential μ_B , and in the end integrating the result again with respect to μ_B .

The integral over the loop momentum in Eq. (60) can be carried out using the integral representation of the Gamma function

$$\frac{1}{a^z} = \frac{1}{\Gamma(z)} \int_0^\infty \frac{d\tau}{\tau} \tau^z e^{-a\tau}. \quad (64)$$

In arbitrary space dimension d this yields, using dimensional regularization:

$$\mathcal{V}_{\text{eff}} = -L_d m^{d/2} \mu_B^{d/2+1}, \quad L_d = \frac{\Gamma(1 - \frac{d}{2}) \Gamma(\frac{d}{2} + \frac{1}{2})}{2\pi^{d/2+1/2} \Gamma(\frac{d}{2} + 2)}. \quad (65)$$

For $d = 2$, the effective potential diverges. To investigate this, we set $d = 2 - \varepsilon$, with ε small and positive, and expand $\mathcal{V}_{\text{eff}}$ around $d = 2$, giving

$$\mathcal{V}_{\text{eff}} = -\frac{m}{4\pi\varepsilon} \frac{\mu_B^2}{\kappa^{\varepsilon/2}}, \quad (66)$$

with κ an arbitrary renormalization group scale parameter which enters for dimensional reasons. If the Bogoliubov spectrum had not been gapless, but had an energy gap instead, this parameter would have appeared in Eq. (66) in the place of κ . As always in dimensional regularization, the divergence shows up as a pole in ε . Comparing the one-loop contribution with the classical contribution (52), we conclude that Eq. (66) leads to a renormalization of the coupling constant λ , yielding the renormalized coupling λ_r [17]

$$\frac{1}{\lambda_r} = \frac{1}{\lambda} + \frac{m}{\pi} \frac{1}{\varepsilon}, \quad (67)$$

where $\hat{\lambda} = \lambda/\kappa^{\varepsilon/2}$ and a similar definition for $\hat{\lambda}_r$. The quantum critical point is approached by letting the renormalized group scale parameter $\kappa \rightarrow 0$. For fixed coupling λ , it then follows that upon approaching the critical point, the renormalized coupling tends to $\hat{\lambda}_r \rightarrow \pi\varepsilon/m$. For $d < 2$, or equivalently $\varepsilon > 0$, the fixed point is nontrivial. In the limit $d \rightarrow 2$, $\hat{\lambda}_r \rightarrow 0$ and the theory becomes Gaussian, identifying $d = 2$ as the upper critical dimension.

Due to quantum fluctuations not all the particles are known to reside in the condensate [18]. Specifically, in d space dimensions, the (constant) particle number density $\bar{n}$ at the one-loop level is given by [19]

$$\bar{n} = |\bar{\phi}_0|^2 - 2^{d/2-2} \frac{d^2 - 4}{d - 1} L_d m^{d/2} \lambda_B^{d/2} |\bar{\phi}_0|^d. \quad (68)$$

Since the quantum-induced term is positive for $1 < d < 4$, the number of particles residing in the condensate given $\bar{n}$ is reduced compared to the classical result $\bar{n} = |\bar{\phi}_0|^2$. This shows that due to quantum fluctuations, particles are knocked out of the condensate.

Exercise: Derive Eq. (68). In doing so, one should not use the mean-field equation (51) too early, and instead work with the more general single-particle spectrum

$$E(\mathbf{k}) = \sqrt{[\epsilon(\mathbf{k}) - \mu_0 + 4\lambda_0|\bar{\phi}|^2]^2 - 4\lambda_0^2|\bar{\phi}|^4}. \quad (69)$$

It reduces to the Bogoliubov spectrum when the mean-field equation is used.

Despite that due to quantum fluctuations not all the particles reside in the condensate, all the particles do in the absence of impurities and at zero temperature participate in the superflow, and move on the average with the superfluid velocity. Put differently, the superfluid mass density ρ_s is given by the total particle number density n : $\rho_s = mn$.

Exercise: Prove this. To this end, assume that the entire system moves with a velocity $\mathbf{u}$ relative to the laboratory system. As in standard hydrodynamics, the time derivative in the frame following the motion of the fluid is $\partial_0 + \mathbf{u} \cdot \nabla$. Also assume that the condensate moves with the superfluid velocity $\mathbf{v}_s$ and boost the field:

$$\phi(x) \rightarrow \phi'(x) = e^{im\mathbf{v}_s \cdot \mathbf{x}} \phi(x). \quad (70)$$

Show that when incorporated in the Lagrangian (49) of the interacting Bose gas, these two changes result in a change of the chemical potential

$$\mu_B \rightarrow \mu_{\text{eff}} := \mu_B - \frac{1}{2}m\mathbf{v}_s \cdot (\mathbf{v}_s - 2\mathbf{u}). \quad (71)$$

Show that the resulting Bogoliubov spectrum and thermodynamic potential are given by the previous results (61) and (65) with this replacement.

The momentum density, or equivalently, the mass current $\mathbf{g}$ of the system is obtained in this approximation by differentiating the effective potential with respect to $-\mathbf{u}$. Show that

$$\mathbf{g} = \bar{\rho}_s \mathbf{v}_s, \quad (72)$$

with $\bar{\rho}_s = m\bar{n}$ the superfluid mass density.

3.5. Impurities

One of the ways to trigger a superconductor–insulator transition is to change the amount of impurities. This means that, *e.g.*, the correlation length ξ diverges as $|\hat{\alpha}^* - \hat{\alpha}|^{-\nu}$ when the parameter $\hat{\alpha}$ characterizing the impurities approaches the critical value $\hat{\alpha}^*$.

To account for impurities, we include a term [8]

$$\mathcal{L}_\alpha = \psi(\mathbf{x}) |\phi(x)|^2 \quad (73)$$

in the bosonic theory (49), where $\psi(\mathbf{x})$ is a space-dependent random field with a Gaussian distribution

$$P[\psi] = \exp \left[-\frac{1}{\alpha} \int_{\mathbf{x}} \psi^2(\mathbf{x}) \right], \quad (74)$$

characterized by the impurity strength α ($\hat{\alpha}$ alluded to above is a rescaled version of α). Notice that the random field does not depend on time. This is because it is introduced to mimic impurities, which are randomly distributed in space, not in time.

We shall treat the impurities in the so-called quenched approximation [8], where the average of an observable $O(\phi^*, \phi)$ is obtained as follows

$$\langle O(\phi^*, \phi) \rangle = \int_{\psi} P[\psi] \langle O(\phi^*, \phi) \rangle_{\psi}, \quad (75)$$

with $\langle O(\phi^*, \phi) \rangle_{\psi}$ indicating the grand-canonical average for a given impurity configuration. That is to say, first the ensemble average is taken for fixed ψ , and only after that the averaging over the random field is carried out.

In terms of the shifted field (54), the random term (73) becomes

$$\mathcal{L}_\alpha = \psi(\mathbf{x}) (|\bar{\phi}_0|^2 + |\bar{\phi}|^2 + \bar{\phi}_0 \bar{\phi}^* + \bar{\phi}_0^* \bar{\phi}). \quad (76)$$

The first two terms lead to an irrelevant change in the chemical potential, so that only the last two terms need to be considered, which can be cast in the matrix form

$$\mathcal{L}_\alpha = \psi(\mathbf{x}) \bar{\Phi}_0^\dagger \tilde{\Phi}, \quad \bar{\Phi}_0 = \begin{pmatrix} \bar{\phi}_0 \\ \bar{\phi}_0^* \end{pmatrix}. \quad (77)$$

The integral over $\tilde{\Phi}$ is Gaussian in the Bogoliubov approximation and therefore easily performed to yield an additional term to the effective action

$$S_\alpha = -\frac{1}{2} \int_{x,y} \psi(\mathbf{x}) \bar{\Phi}_0^\dagger G_0(x-y) \bar{\Phi}_0 \psi(\mathbf{y}), \quad (78)$$

where the correlation function G_0 is the inverse of the matrix M_0 introduced in Eq. (56). To proceed, we pass to the Fourier representation:

$$G_0(x-y) = \int_k e^{-ik \cdot (x-y)} G_0(k), \quad (79)$$

$$\psi(\mathbf{x}) = \int_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \psi(\mathbf{k}). \quad (80)$$

The contribution to the effective action then appears in the form

$$S_\alpha = -\frac{1}{2} \int_{\mathbf{k}} |\psi(\mathbf{k})|^2 \bar{\Phi}_0^\dagger G(0, \mathbf{k}) \bar{\Phi}_0. \quad (81)$$

Since the random field is Gaussian distributed, the average over this field representing quenched impurities yields:

$$\langle |\psi(\mathbf{k})|^2 \rangle = \frac{1}{2} \Omega \alpha, \quad (82)$$

with Ω the volume of the system. The remaining integral over the loop momentum in Eq. (81) is readily carried out to yield in arbitrary space dimensions the contribution to the Lagrangian

$$\langle \mathcal{L}_\alpha \rangle = \frac{1}{2} \Gamma \left(1 - \frac{d}{2} \right) \left(\frac{m}{2\pi} \right)^{d/2} |\bar{\phi}_0|^2 (6\lambda_B |\bar{\phi}_0|^2 - \mu_B)^{d/2-1} \alpha. \quad (83)$$

The divergence in the limit $d \rightarrow 2$ shows that also in the presence of impurities, the two-dimensional case is special. This expression can be used to obtain the additional depletion due to impurities. To this end, we differentiate it with respect to the chemical potential, giving [20, 21]

$$\bar{n}_\alpha = \frac{\partial \langle \mathcal{L}_\alpha \rangle}{\partial \mu_B} = \frac{2^{d/2-5} \Gamma(2 - \frac{d}{2})}{\pi^{d/2}} m^{d/2} \lambda_B^{d/2-2} \bar{n}_0^{d/2-1} \alpha, \quad (84)$$

where we recall that $\bar{n}_0 = |\bar{\phi}_0|^2$ denotes the (constant) number density of particles residing in the condensate. Because this contribution is positive, it amounts to an additional depletion of the condensate. The divergence in the limit $\lambda_B \rightarrow 0$ for $d < 4$ signals the collapse of the system with impurities when the interparticle repulsion is removed.

To determine the superfluid mass density $\bar{\rho}_s$ in the presence of impurities, we replace, as in the last exercise of Sec. 3.4, μ_B with μ_{eff} defined in Eq. (71) and $i\partial_0$ with $i\partial_0 - (\mathbf{u} - \mathbf{v}_s) \cdot (-i\nabla)$ in the contribution (81) to the effective action. Differentiating it with respect to the externally imposed velocity, $-\mathbf{u}$, we find to linear order in the difference $\mathbf{u} - \mathbf{v}_s$:

$$\mathbf{g} = \bar{\rho}_s \mathbf{v}_s + \bar{\rho}_n \mathbf{u}, \quad (85)$$

with the superfluid and normal mass density [21]

$$\bar{\rho}_s = m \left(\bar{n} - \frac{4}{d} \bar{n}_\alpha \right), \quad \bar{\rho}_n = \frac{4}{d} m \bar{n}_\alpha. \quad (86)$$

As expected, $\bar{\rho}_s \neq m\bar{n}$ in the presence of impurities. Moreover, the normal mass density is a factor $4/d$ larger than the mass density $m\bar{n}_\alpha$ knocked out of the condensate by the impurities. For $d = 3$ this gives the factor $\frac{4}{3}$ first found in Ref. [22]. As argued there, this indicates that part of the zero-momentum state belongs not to the condensate, but to the normal fluid. Being trapped by the impurities, the fraction $(4 - d)/d \times \bar{n}_\alpha$ of the zero-momentum state is localized.

This is an important conclusion as it shows that the phenomenon of localization can be accounted for in the Bogoliubov theory of superfluidity by including a random field, without necessarily destroying that state.

4. Phase-only theory

In this section we show that the Bogoliubov theory, which we obtained in the composite boson limit after integrating out the fermionic degrees of freedom from the theory of superconductivity, contains only one degree of freedom, viz. the phase of the order parameter. Physically, it describes the Goldstone mode of the spontaneously broken global $U(1)$ symmetry. In the context of superconductivity, this mode is called Anderson–Bogoliubov mode. The Bogoliubov theory may therefore, at least in the superfluid state, be represented by a phase-only effective theory. We continue to account for the $1/r$ Coulomb potential in the effective theory and give general scaling arguments for the physical quantities represented by that theory.

4.1. Derivation

It was first shown by Beliaev [23] that the gaplessness of the single-particle spectrum first found by Bogoliubov at the classical level persists at the one-loop order and later proven by Hugenholtz and Pines [24] to hold to all orders in perturbation theory. In fact, as was proven by Gavoret and Nozières [25], the Bogoliubov spectrum is identical to that of the Goldstone mode accompanying the spontaneous breakdown of the global U(1) symmetry, thus explaining its gaplessness.

Also from the perspective of degrees of freedom, this conclusion makes sense. Although the normal phase is described by a complex ϕ -field, having two components, it contains only one degree of freedom [26]. This is because the energy $E(\mathbf{k}) \sim \mathbf{k}^2$ is always positive. As a result, only positive energies appear in the Fourier decomposition of the field, and one needs — as is well known from standard quantum mechanics — a complex field to describe a single spinless particle. In the superfluid phase, on the other hand, where $E^2(\mathbf{k}) \sim \mathbf{k}^2$, the Fourier decomposition contains positive as well as negative energies so that a single real field suffices to describe this mode. In other words, although the number of fields is different, the number of degrees of freedom is the same in both phases. This implies that the superfluid state can be described by a phase-only theory as it captures all the degrees of freedom, ignoring vortices for the moment which are easily incorporated in the theory as will be discussed in the next section.

To obtain the phase-only theory, we set, *cf.* Eq. (54)

$$\phi(x) = e^{i\varphi(x)} (\bar{\phi}_0 + \tilde{\phi}), \quad (87)$$

with $\varphi(x)$ a background field representing the Goldstone mode accompanying the spontaneous symmetry breakdown of the global U(1) symmetry. Inserting this in the Lagrangian (49) and expanding it, we obtain

$$\mathcal{L}^{(2)} = -\mathcal{V}_0 - |\bar{\phi}_0|^2 U - U(\bar{\phi}_0 \tilde{\phi}^* + \bar{\phi}_0^* \tilde{\phi}) - \lambda_B |\bar{\phi}_0|^2 (\bar{\phi}_0 \tilde{\phi}^* + \bar{\phi}_0^* \tilde{\phi})^2, \quad (88)$$

where the field $U(x)$ stands for the combination

$$U(x) = \partial_0 \varphi(x) + \frac{1}{2m} [\nabla \varphi(x)]^2. \quad (89)$$

In deriving Eq. (88), we used the mean-field equation $\mu_B = 2\lambda_B |\bar{\phi}_0|^2$. We continue to integrate out the tilde field (which is tantamount to substituting its field equation back into the Lagrangian) to obtain the phase-only theory

$$\mathcal{L}_{\text{eff}} = -\bar{n}U(x) + \frac{1}{4}U(x)\frac{1}{\lambda_B}U(x), \quad (90)$$

where we ignored the irrelevant constant term $\mathcal{V}_0$ and substituted $|\bar{\phi}_0|^2 (= \bar{n}_0) = \bar{n}$ to this order. Using the mean-field equation again, we can write the coefficient of the last term as:

$$\frac{1}{4} \frac{1}{\lambda_B} = \frac{1}{2} \frac{\bar{n}}{m c^2} = \frac{1}{2} \bar{n}^2 \kappa, \quad (91)$$

with c the speed of sound introduced in Eq. (63). Standard thermodynamics relates c to the compressibility κ via

$$\kappa = \frac{1}{m \bar{n} c^2}. \quad (92)$$

The phase-only theory (90) can thus be cast in the equivalent form

$$\mathcal{L}_{\text{eff}} = -\bar{n} \left[\partial_0 \varphi + \frac{1}{2m} (\nabla \varphi)^2 \right] + \frac{1}{2} \bar{n}^2 \kappa \left[\partial_0 \varphi + \frac{1}{2m} (\nabla \varphi)^2 \right]^2, \quad (93)$$

which turns out to be exact [27].

The theory describes a sound wave, with the dimensionless phase field φ representing the Goldstone mode of the spontaneously broken global U(1) symmetry. It has the gapless spectrum $E^2(\mathbf{k}) = c^2 \mathbf{k}^2$. The effective theory gives, ignoring vortices for the moment, a complete description of the superfluid at low energies and small momenta. When one goes to higher energies and momenta, additional terms with higher-order derivatives should be included in the effective theory, but it remains a phase-only theory.

4.2. Coulomb potential

It is straightforward to generalize the result (90) to include long-ranged interactions. A case of particular interest to us is the 3-dimensional Coulomb potential

$$V(\mathbf{x}) = \frac{q^2}{|\mathbf{x}|}, \quad (94)$$

whose Fourier transform in d space dimensions reads

$$V(\mathbf{k}) = 2^{d-1} \pi^{(d-1)/2} \Gamma\left[\frac{1}{2}(d-1)\right] \frac{q^2}{|\mathbf{k}|^{d-1}}. \quad (95)$$

Here, q stands for the electric charge, which in the case of Cooper pairs is twice the electron charge. The simple contact interaction $L_{\text{int}} = -\lambda_B \int_{\mathbf{x}} |\phi(\mathbf{x})|^4$ in Eq. (49) is now replaced by

$$L_{\text{int}} = -\frac{1}{2} \int_{\mathbf{x}, \mathbf{y}} |\phi(t, \mathbf{x})|^2 V(\mathbf{x} - \mathbf{y}) |\phi(t, \mathbf{y})|^2. \quad (96)$$

The rationale for using the 3-dimensional Coulomb potential, even when considering charges confined to move in a lower dimensional space, is that the electromagnetic interaction remains 3-dimensional. The effective theory then becomes after passing over to the Fourier representation

$$\mathcal{L}_{\text{eff}} = -\bar{n}U(k) + \frac{1}{2}U(k_0, \mathbf{k})\frac{1}{V(\mathbf{k})}U(k_0, -\mathbf{k}), \quad (97)$$

and leads to the spectrum

$$E^2(\mathbf{k}) = 2^{d-1}\pi^{(d-1)/2}\Gamma\left[\frac{1}{2}(d-1)\right]\frac{\bar{n}q^2}{m}|\mathbf{k}|^{3-d}. \quad (98)$$

For $d = 3$, this yields the famous plasma mode, with an energy gap given by the plasma frequency $\omega_p^2 = 4\pi\bar{n}q^2/m$. For $d = 2$ on the other hand, the spectrum behaves as $E(\mathbf{k}) \propto \sqrt{|\mathbf{k}|}$, implying that the mode it describes is much harder than the sound wave with the spectrum $E(\mathbf{k}) \propto |\mathbf{k}|$ obtained for the system without the $1/r$ Coulomb interaction included.

To appreciate under which circumstances the Coulomb interaction becomes important, we note that for electronic systems $1/|\mathbf{x}| \sim k_F$ for dimensional reasons and the fermion number density $\bar{n} \sim k_F^d$, where k_F is the Fermi momentum. The ratio of the Coulomb interaction energy to the Fermi energy $\epsilon_F = k_F^2/2m$ is therefore proportional to $\bar{n}^{-1/d}$. This means that the lower the electron number density is, the more important the Coulomb interaction becomes.

4.3. Hyperscaling

Let us consider the two terms in the effective theory (93) quadratic in the Goldstone field φ and write them in the most general form [28]

$$\mathcal{L}_{\text{eff}}^{(2)} = -\frac{1}{2}\frac{\rho_s}{m^2}(\nabla\varphi)^2 + \frac{1}{2}\bar{n}^2\kappa(\partial_0\varphi)^2. \quad (99)$$

The coefficient ρ_s is the superfluid mass density, which is, as we saw in the previous section, a response function and in general does not equal $m\bar{n}$. The other coefficient,

$$\bar{n}^2\kappa = \frac{\partial\bar{n}}{\partial\mu_B}, \quad (100)$$

can be related to the (0,0)-component of the polarization tensor Π_{00} . This can be understood by noting that an electromagnetic field is included via the minimal substitution $\hat{\partial}_\mu \rightarrow \hat{\partial}_\mu + qA_\mu$, with A_μ the electromagnetic vector

potential. Since the polarization tensor (times q^2) is obtained by differentiating the effective theory twice with respect to the vector potential, we obtain

$$\lim_{|\mathbf{k}| \rightarrow 0} \Pi_{00}(0, \mathbf{k}) = \bar{n}^2 \kappa, \quad (101)$$

where, as is typical for response functions, the energy transfer is put to zero before the momentum transfer $\mathbf{k}$ is. Equation (99) leads to the general expression for the speed of sound

$$c^2 = \frac{\rho_s}{m^2 \bar{n}^2 \kappa}. \quad (102)$$

The singular behavior of the system close to the critical point is encoded in the phase-only theory. Simple dimensional analysis shows that near the phase transition it scales as

$$\mathcal{L}_{\text{eff}} \sim \xi^{-(d+z)}, \quad (103)$$

while

$$(\nabla \varphi)^2 \sim \xi^{-2}, \quad (\partial_0 \varphi)^2 \sim \xi_t^{-2} \sim \xi^{-2z}, \quad (104)$$

with ξ_t the correlation time and z the dynamic exponent. Combining these hyperscaling arguments, and remembering that the mass parameter is inessential with regards to the critical behavior, one arrives [28] at the scaling laws for the two coefficients appearing in the effective theory (99):

$$\rho_s \sim \xi^{-(d+z-2)}, \quad \kappa \sim \xi^{-(d-z)}. \quad (105)$$

The first conclusion is consistent with the universal jump predicted by Nelson and Kosterlitz [29] which corresponds to taking $z = 0$ and $d = 2$.

In the presence of impurities it is believed that the compressibility stays finite at the critical point, implying $z = d$ [28]. This remarkably simple argument thus predicts an exact and nontrivial value for the dynamic exponent.

Without impurities, the dynamic exponent is $z = 2$ [17]. This agrees with what one naively expects, given that in the nonrelativistic theory (49) we started with, one time derivative appears in combination with two space derivatives, $i\partial_0 + \nabla^2/2m$. This last argument should, however, be treated with care when applied to the phase-only theory (93). In that theory, the time and space derivatives appear in a symmetrical form, yet z is in general not unity, as we just saw. The difference is that in the effective theory, the relative coefficient c^2 scales according to Eq. (102) with the scaling laws (105) as

$$c^2 \sim \xi^{2(1-z)}, \quad (106)$$

while the relative coefficient m in the microscopic theory does not scale. Incidentally, the (quantum) XY model has a dynamic exponent $z = 1$, so that c in that case does not scale.

In experiments on charged systems, instead of the superfluid mass density, usually the conductivity σ is measured. To see the relation between the two, we introduce a vector potential in the effective theory by replacing $\nabla\varphi$ with $\nabla\varphi - q\mathbf{A}$ in Eq. (99), and allow the superfluid mass density to vary in space and time. The term in the action quadratic in $\mathbf{A}$ then becomes after passing to the Fourier representation

$$S_\sigma = -\frac{1}{2} \frac{q^2}{m^2} \int_k \mathbf{A}(-k) \rho_s(k) \mathbf{A}(k). \quad (107)$$

The electromagnetic current,

$$\mathbf{j}(k) = \frac{\delta S_\sigma}{\delta \mathbf{A}(-k)} \quad (108)$$

obtained from this action can be written as

$$\mathbf{j}(k) = \sigma(k) \mathbf{E}(k), \quad (109)$$

with the conductivity

$$\sigma(k) = i \frac{q^2}{m^2} \frac{\rho_s(k)}{k_0} \quad (110)$$

essentially given by the superfluid mass density. So if we know the scaling of the electric charge, we can determine the scaling of the conductivity.

With the $1/r$ Coulomb potential included, the quadratic terms in the effective theory (97) may, after passing to the Fourier representation, be cast in the general form

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{1}{2} \left(\frac{\rho_s}{m^2} \mathbf{k}^2 - \frac{|\mathbf{k}|^{d-1}}{q'^2} k_0^2 \right) |\varphi(k)|^2, \quad (111)$$

where q' is the redefined charge parameter

$$q'^2 = 2^{d-1} \pi^{(d-1)/2} \Gamma\left[\frac{1}{2}(d-1)\right] q^2. \quad (112)$$

The charge is connected to the $(0, 0)$ -component of the polarization tensor via

$$q'^2 = \lim_{|\mathbf{k}| \rightarrow 0} \frac{|\mathbf{k}|^{d-1}}{\Pi_{00}(0, \mathbf{k})}. \quad (113)$$

A simple hyperscaling argument like the one given above for the case without Coulomb interaction shows that near the transition, the charge scales as [30]

$$q'^2 \sim \xi^{1-z}, \quad (114)$$

independent of the number of space dimensions d . It then follows from Eq. (110) that the conductivity scales as

$$\sigma \sim \xi^{3-(d+z)}. \quad (115)$$

Exercise: Give an alternative derivation of the result (114), using Eq. (101).

In the presence of random impurities, the charge is expected to be finite at the transition, so that $z = 1$ [30]. This is again an exact result, which replaces the value $z = d$ of an impure system without Coulomb interaction. The prediction was first confirmed for impure superconducting films [2], and has subsequently also been observed in other 2-dimensional systems such 2-dimensional Josephson-junction arrays [31], quantum Hall systems [32], and 2-dimensional electron systems [33]. We will refer to a Quantum Critical Point with a $1/r$ Coulomb interaction as CQCP. In the vicinity of such a critical point, the conductivity scales as [34]

$$\sigma \sim \xi^{2-d}, \quad (116)$$

implying that in two space dimensions, the conductivity is a marginal operator which remains finite at the CQCP.

4.4. Scaling of magnetic vector potential

Let us finish this section by determining the scaling of the magnetic vector potential. We start with the observation that close to a CQCP, the electric field E scales as $E \sim \xi_t^{-1} \xi^{-1} \sim \xi^{-(z+1)}$ (for a review, see Ref. [35]). Thus conductivity measurements [33, 36] close to a CQCP collapse onto a single curve when plotted as function of the dimensionless combination $|\delta|^{\nu(z+1)}/E$, where as before $\delta = K - K_c$ measures the distance from the critical point K_c , and ν is the correlation length exponent, $\xi \sim |\delta|^{-\nu}$. (For a field-controlled transition, K stands for the applied magnetic field, while for a density-controlled transition it stands for the charge-carrier density.) The scaling of the electric field with the correlation length expresses the more fundamental result that the anomalous scaling dimension $d_{\mathbf{A}}$ of the magnetic vector potential $\mathbf{A}$ is unity, $d_{\mathbf{A}} = 1$.

Because the magnetic vector potential always appears in the gauge-invariant combination $\nabla - q\mathbf{A}$, the anomalous scaling dimension of the electric charge q of the charge carriers times the vector potential is unity too, $d_q\mathbf{A} = 1$. Writing the anomalous scaling dimension of the vector potential as a sum $d_{\mathbf{A}} = d_{\mathbf{A}}^0 + \frac{1}{2}\eta_{\mathbf{A}}$ of its canonical scaling dimension $d_{\mathbf{A}}^0 = \frac{1}{2}(d + z - 2)$, obtained by simple power counting, and (half) the critical exponent $\eta_{\mathbf{A}}$, describing how the correlation function decays at the critical point, we conclude that $d_q = d_q^0 - \frac{1}{2}\eta_{\mathbf{A}}$. Here, $d_q^0 = 1 - d_{\mathbf{A}}^0$ stands for the canonical scaling dimension of the electric charge. Now, for a $1/r$ Coulomb potential, the charge scales according to Eq. (114) as $q^2 \sim \xi^{1-z}$ independent of the number d of space dimensions [30]. Combined with the previous result, this fixes the value of the exponent $\eta_{\mathbf{A}}$ in terms of the number of space dimensions and the dynamic exponent:

$$\eta_{\mathbf{A}} = 5 - d - 2z. \quad (117)$$

Since in the presence of impurities, the electric charge is finite at a CQCP, leading to $z = 1$, it follows that $\eta_{\mathbf{A}} = 1$ in two space dimensions.

As we shall see in the next section, this exponent becomes important when considering the interaction between vortices close to the CQCP.

4.5. *Experimental status*

For a critical discussion of the experimental status of the phase-only theory, see Ref. [37]. A more recent discussion can be found in Ref. [38].

According to the phase-only theory discussed here, no electronic excitations exist in the critical region. However, electron tunneling measurements on superconducting films of varying thickness apparently probed the energy gap of these excitations [39]. Moreover, the gap was found to approach zero as the transition to the insulating state is approached. Similar experiments [40] for the field-tuned transition showed the presence of a large number of electronic excitations near the Fermi energy, thus raising doubts about the applicability of the phase-only theory.

Experimental support for the presence of electron pairs in the insulating state comes from Hall effect studies on superconducting films, which show two critical fields [41]. The lower critical field is seen in the longitudinal resistance and is believed to mark the superconductor–insulator transition. The higher critical field is seen in the transverse or Hall resistance and is believed to signal the crossover from a bosonic to a fermionic insulator without pairing. At the higher critical field, the longitudinal resistance has its maximum.

The critical exponents determined in earlier experiments on the superconductor–insulator transition [2, 36] had the value $z = 1$ for the dynamic

exponent, in accord with the prediction in Ref. [1], and $\nu = 1.3$ for the correlation length exponent. More recent studies [42], however, find agreement with these results only for the transition tuned by changing the film thickness. For the field-tuned transition the value $z\nu = 0.7$ was found instead, which is about half the value one expects. The cause for this discrepancy is not clear. It implies that, contrary to common believe, the critical exponents depend on how the phase transition is crossed, by tuning the field or the film thickness.

Clearly, more experimental and theoretical studies are required to fully understand the superconductor–insulator transition, and to establish to what extent the phase-only theory is applicable.

5. Duality

One of the most intriguing results found in experiments on quantum phase transitions in superconducting films, as well as in 2-dimensional Josephson-junction arrays [31], quantum Hall systems [43], and 2-dimensional electron systems [33] is the striking similarity in the current–voltage (I – V) characteristics on both sides of the transition. By interchanging the I and V axes in one phase, an I – V characteristic of that phase at a given value of the applied magnetic field (in superconducting films, 2-dimensional Josephson-junction arrays, and quantum Hall systems) or charge carrier density (in 2-dimensional electron systems) can be mapped onto an I – V characteristic of the other phase at a different value of the magnetic field or charge-carrier density. This reflection symmetry hints at a deep connection between the conduction mechanisms in the two phases that can be understood by invoking a duality transformation [1, 44]. Whereas the conducting phase is most succinctly described in terms of charge carriers of the system, the insulating phase is best formulated in terms of vortices. At zero temperature, these topological defects should, just like the charge carriers, be thought of as quantum point particles. The duality transformation links the two descriptions, which turn out to be very similar.

5.1. Vortices

Let us now include vortices in the phase-only theory. This is achieved by introducing the so-called plastic field φ_μ^P via the minimal substitution $\hat{\partial}_\mu \varphi \rightarrow \hat{\partial}_\mu \varphi + \varphi_\mu^P$ [45]. The plastic field is defined such that its curl gives a delta function at the location of the vortices. Specifically, in two space dimensions, where vortices are point objects, located at the positions $\mathbf{x}^\alpha$ say:

$$\nabla \times \varphi^P = -2\pi \sum_\alpha \delta(\mathbf{x} - \mathbf{x}^\alpha), \quad (118)$$

while in three dimensions, where vortices are line objects, located along the curves C_α say:

$$\nabla \times \boldsymbol{\varphi}^P = -2\pi \sum_\alpha \int_{C_\alpha} d\mathbf{x}^\alpha \delta(\mathbf{x} - \mathbf{x}^\alpha). \quad (119)$$

Let us concentrate on static phenomena so that we can ignore the time derivatives in the effective theory (99). When besides vortices also the magnetic vector potential is included, the effective theory becomes in three dimensions

$$\mathcal{L}_{\text{eff}}^{(2)} = -\frac{1}{2} \frac{\rho_s}{m^2} (\nabla \varphi - \boldsymbol{\varphi}^P - q\mathbf{A})^2 - \frac{1}{2} (\nabla \times \mathbf{A})^2, \quad (120)$$

or after the canonical transformation $q\mathbf{A} \rightarrow q\mathbf{A} - \boldsymbol{\varphi}^P$:

$$\mathcal{L}_{\text{eff}}^{(2)} = -\frac{1}{2} \frac{\rho_s}{m^2} (\nabla \varphi - q\mathbf{A})^2 - \frac{1}{2} (\nabla \times \mathbf{A} - \mathbf{B}^P)^2, \quad (121)$$

where the plastic field $\mathbf{B}^P$ stands for

$$\mathbf{B}^P = -\Phi_0 \sum_\alpha \int_{C_\alpha} d\mathbf{x}^\alpha \delta(\mathbf{x} - \mathbf{x}^\alpha), \quad (122)$$

with $\Phi_0 = 2\pi/q$ the magnetic flux quantum in units where the speed of light and Planck's constant $\hbar$ are set to unity. [In two dimensions, this plastic field is a scalar and stands for

$$B^P = -\Phi_0 \sum_\alpha \delta(\mathbf{x} - \mathbf{x}^\alpha), \quad (123)$$

as follows from Eq. (118).]

After integrating out the phase field φ in Eq. (121), we obtain the magnetic part of the effective action S_{mag} . Written as a functional integral over the magnetic vector potential, it reads in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$

$$e^{iS_{\text{mag}}} = \int_{\mathbf{A}} \exp \left\{ i \int_x \left[-\frac{1}{2} (\nabla \times \mathbf{A} - \mathbf{B}^P)^2 - \frac{1}{2} \frac{1}{\lambda^2} \mathbf{A}^2 \right] \right\}, \quad (124)$$

with λ the magnetic penetration depth. The mass term, with $\lambda^{-2} = q^2 \rho_s / m^2$, is generated through the Anderson-Higgs mechanism in the process of integrating out the phase mode φ .

With this construction, we can now calculate the interaction between two vortices. To facilitate the calculation in the case of a superconducting film

below, we linearize the first term in Eq. (124) by introducing an auxiliary field $\tilde{\mathbf{h}}$ via a Hubbard–Stratonovich transformation:

$$-\frac{1}{2}(\nabla \times \mathbf{A} - \mathbf{B}^P)^2 \rightarrow i(\nabla \times \mathbf{A} - \mathbf{B}^P) \cdot \tilde{\mathbf{h}} - \frac{1}{2}\tilde{\mathbf{h}}^2. \quad (125)$$

The original form is regained after integrating out the auxiliary field again. After integrating out the magnetic vector potential, we arrive at a form appropriate for a dual description in terms of magnetic vortices rather than electric charges [46]

$$e^{iS_{\text{mag}}} = \int_{\tilde{\mathbf{h}}} \exp \left\{ i \int_x \left[-\frac{1}{2}\lambda^2(\nabla \times \tilde{\mathbf{h}})^2 - \frac{1}{2}\tilde{\mathbf{h}}^2 - i\tilde{\mathbf{h}} \cdot \mathbf{B}^P \right] \right\}. \quad (126)$$

Physically, $\tilde{\mathbf{h}}$, which satisfies the condition $\nabla \cdot \tilde{\mathbf{h}} = 0$, represents (i times) the fluctuating local induction. The vortices described by $\mathbf{B}^P$ couple to $\tilde{\mathbf{h}}$ with a coupling constant $g = \Phi_0/\lambda$ independent of the electric charge. Observe the close similarity between the original (124) and the dual form (126). This becomes even more so when an external electric current $\mathbf{j}^P$ is coupled to the $\mathbf{A}$ field by including a term $-\mathbf{A} \cdot \mathbf{j}^P$ in Eq. (124), and $\mathbf{B}^P$ describing the vortices is set to zero there.

Finally, also integrating out the local induction, one obtains the well-known Biot–Savart law for the interaction potential $S_{\text{mag}} = -\int_t V$ between two static vortices in a bulk superconductor [47],

$$\begin{aligned} V(r) &= \frac{1}{2\lambda^2} \int_{\mathbf{x}, \mathbf{y}} B_i^P(\mathbf{x}) G(\mathbf{x} - \mathbf{y}) B_i^P(\mathbf{y}) \\ &= \frac{g^2}{4\pi} \int_{C_1} \int_{C_2} d\mathbf{l}^1 \cdot d\mathbf{l}^2 \frac{e^{-R/\lambda}}{R} \\ &= -\frac{g^2}{2\pi} L \left[\ln \left(\frac{r}{2\lambda} \right) + \gamma \right] + \mathcal{O} \left(\frac{r}{\lambda} \right)^2, \end{aligned} \quad (127)$$

where we ignored the self-interaction. In Eq. (127), $G(\mathbf{x})$ is the vortex–vortex correlation function with Fourier transform $G(\mathbf{k}) = 1/(\mathbf{k}^2 + \lambda^{-2})$, R denotes the distance between the differential lengths $d\mathbf{l}^1$ and $d\mathbf{l}^2$, L is the length of each of the two vortices, and γ is Euler’s constant. For distances smaller than the magnetic penetration depth, which is the length scale for variations in the electric current and the magnetic field, the interaction is logarithmic as in a superfluid. If the system size is smaller than λ , it will replace the penetration depth as infra-red cutoff in the logarithm, and there will be no reference to the electric charge anymore.

To describe magnetic vortices in a film of thickness w [48], the bulk result (126) has to be adjusted in two ways to account for the fact that both the vortices and the screening currents, which produce the second term in (126), are confined to the plane. This is achieved by including a Dirac delta function $w\delta(x_3)$ in the second and third term. Instead of Eq. (126), we then arrive at the interaction potential [47, 48]

$$\begin{aligned} V_{\perp}(r) &= \frac{1}{2\lambda_{\perp}} \int_{\mathbf{x}_{\perp}, \mathbf{y}_{\perp}} B_{\perp}^{\text{P}}(\mathbf{x}_{\perp}) G_{\perp}(\mathbf{x}_{\perp} - \mathbf{y}_{\perp}) B_{\perp}^{\text{P}}(\mathbf{y}_{\perp}) \\ &= -\frac{g_{\perp}^2}{2\pi} \left[\ln \left(\frac{r}{4\lambda_{\perp}} \right) + \gamma \right] + \mathcal{O} \left(\frac{r}{\lambda_{\perp}} \right)^2, \end{aligned} \quad (128)$$

where $B_{\perp}^{\text{P}} = -\Phi_0 \sum_{\alpha} \delta(\mathbf{x}_{\perp} - \mathbf{x}_{\perp}^{\alpha})$ describes the vortices in the film with coordinates $\mathbf{x}_{\perp}$, $\lambda_{\perp} = \lambda^2/w$ is the transverse magnetic penetration depth, $g_{\perp}^2 = \Phi_0^2/\lambda_{\perp}$ the coupling constant squared, and

$$\begin{aligned} G_{\perp}(\mathbf{x}_{\perp}) &= \int_{x_3} G_{\perp}(\mathbf{x}_{\perp}, x_3) \\ &= \int_{\mathbf{k}_{\perp}} e^{-i\mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} G_{\perp}(\mathbf{k}_{\perp}, 0), \end{aligned} \quad (129)$$

with $G_{\perp}(\mathbf{k}_{\perp}, 0) = 2/k_{\perp}(2k_{\perp} + \lambda_{\perp}^{-1})$. For small distances, the interaction is seen to be identical to that in a bulk superconductor [48], and also to that in a superfluid film. As in the bulk, the vortex coupling constant $g_{\perp}$ in the film is independent of the electric charge.

The logarithmic interaction between vortices we found in Eq. (128) appears to pose a severe problem to the duality picture we alluded to in the introduction of this section as the charges interact via a $1/r$ Coulomb potential. The difference should spoil the experimentally observed reflection symmetry in the I - V characteristics. However, it should be realized that the results derived in this subsection are valid only in the mean-field region, where $\eta_{\text{A}} = 0$. In the critical region governed by a CQCP, the value of this exponent was found in Sec. 4.4 to be unity. As we will now demonstrate, this leads to a qualitative change in the interaction potential between two vortices from logarithmic in the mean-field region to $1/r$ in the vicinity of the CQCP.

5.2. Changing vortex interaction

Close to a CQCP we have to include the field renormalization factor $Z_{\mathbf{A}}$ in the vortex–vortex correlation function $G_{\perp}$ appearing in the expression (128) for the vortex interaction. It then becomes

$$G_{\perp}(\mathbf{k}_{\perp}, 0) = \frac{2}{k_{\perp}} \frac{Z_{\mathbf{A}}}{2k_{\perp} + \lambda_{\perp}^{-1}}, \quad (130)$$

with $Z_{\mathbf{A}} \sim k_{\perp}^{\eta_{\mathbf{A}}}$. Because the magnetic vector potential and the local induction renormalize in the same way, their renormalization factor is identical. Due to this extra factor, the interaction between two vortices in the film takes the form of a $1/r$ Coulomb potential [49]

$$V_{\perp}(r) = \frac{g_{\perp}^2}{2\pi} \frac{a}{r}, \quad (131)$$

where a is some microscopic length scale which accompanies the renormalization factor $Z_{\mathbf{A}}$ for dimensional reasons [50].

The absence of any reference to the electric charge in the renormalized and bare interaction (at least for small enough systems) implies that the same results should be derivable from our starting theory (120) with q set to zero. By directly integrating out the Anderson–Bogoliubov mode, and ignoring the momentum dependence of ρ_s , which is valid outside the critical region, one easily reproduces the bare interaction potential (128). The renormalized interaction (131) is obtained by realizing that according to Eq. (105), $\rho_s \sim k_{\perp}$ for $d = 2$ and $z = 1$. In other words, the extra factor of $k_{\perp}$ that came in via the renormalization factor $Z_{\mathbf{A}}$ in our first calculation to produce the $1/r$ potential, comes in via ρ_s here.

One might wonder if perhaps also the Coulomb interaction between electric charges changes in the vicinity of a CQCP. We do not expect this to happen. Since the $1/r$ Coulomb interaction is genuine 3-dimensional, this interaction cannot be affected too much by what happens in the film, which constitutes a mere slice of 3-dimensional space. The reason that the interaction between vortices is susceptible to the presence of a CQCP, is that this interaction is a result of currents around the vortex cores which are confined to the plane.

A similar change in the r -dependence of the interaction between two vortices upon entering a critical region has been observed numerically in the 3-dimensional Ginzburg–Landau model [51]. Near the charged fixed point of that theory, $\eta_{\mathbf{a}} = 1$ [52], as in our case.

This is a very pleasing coincidence as the $(2 + 1)$ -dimensional Ginzburg–Landau model constitutes the dual formulation of the system.

5.3. Dual theory

The *dynamics* of the charged degrees of freedom is described by the effective Lagrangian (99) with the speed of sound given by Eq. (102). In accord with the above findings, we have ignored the coupling to the magnetic vector potential, so that the effective theory essentially describes a superfluid.

In the dual formulation, the roles of charges and vortices are interchanged. And the Anderson–Bogoliubov mode mediating the interaction between two vortices is represented as a photon associated with a fictitious gauge field a_μ , *i.e.*, $\tilde{\partial}_\mu \varphi \sim \epsilon_{\mu\nu\lambda} \tilde{\partial}_\nu a_\lambda$. In $2 + 1$ dimensions, this identification makes sense as a photon has only one transverse direction and thus only one degree of freedom. It therefore represents the same number of degrees of freedom as does the Anderson–Bogoliubov mode.

The elementary excitations of the dual theory are the vortices, described by a complex scalar field ψ . Specifically, the dual theory of Eq. (99) turns out to be the Ginzburg–Landau model [44–46, 53]

$$\mathcal{L}_{\text{dual}} = -\frac{1}{4}f_{\mu\nu}^2 + |(\partial_\mu - ig a_\mu)\psi|^2 - m_\psi^2|\psi|^2 - u|\psi|^4, \quad (132)$$

with $f_{\mu\nu} = \tilde{\partial}_\mu a_\nu - \tilde{\partial}_\nu a_\mu$, m_ψ a mass parameter, and u the strength of the self-coupling. Both the gauge part as well as the matter part of the dual theory are of a relativistic form. The gauge part is because the effective theory (99), obtained after ignoring nonlinear terms, is Lorentz invariant. The matter part is because vortices of positive and negative circulation can annihilate, and can also be created. In this sense they behave as relativistic particles. As was pointed out in Ref. [44], the speed of “light” in the gauge and matter part need not to be identical and will in general differ.

The interaction potential (128) between two external vortices is now being interpreted as the 2-dimensional Coulomb potential between charges. The observation concerning the critical behavior of the Ginzburg–Landau model implies that the qualitative change in $V(r)$ upon entering the critical region is properly represented in the dual formulation.

Whereas in the conducting phase, the charges are condensed, in the insulating phase, the vortices are condensed [1]. In the dual theory, the vortex condensate is represented by a nonzero expectation value of the ψ field, which in turn leads via the Anderson–Higgs mechanism to a mass term for the gauge field a_μ . Because $(\epsilon_{\mu\nu\lambda} \tilde{\partial}_\nu a_\lambda)^2 \sim (\tilde{\partial}_\mu \varphi)^2$, the mass term a_μ^2 with two derivatives less implies that the Anderson–Bogoliubov mode has acquired an energy gap. That is to say, the phase where the vortices are condensed is incompressible and indeed an insulator. Since electric charges are seen by the dual theory as flux quanta, they are expelled from the system as long as the dual theory is in the Meissner state. Above the critical field $h = \nabla_\perp \times \mathbf{a} = h_{c1}$ they start penetrating the system and form an Abrikosov

lattice. In the original formulation, this corresponds to a Wigner crystal of the charges. Finally, when more charges are added and the dual field reaches the critical value h_{c2} , the lattice melts and the charges condense in the superfluid phase described by the effective theory (99).

It is a pleasure to thank J. Spalek and the other organizers of the XL Cracow School of Theoretical Physics for inviting me to lecture at the renowned summer school at Zakopane, Poland. I am indebted to M. Krusius for the hospitality at the Low Temperature Laboratory, Helsinki University of Technology, Finland where the lectures were prepared, and to B. Rosenstein for the hospitality at the Department of Electrophysics, National Chiao Tung University, Hsinchu, Taiwan, where these notes were written. I would also like to acknowledge helpful discussions with G. Honig and M. Paalanen.

This work was funded in part by the EU sponsored program Transfer and Mobility of Researchers under contract No. ERBFMGECT980122, and by the National Science Council (NCS) of Taiwan.

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