REMARKS ON DEFORMATION QUANTIZATION ON THE CYLINDER

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Some problems of the deformation quantization for the particle on the circle are considered. It is argued that, from the physical point of view, it seems to be necessary to deal with "quantized" classical phase space. The compact form of the Moyal $*$-product is given.

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In memory of Professor Moshé Flato and Professor André Lichnerowicz

1. Introduction

The idea of deformation quantization was introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer, in their beautiful papers [1] where, starting from the old ideas of Weyl [2], Wigner [3], Moyal [4], Gerstenhaber [5] and Vey [6], they "... suggest that quantization be understood as a deformation of the structure of the algebra of classical observables, rather than a radical change in the nature of the observables." This quantization arises as a deformation of the usual product algebra of the smooth functions on the classical phase space and then as a deformation of the Poisson bracket algebra. The deformed product is called the *$*$-product* (in our paper we call it the *Moyal $*$-product*) and it has been proved that such a product exists for any symplectic manifold [7,8,9]. (Recently it has been shown by

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Kontsevich [10] that a deformation quantization exists also for any Poisson manifold). This result evidently justifies the suggestion of Bayen, Flato et al. However, the crucial point is to make that formal deformation quantization be a physical theory corresponding to quantum mechanics. Although it appears to be an easy problem in the case of Cartesian phase space $\mathbb{R}^{2n}$, it is not so in more general cases.

In this paper we deal with the deformation quantization on the cylinder and we show that it seems to be necessary to reduce the classical phase space $\mathbb{R}\times S^1$ to the following subset: $\hbar\mathbb{Z}\times S^1$. This result is in agreement with the previous results by Mukunda [11], Berry [12] and Kasperkovitz and Peev [13] and we suppose that similar reduction or quantization of the classical phase space will be observed in other cases when the topology of the coordinate space is non-trivial.

Our paper is organized as follows. In Section 2 we consider the deformation quantization on $\mathbb{R}\times\mathbb{R}$. We deal with operator bases: $\widehat{U}(\mu,\lambda)$ (unitary basis), $\widehat{\Omega}(p,q)$ (the Stratonovich-Weyl quantizer) and $\widehat{I}(\mu,\lambda)$ ($\widehat{I}(\mu,\lambda) := \widehat{U}(\mu,\lambda)\widehat{P}$ with $\widehat{P}$ being the parity operator) and we find the relations between these bases. Then the Moyal $*$-product and the Moyal bracket are presented.

In Section 3 the deformation quantization on the cylinder $\mathbb{R}\times S^1$ is investigated. We argue that from the physical point of view one should use the *discrete Stratonovich-Weyl quantizer* given by Mukunda [11], rather than the usual Stratonovich-Weyl quantizers considered by Gadella et al [14] or Arratia and del Olmo [15]. Finally, the compact form of the Moyal $*$-product on the cylinder is found and it is pointed out that in the case of (physical) deformation quantization on the cylinder the classical phase space $\mathbb{R}\times S^1$ should be quantized to be $\hbar\mathbb{Z}\times S^1$.

2. Deformation quantization on $\mathbb{R}\times\mathbb{R}$

We recall some results concerning the deformation quantization on the Cartesian phase space (for details see [1-4,13,16-20]). The symplectic 2-form ω is defined by

$$\omega = dp\wedge dq, \quad (p,q)\in\mathbb{R}^2\,. \tag{2.1}$$

Let $f=f(p,q)$ be a tempered distribution on $\mathbb{R}^2$ and $\widetilde{f}=\widetilde{f}(\lambda,\mu)$

$$\widetilde{f}=\widetilde{f}(\lambda,\mu) := \int\limits_{\mathbb{R}^2} f(p,q)\exp\{-i(\lambda p+\mu q)\}\,dpdq \tag{2.2}$$

its Fourier transform. Then, according to the Weyl rule, the quantum operator corresponding to f is defined as follows

$$\widehat{f} = W(f(p,q)) := \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2} \widetilde{f}(\lambda,\mu)\widehat{U}(\mu,\lambda)d\lambda d\mu\,, \tag{2.3}$$

where the family of unitary operators $\left\{\widehat{U}(\mu,\lambda) : (\mu,\lambda) \in \mathbb{R}^2\right\}$ is given by

$$\widehat{U}(\mu,\lambda) := \exp\left\{i(\lambda\widehat{p} + \mu\widehat{q})\right\}, \quad (\mu,\lambda) \in \mathbb{R}^2 \tag{2.4}$$

with $\widehat{p}$ and $\widehat{q}$ being the momentum and position operators, respectively.

Using the Baker-Campbell-Hausdorff formula we get

$$\begin{aligned}\widehat{U}(\mu,\lambda) &= \exp\left(-\frac{i}{2}\hbar\lambda\mu\right)\exp(i\lambda\widehat{p})\exp(i\mu\widehat{q}) \\ &= \exp\left(\frac{i}{2}\hbar\lambda\mu\right)\exp(i\mu\widehat{q})\exp(i\lambda\widehat{p})\,.\end{aligned} \tag{2.5}$$

Using also the relations

$$\begin{aligned}\exp(i\lambda\widehat{p}) \mid q\rangle &= \mid q - \hbar\lambda\rangle\,, \\ \exp(i\mu\widehat{q}) \mid p\rangle &= \mid p + \hbar\mu\rangle\end{aligned} \tag{2.6}$$

one quickly finds that the unitary operator $\widehat{U}(\mu,\lambda)$ can also be written in the following equivalent forms

$$\begin{aligned}\widehat{U}(\mu,\lambda) &= \int\limits_{-\infty}^{+\infty} \widehat{U}(\mu,\lambda) \mid q\rangle dq\langle q \mid \\ &= \int\limits_{-\infty}^{+\infty} \exp(i\mu q) \mid q - \tfrac{\hbar\lambda}{2}\rangle dq\langle q + \tfrac{\hbar\lambda}{2} \mid\,, \\ \widehat{U}(\mu,\lambda) &= \int\limits_{-\infty}^{+\infty} \widehat{U}(\mu,\lambda) \mid p\rangle dp\langle p \mid \\ &= \int\limits_{-\infty}^{+\infty} \exp\left(i\lambda p\right) \mid p + \tfrac{\hbar\mu}{2}\rangle dp\langle p - \tfrac{\hbar\mu}{2} \mid\,.\end{aligned} \tag{2.7}$$

Employing (2.7) we easily obtain the important relations

$$\mathrm{Tr}\left\{\widehat{U}(\mu,\lambda)\right\} = \frac{2\pi}{\hbar}\delta(\mu)\delta(\lambda) \tag{2.8}$$

and

$$\mathrm{Tr}\left\{\widehat{U}^{+}(\mu,\lambda)\widehat{U}(\mu^{'},\lambda^{'})\right\}=\frac{2\pi}{\hbar}\delta(\mu-\mu^{'})\delta(\lambda-\lambda^{'})\,. \tag{2.9}$$

Multiplying the both sides of (2.3) by $\widehat{U}^{+}(\mu,\lambda)$, taking the trace and using (2.9) one gets

$$\widetilde{f}=\widetilde{f}(\lambda,\mu)=2\pi\hbar\mathrm{Tr}\left\{\widehat{U}^{+}(\mu,\lambda)\widehat{f}\right\}. \tag{2.10}$$

From (2.2) and (2.3) it is evident that we can relate $f=f(p,q)$ and $\widehat{f}$ directly according to the following rule

$$\widehat{f}=\int\limits_{\mathbb{R}^2}f(p,q)\widehat{\Omega}(p,q)\frac{dpdq}{2\pi\hbar}\,, \tag{2.11}$$

where $\widehat{\Omega}=\widehat{\Omega}(p,q)$ is the operator-valued distribution

$$\widehat{\Omega}=\widehat{\Omega}(p,q):=\frac{\hbar}{2\pi}\int\limits_{\mathbb{R}^2}\exp\{-i(\lambda p+\mu q)\}\widehat{U}(\mu,\lambda)d\lambda d\mu \tag{2.12}$$

called the *Stratonovich–Weyl quantizer* [18–20].

From (2.7) and (2.12) we get

$$\begin{aligned}\widehat{\Omega}(p,q)&=\int\limits_{-\infty}^{+\infty}\exp\left(\frac{i\xi p}{\hbar}\right)\left|q+\frac{\xi}{2}\right\rangle d\xi\left\langle q-\frac{\xi}{2}\right|\\ \widehat{\Omega}(p,q)&=\int\limits_{-\infty}^{+\infty}\exp\left(-\frac{i\xi q}{\hbar}\right)\left|p+\frac{\xi}{2}\right\rangle d\xi\left\langle p-\frac{\xi}{2}\right|\end{aligned} \tag{2.13}$$

Then one quickly finds the following important properties of $\widehat{\Omega}(p,q)$:

$$\widehat{\Omega}^{+}(p,q)=\widehat{\Omega}(p,q) \tag{2.14}$$

$$\mathrm{Tr}\left\{\widehat{\Omega}(p,q)\right\}=1 \tag{2.15}$$

$$\mathrm{Tr}\left\{\widehat{\Omega}(p,q)\widehat{\Omega}(p^{'},q^{'})\right\}=2\pi\hbar\delta(p-p^{'})\delta(q-q^{'}) \tag{2.16}$$

Consequently, from (2.11) and (2.16) we have

$$f=f(p,q)=\mathrm{Tr}\left\{\widehat{\Omega}(p,q)\widehat{f}\right\}. \tag{2.17}$$

Therefore (2.15) which can be rewritten in the form $\mathrm{Tr}\left\{\widehat{\Omega}(p,q)\widehat{1}\right\} = 1$ means that under the quantization process the obvious correspondence

$$1 \leftrightarrow \widehat{1} \tag{2.18}$$

holds.

Then one gets also the useful conditions

$$\begin{aligned} \mathrm{Tr}\left\{\widehat{U}^{+}(\mu,\lambda)\widehat{\Omega}(p,q)\right\} &= \exp\left\{-i(\lambda p+\mu q)\right\}, \\ \mathrm{Tr}\left\{\widehat{\Omega}(p,q)\widehat{U}(\mu,\lambda)\right\} &= \exp\{i(\lambda p+\mu q)\}, \end{aligned} \tag{2.19}$$

which enable us to write the correspondence between two operator bases $\widehat{U}(\mu,\lambda)$ and $\widehat{\Omega}(p,q)$ in the following form

$$\begin{aligned} \widehat{\Omega}(p,q) &= \frac{\hbar}{2\pi}\int\limits_{\mathbb{R}^2} \mathrm{Tr}\left\{\widehat{U}^{+}(\mu,\lambda)\widehat{\Omega}(p,q)\right\}\widehat{U}(\mu,\lambda)d\lambda d\mu, \\ \widehat{U}(\mu,\lambda) &= \int\limits_{\mathbb{R}^2} \mathrm{Tr}\left\{\widehat{\Omega}(p,q)\widehat{U}(\mu,\lambda)\right\}\widehat{\Omega}(p,q)\frac{dpdq}{2\pi\hbar}. \end{aligned} \tag{2.20}$$

It is well known that there exists a close relation between the Stratonovich-Weyl quantizer $\widehat{\Omega}(p,q)$ and the parity operator $\widehat{P}$ defined by

$$\widehat{P} := \int\limits_{-\infty}^{+\infty} \mid q\rangle dq\langle -q \mid = \int\limits_{-\infty}^{+\infty} \mid p\rangle dp\langle -p \mid \tag{2.21}$$

(see [21] and [13] for details).

Define

$$\widehat{I} = \widehat{I}(\mu,\lambda) := \widehat{U}(\mu,\lambda)\widehat{P}. \tag{2.22}$$

Of course $\widehat{I}(0,0) = \widehat{P}$. Substituting (2.7) and (2.21) into (2.22) one gets

$$\begin{aligned} \widehat{I}(\mu,\lambda) &= \int\limits_{-\infty}^{+\infty} \exp(i\mu q)\Big|q-\frac{\hbar\lambda}{2}\Big\rangle dq\Big\langle -q-\frac{\hbar\lambda}{2}\Big| \\ &= \int\limits_{-\infty}^{+\infty} \exp(i\lambda p)\Big|p+\frac{\hbar\mu}{2}\Big\rangle dp\Big\langle -p+\frac{\hbar\mu}{2}\Big|. \end{aligned} \tag{2.23}$$

From (2.23) we easily obtain

$$\widehat{I}^{+}(\mu,\lambda)=\widehat{I}(\mu,\lambda)=\widehat{I}^{-1}(\mu,\lambda) \tag{2.24}$$

i.e., the operator $\widehat{I}(\mu,\lambda)$ is both, self-adjoint and unitary for every $(\mu,\lambda)\in\mathbb{R}^2$.

Then from (2.22) with (2.24) one has

$$\widehat{I}(\mu,\lambda)=\widehat{U}\left(\frac{\mu}{2},\frac{\lambda}{2}\right)\widehat{P}\widehat{U}\left(-\frac{\mu}{2},-\frac{\lambda}{2}\right). \tag{2.25}$$

Finally, comparing (2.13) with (2.23) one arrives at the following important relation

$$\widehat{\Omega}(p,q)=2\widehat{I}\left(\frac{2p}{\hbar},-\frac{2q}{\hbar}\right)\Longrightarrow\widehat{\Omega}(0,0)=2\widehat{I}(0,0)=2\widehat{P}\,. \tag{2.26}$$

Consequently, the operator bases $\widehat{\Omega}(p,q)$ and $2\widehat{I}(\mu,\lambda)$ overlap for the $\mathbb{R}^2$ case.

As we will see in the next section this situation changes drastically when the cylindrical phase space is considered.

Let the functions $f_1=f_1(p,q)$ and $f_2=f_2(p,q)$ correspond to the operator $\widehat{f_1}$ and $\widehat{f_2}$, respectively, *i.e.*, $f_1(p,q)=W^{-1}(\widehat{f_1})=\mathrm{Tr}\,\{\widehat{\Omega}(p,q)\widehat{f_1}\}$ and $f_2(p,q)=W^{-1}(\widehat{f_2})=\mathrm{Tr}\,\{\widehat{\Omega}(p,q)\widehat{f_2}\}$. The question is what a function corresponds to the product of operators $\widehat{f_1}\widehat{f_2}$. Denoting this function by $(f_1*f_2)(p,q):=W^{-1}(\widehat{f_1}\widehat{f_2})$ and employing (2.11) and (2.17) one quickly finds

$$\begin{aligned}(f_1*f_2)(p,q)&=\mathrm{Tr}\left\{\widehat{\Omega}(p,q)\widehat{f_1}\widehat{f_2}\right\}\\&=\int\limits_{\mathbb{R}^4}f_1(p',q')\mathrm{Tr}\left\{\widehat{\Omega}(p,q)\widehat{\Omega}(p',q')\widehat{\Omega}(p'',q'')\right\}f_2(p'',q'')\tfrac{dp'dq'dp''dq''}{(2\pi\hbar)^2}\end{aligned} \tag{2.27}$$

using (2.13) we have

$$\begin{aligned}&\mathrm{Tr}\left\{\widehat{\Omega}(p,q)\widehat{\Omega}(p',q')\widehat{\Omega}(p'',q'')\right\}\\&=4\exp\left\{\tfrac{2i}{\hbar}\left[(q-q')(p-p'')-(q-q'')(p-p')\right]\right\}.\end{aligned} \tag{2.28}$$

Substituting (2.28) into (2.27) one gets the *Moyal $*$-product of f_1 and f_2* [4,16-20]

$$(f_1 * f_2)(p,q) = 4 \int\limits_{\mathbb{R}^4} \{f_1(p+p', q+q') \times \exp\left\{\tfrac{2i}{\hbar}(q'p'' - q''p')\right\} f_2(p+p'', q+q'')\} \tfrac{dp'\,dq'\,dp''\,dq''}{(2\pi\hbar)^2}\,. \tag{2.29}$$

Assuming that $f_1,\ f_2 \in C^\infty(\mathbb{R}^2)$ we can expand $f_1(p+p', q+q')$ and $f_2(p+p'', q+q'')$ in the formal Taylor series at the point $(p,q) \in \mathbb{R}^2$. Then performing some simple manipulations (see [20] for details) we obtain the following formal result (also called the *Moyal $*$-product of f_1 and f_2*)

$$(f_1 * f_2)(p,q) = f_1 \exp\left(\frac{i\hbar}{2} \overleftrightarrow{\mathcal{P}}\right) f_2\,, \tag{2.30}$$

where $\overleftrightarrow{\mathcal{P}}$ is the Poisson operator

$$\overleftrightarrow{\mathcal{P}} := \overleftarrow{\frac{\partial}{\partial q}} \overrightarrow{\frac{\partial}{\partial p}} - \overleftarrow{\frac{\partial}{\partial p}} \overrightarrow{\frac{\partial}{\partial q}} = \omega^{ij} \overleftarrow{\frac{\partial}{\partial x^i}} \overrightarrow{\frac{\partial}{\partial x^j}}, \qquad i,j = 1,2$$
$$(x^1, x^2) \equiv (p,q); \tag{2.31}$$

ω^{ij} stands for the inverse tensor to the symplectic form ω defined by (2.1), *i.e.*, $\omega^{ij}\omega_{jk} = \delta^i_k$.

Then the *Moyal bracket* is defined by

$$\{f_1, f_2\}_M := \frac{1}{i\hbar}(f_1 * f_2 - f_2 * f_1). \tag{2.32}$$

One quickly finds that

$$\lim_{\hbar \longrightarrow 0} \{f_1, f_2\}_M = f_1 \overleftrightarrow{\mathcal{P}} f_2 := \{f_1, f_2\}_P \tag{2.33}$$

(assuming that $\frac{\partial f_1}{\partial \hbar} = 0 = \frac{\partial f_2}{\partial \hbar}$).

Gathering, we are led to the following conclusion.

Let $C^\infty(\mathbb{R}^2)[[\hbar]]$ be a linear space of formal power series

$$a(p,q;\hbar) = \sum_{k=0}^{\infty} \hbar^k a_k(p,q) \tag{2.34}$$

with $a_k(p,q) \in C^\infty(\mathbb{R}^2)$. Then the associative, non-commutative algebra $(C^\infty(\mathbb{R}^2)[[\hbar]], *)$ where the Moyal $*$-product is given by (2.30) is called the *Moyal $*$-algebra* and this algebra is the first known example of the so called *deformation quantization* [1,8,9,22].

The Lie algebra $(C^\infty(\mathbb{R}^2)[[\hbar]], \{\cdot,\cdot\}_M)$ known as the *Moyal bracket algebra,* is a deformation of the Poisson bracket algebra $(C^\infty(\mathbb{R}^2), \{\cdot,\cdot\}_P)$.

Now we find the Fourier transform $\widetilde{f_1 * f_2}$ of $f_1 * f_2$. From (2.3) and (2.10) one gets

$$\begin{aligned}
&\left(\widetilde{f}_1 \square \widetilde{f}_2\right)(\lambda,\mu) := \left(\widetilde{f_1 * f_2}\right)(\lambda,\mu) = 2\pi\hbar \mathrm{Tr}\left\{\widehat{U}^+(\mu,\lambda)\widehat{f}_1\widehat{f}_2\right\} \\
&= \tfrac{\hbar}{(2\pi)^3}\int\limits_{\mathbb{R}^4}\Big[\widetilde{f}_1(\lambda',\mu')\mathrm{Tr}\left\{\widehat{U}^+(\mu,\lambda)\widehat{U}(\mu',\lambda')\widehat{U}(\mu'',\lambda'')\right\} \\
&\times f_2(\lambda'',\mu'')\Big] d\lambda' d\mu' d\lambda'' d\mu'' \,. \qquad (2.35)
\end{aligned}$$

But

$$\begin{aligned}
&\mathrm{Tr}\left\{\widehat{U}^+(\mu,\lambda)\widehat{U}(\mu',\lambda')\widehat{U}(\mu'',\lambda'')\right\} \\
&= \tfrac{2\pi}{\hbar}\exp\left\{\tfrac{i\hbar}{2}\left(\lambda'\mu'' - \lambda''\mu'\right)\right\}\delta\left(\mu' + \mu'' - \mu\right)\delta\left(\lambda' + \lambda'' - \lambda\right) \,. \qquad (2.36)
\end{aligned}$$

Substituting (2.36) into (2.35) we obtain

$$\begin{aligned}
&\left(\widetilde{f}_1 \square \widetilde{f}_2\right)(\lambda,\mu) \\
&= \tfrac{1}{(2\pi)^2}\int\limits_{\mathbb{R}^2}\widetilde{f}_1(\lambda',\mu')\exp\left\{\tfrac{i\hbar}{2}\left(\lambda'\mu - \lambda\mu'\right)\right\}\widetilde{f}_2(\lambda-\lambda',\mu-\mu')d\lambda' d\mu' \qquad (2.37)
\end{aligned}$$

(compare with [13]).

The Weyl correspondence between the tempered distributions on the phase space $\mathbb{R}^2$ and the operators in the rigged Hilbert space (or the Gelfand triplet)

$$S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R}), \qquad (2.38)$$

where $S(\mathbb{R})$ is the Schwartz space and $S'(\mathbb{R})$ is the space of tempered distributions on $\mathbb{R}$, has two natural properties. Namely, if $f = f(p)$ then from (2.11) and (2.13) one has

$$\begin{aligned}
\widehat{f} &= \int\limits_{\mathbb{R}^3} f(p)\exp\left(-\frac{i\xi q}{\hbar}\right)\left|p + \frac{\xi}{2}\right\rangle\frac{d\xi dp dq}{2\pi\hbar}\left\langle p - \frac{\xi}{2}\right| \\
&= \int\limits_{-\infty}^{+\infty} f(p)\mid p\rangle dp\langle p \mid = f(\widehat{p}); \qquad (2.39)
\end{aligned}$$

if $g = g(q)$ then

$$\widehat{g} = \int\limits_{\mathbb{R}^3} g(q) \exp\left(\frac{i\xi p}{\hbar}\right) \Big|q + \frac{\xi}{2}\Big\rangle \frac{d\xi dp dq}{2\pi\hbar} \Big\langle q - \frac{\xi}{2}\Big|$$
$$= \int\limits_{-\infty}^{+\infty} g(q) \mid q\rangle dq \langle q \mid = g(\widehat{q})\,. \tag{2.40}$$

Another important question which should be also considered is the one concerning the transformation law for the Stratonovich-Weyl quantizer $\widehat{\Omega} = \widehat{\Omega}(p,q)$ under the characteristic group for our physical system *i.e.*, the Galilei group

$$\begin{aligned} q' &= q + vt + q_0\,, \\ t' &= t + b\,, \\ p' &= p + m_0 v, \quad b, q_0 \in \mathbb{R}, \end{aligned} \tag{2.41}$$

where t, v and m_0 denote time, velocity and mass, respectively.

Inserting (2.41) into (2.13), using also (2.6) one quickly finds

$$\begin{aligned} \widehat{\Omega}(p',q') \quad &= \int\limits_{-\infty}^{+\infty} \exp\left\{\tfrac{i\xi(p+m_0 v)}{\hbar}\right\} \Big|q + vt + q_0 + \tfrac{\xi}{2}\Big\rangle d\xi \Big\langle q + vt + q_0 - \tfrac{\xi}{2}\Big| \\ &= \int\limits_{-\infty}^{+\infty} \{\exp\left(\tfrac{i\xi p}{\hbar}\right) \exp\left(\tfrac{i\xi m_0 v}{\hbar}\right) \exp\left\{-\tfrac{i}{\hbar}(vt + q_0)\widehat{p}\right\} \\ &\times \mid q + \tfrac{\xi}{2}\rangle d\xi \langle q - \tfrac{\xi}{2} \mid \exp\left\{\tfrac{i}{\hbar}(vt + q_0)\widehat{p}\right\}\} \\ &= \int\limits_{-\infty}^{+\infty} \{\exp\left(\tfrac{i\xi p}{\hbar}\right) \exp\left\{-\tfrac{i}{\hbar}(vt + q_0)\widehat{p}\right\} \exp\left\{\tfrac{i}{\hbar} m_0 v \widehat{q}\right\} \\ &\times \mid q + \tfrac{\xi}{2}\rangle d\xi \langle q - \tfrac{\xi}{2} \mid \exp\left\{-\tfrac{i}{\hbar} m_0 v \widehat{q}\right\} \exp\left\{\tfrac{i}{\hbar}(vt + q_0)\widehat{p}\right\}\} \\ &= \{\exp\left\{\tfrac{i}{\hbar}\left[-(vt + q_0)\widehat{p} + m_0 v \widehat{q}\right]\right\} \widehat{\Omega}(p,q) \\ &\times \exp\left\{-\tfrac{i}{\hbar}\left[-(vt + q_0)\widehat{p} + m_0 v \widehat{q}\right]\right\}\}\,. \end{aligned} \tag{2.42}$$

The formula (2.42) says that $\widehat{\Omega}(p,q)$ transforms according to the irreducible projective unitary representation of the Galilei group.

Finally, we consider some facts concerning the Wigner function and the evolution equations.

If $\widehat{\rho} = \widehat{\rho}^{+}$ is the density operator representing the state of our system then according to Weyl correspondence (2.17)

$$\rho = \rho(p,q) := W^{-1}(\widehat{\rho}) = \mathrm{Tr}\left\{\widehat{\Omega}(p,q)\widehat{\rho}\right\}. \tag{2.43}$$

Wigner function $w = w(p,q)$ is defined to be

$$w = w(p,q) = \frac{1}{2\pi\hbar}\rho(p,q). \tag{2.44}$$

It is an easy matter to show the following relations

$$\widehat{\rho} = \widehat{\rho}^{+} \Longleftrightarrow w = \overline{w}, \tag{2.45}$$

$$\mathrm{Tr}\,(\widehat{\rho}) = 1 \Longleftrightarrow \int\limits_{\mathbb{R}^2} w(p,q)dpdq = 1, \tag{2.46}$$

$$\mathrm{Tr}\,\left(\widehat{\rho}\widehat{g}^{+}\widehat{g}\right) \geq 0 \Longleftrightarrow \int\limits_{\mathbb{R}^2} (\overline{g} * g)\,(p,q)w(p,q)dpdq \geq 0, \tag{2.47}$$

where the overbar stands for the complex conjugation.

[Note that by (2.47) the states in terms of deformation quantization, are defined as the positive functionals on the Moyal $*$-algebra *i.e.*, the functionals $w = w(p,q)$ which satisfy the condition

$$\int\limits_{\mathbb{R}^2} (\overline{g} * g)\,(p,q)w(p,q)dpdq \geq 0 \tag{2.48}$$

for every $g = g(p,q)$ (compare with [23])].

The expected value of an observable $\widehat{f}$ reads

$$\left\langle \widehat{f} \right\rangle = \frac{\int\limits_{\mathbb{R}^2} f(p,q)w(p,q)dpdq}{\int\limits_{\mathbb{R}^2} w(p,q)dpdq}. \tag{2.49}$$

The Liouville-von Neumann evolution equation takes the form of

$$\frac{dw}{dt} = \{H, w\}_M \tag{2.50}$$

where H is the Hamiltonian of our system.

Finally, the Heisenberg equation can be written as follows

$$\frac{df}{dt} = \{f, H\}_M \ . \tag{2.51}$$

3. Quantization on the cylinder

Here we deal with the quantization of the system consisting of a spinless particle on the circle S^1. The phase space is now the cylinder $\mathbb{R}\times S^1$. We denote the angle coordinate by $\theta \in [-\pi, \pi)$ and the momentum conjugated with θ by $p \in (-\infty, \infty)$. The symplectic form ω is

$$\omega = dp \wedge d\theta \, . \tag{3.1}$$

First we recall the quantization procedure given by Mukunda [11] and Berry [12] and we also consider some results obtained by Kasperkovitz and Peev [13] although we don't use their formalism in which the extension of the period to 4π is assumed.

By analogy to the $\mathbb{R}^2$ case consider the family of unitary operators

$$\widehat{U}(\nu, \tau) = \exp\left\{ i \left(\frac{\tau}{\hbar}\widehat{p} + \nu\widehat{\theta} \right) \right\}, \qquad (\nu, \tau) \in \mathbb{R}^2 \, . \tag{3.2}$$

However, as the operators (3.2) act on the Hilbert space $L^2(S^1)$, ν must be an *integer.*

Moreover, for any $k, l, m \in \mathbb{Z}$

$$\begin{aligned} \widehat{U}\,(m, \tau + 2k\pi) \mid l\rangle \quad &= \{\exp\left\{-\tfrac{i}{2}(\tau + 2k\pi)m\right\} \exp\left\{\tfrac{i}{\hbar}(\tau + 2k\pi)\widehat{p}\right\} \\ &\times \ \exp\left(im\widehat{\theta}\right) \mid l\rangle\} \\ &= (-1)^{mk} \exp\left(-\tfrac{i}{2}\tau m\right) \exp\left(\tfrac{i}{\hbar}\tau\widehat{p}\right) \mid l + m\rangle \\ &= (-1)^{mk} \widehat{U}\,(m, \tau) \mid l\rangle \, , \end{aligned} \tag{3.3}$$

where $\mid l\rangle$ denotes the eigenket of $\widehat{p}$

$$\widehat{p} \mid l\rangle = l\hbar \mid l\rangle, \qquad l \in \mathbb{Z} \, . \tag{3.4}$$

Hence,

$$\widehat{U}\,(m, \tau + 2k\pi) = (-1)^{mk} \widehat{U}\,(m, \tau) \tag{3.5}$$

and therefore, in order to have the family of independent unitary operators one can take $-\pi \leq \tau < \pi$. Gathering, in the present case the analogue of the family (2.4) reads

$$\left\{\widehat{U}\left(m,\tau\right) : m \in \mathbb{Z}, -\pi \leq \tau < \pi\right\}$$
$$\widehat{U}\left(m,\tau\right) = \exp\left\{i\left(\tfrac{\tau}{\hbar}\widehat{p} + m\widehat{\theta}\right)\right\}. \tag{3.6}$$

Then $\widehat{U}\left(m,\tau\right)$ can be written in the form of

$$\begin{aligned}\widehat{U}\left(m,\tau\right) &= \sum_{k=-\infty}^{\infty} \widehat{U}\left(m,\tau\right) \mid k\rangle\langle k \mid \\ &= \sum_{k=-\infty}^{\infty} \exp\left(-\tfrac{i}{2}\tau m\right) \exp\left(\tfrac{i}{\hbar}\tau\widehat{p}\right) \exp\left(im\widehat{\theta}\right) \mid k\rangle\langle k \mid \\ &= \sum_{k=-\infty}^{\infty} \exp\left\{i\tau\left(k + \tfrac{m}{2}\right)\right\} \mid k+m\rangle\langle k \mid \end{aligned} \tag{3.7}$$

or in terms of the coordinate basis $\mid \theta\rangle$

$$\int_{-\pi}^{+\pi} \mid \theta\rangle d\theta\langle\theta \mid = \widehat{1}, \qquad \langle\theta \mid \theta\rangle = \delta^{(S)}\left(\theta - \theta'\right), \tag{3.8}$$

(where $\delta^{(S)}\left(\theta\right)$ is the Dirac delta on the circle S^1), in the form of

$$\begin{aligned}\widehat{U}\left(m,\tau\right) &= \int_{-\pi}^{+\pi} \widehat{U}\left(m,\tau\right) \mid \theta\rangle d\theta\langle\theta \mid \\ &= \int_{-\pi}^{+\pi} \exp\left\{im\left(\theta - \frac{\tau}{2}\right)\right\} \mid [\theta-\tau]\rangle d\theta\langle\theta \mid \\ &= \int_{-\pi}^{+\pi} \exp\left(im\theta\right) \left|\left[\theta - \frac{\tau}{2}\right]\right\rangle d\theta \left\langle\left[\theta + \frac{\tau}{2}\right]\right|, \end{aligned} \tag{3.9}$$

where the symbol $[\theta - \tau]$ means that $[\theta - \tau] := \theta - \tau + 2k\pi$ with such a $k \in \mathbb{Z}$ that $[\theta - \tau] \in [-\pi, \pi)$.

Then one finds the following relations

$$\mathrm{Tr}\left\{\widehat{U}\left(m,\tau\right)\right\} = \sum_{k=-\infty}^{\infty} \exp\left(ik\tau\right)\delta_{m,0} = 2\pi\delta_{m,0}\delta^{(S)}\left(\tau\right) \tag{3.10}$$

and

$$\begin{aligned}
&\mathrm{Tr}\left\{\widehat{U}^{+}(m,\tau)\,\widehat{U}(m',\tau')\right\}\\
&=\delta_{m,m'}\sum_{k=-\infty}^{\infty}\exp\left\{\tfrac{i}{2}m(\tau'-\tau)\right\}\exp\{ik(\tau'-\tau)\}\\
&=2\pi\delta_{m,m'}\delta^{(S)}(\tau-\tau')\,. \qquad (3.11)
\end{aligned}$$

Now we are going to find the Weyl rule of quantization on the cylinder. Let $f=f(p,\theta)$ be a function on the cylinder and let $\widehat{f}=W(f(p,\theta))$ be the corresponding operator.

We can expand $\widehat{f}$ with respect to $\widehat{U}(m,\tau)$ basis

$$\widehat{f}=W(f(p,\theta))=\frac{1}{(2\pi)^2}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{+\pi}\widetilde{F}(\tau,m)\widehat{U}(m,\tau)\,d\tau\,, \qquad (3.12)$$

where $\widetilde{F}=\widetilde{F}(\tau,m)$ is some function. By the analogy to the $\mathbb{R}^2$ case one can expect that $\widetilde{F}(\tau,m)$ should be assumed to be the Fourier transform of $f(p,\theta)$. But if so then a severe problem arises. From (3.12) using (3.11) we get

$$\widetilde{F}(\tau,m)=2\pi\mathrm{Tr}\left\{\widehat{U}^{+}(m,\tau)\,\widehat{f}\right\},\qquad m\in\mathbb{Z}\quad and-\pi\le\tau<\pi\,. \qquad (3.13)$$

Thus (3.13) shows that the operator $\widehat{f}$ given by (3.12) defines the function $\widetilde{F}(\tau,m)$, $m\in\mathbb{Z}$, only for $\tau\in[-\pi,\pi)$. Consequently, if we want $\widetilde{F}=\widetilde{F}(\tau,m)$ to be the Fourier transform of $f=f(p,\theta)$ we are not able to extract all the information about this transform from the formula (3.12) but only its values for $-\pi\le\tau<\pi$ (and not for all $\tau\in(-\infty,\infty)$). Thus one cannot also reconstruct the function $f=f(p,\theta)$ from the corresponding operator $\widehat{f}$.

Therefore we propose another interpretation of the expansion coefficients $\widetilde{F}(\tau,m)$ (This approach within slightly different formalism was given by Mukunda [11]).

Consider the following distribution on the cylinder $\mathbb{R}\times S^1$

$$F=F(p,\theta):=\sum_{n=-\infty}^{\infty}f(n\hbar,\theta)\delta(p-n\hbar). \qquad (3.14)$$

Then we define $\widetilde{F} = \widetilde{F}(\tau, m)$ to be the Fourier transform of F

$$\widetilde{F} = \widetilde{F}(\tau, m) := \int_{-\infty}^{\infty} dp \int_{-\pi}^{+\pi} d\theta F(p,\theta) \exp\left\{-i\left(\frac{\tau}{\hbar}p + m\theta\right)\right\}$$
$$= \sum_{n=-\infty}^{\infty} \int_{-\pi}^{+\pi} d\theta f(n\hbar, \theta) \exp\left\{-i\left(\tau n + m\theta\right)\right\} . \tag{3.15}$$

From the definition (3.15) it follows that $\widetilde{F}(\tau + 2k\pi, m) = \widetilde{F}(\tau, m)$ for every $\tau \in (-\infty, \infty)$ and $m \in \mathbb{Z}$. Hence, the formula (3.13) gives all the information about $\widetilde{F}$ and, consequently, about F given by (3.14). But of course F is completely defined by the values of the function $f = f(p, \theta)$ on the following set

$$\hbar\mathbb{Z}\times S^1 \subset \mathbb{R}\times S^1 . \tag{3.16}$$

Concluding, the Weyl rule of quantization given by (3.12), (3.14) and (3.15) gives the *one to one correspondence between functions on* $\hbar\mathbb{Z}\times S^1 \subset \mathbb{R}\times S^1$ *and the operators in the Hilbert space* $L^2(S^1)$.

Inserting (3.15) into (3.12) one gets

$$\widehat{f} = W\left(f(p,\theta)\right) = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{+\pi} f(n\hbar, \theta)\widehat{\Omega}(n,\theta)\frac{d\theta}{2\pi} , \tag{3.17}$$

where

$$\widehat{\Omega}(n,\theta) := \tfrac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{+\pi} \exp\left\{-i\left(\tau n + m\theta\right)\right\} \widehat{U}(m,\tau) d\tau, \qquad n \in \mathbb{Z}, -\pi \le \theta < \pi. \tag{3.18}$$

Comparing (3.17) and (3.18) with (2.11) and (2.12) we find that $\widehat{\Omega}(n,\theta)$ resembles very much the Stratonovih–Weyl quantizer $\widehat{\Omega}(p,q)$. The important difference is that $\widehat{\Omega}(n,\theta)$ is defined on $\mathbb{Z}\times S^1$ and not on all the classical phase space $\mathbb{R}\times S^1$. Therefore we call it the *discrete Stratonovich-Weyl quantizer for the cylinder.*

From (3.18) with (3.10) and (3.11) one quickly gets the following properties of $\widehat{\Omega}(n,\theta)$

$$\widehat{\Omega}^{+}(n,\theta) = \widehat{\Omega}(n,\theta) , \tag{3.19}$$

$$\mathrm{Tr}\left\{\widehat{\Omega}(n,\theta)\right\} = 1 , \tag{3.20}$$

$$\mathrm{Tr}\left\{\widehat{\Omega}(n,\theta)\widehat{\Omega}(n',\theta')\right\} = 2\pi\delta_{n,n'}\delta^{(S)}(\theta - \theta') . \tag{3.21}$$

Then using (3.7) or (3.9) we have

$$\widehat{\Omega}(n,\theta) = \exp(-2in\theta) \sum_{k=-\infty}^{\infty} \{\exp(2ik\theta)\,[\mid 2n-k\rangle\langle k \mid + \tfrac{2}{\pi} \sum_{l=-\infty}^{\infty} \tfrac{(-1)^l}{(2l+1)} \exp\{-i(2l+1)\theta\} \mid 2(n+l)-k+1\rangle\langle k \mid]\} \tag{3.22}$$

or (compare with [11])

$$\widehat{\Omega}(n,\theta) = \int_{-\pi}^{+\pi} \exp(i\tau n) \Big| \Big[\theta + \frac{\tau}{2}\Big] \Big\rangle d\tau \Big\langle \Big[\theta - \frac{\tau}{2}\Big] \Big| . \tag{3.23}$$

Employing (3.21) one can easily extract $f(n\hbar,\theta)$ from (3.17) to be

$$f(n\hbar,\theta) = \mathrm{Tr}\left\{\widehat{\Omega}(n,\theta)\widehat{f}\right\} . \tag{3.24}$$

Note also that $\widehat{\Omega}(n,\theta)$ leads to two natural relations which hold also in the $\mathbb{R}^2$ case (see (2.39) and (2.40)). Namely if $f = f(p)$ then by (3.17) with (3.22)

$$\widehat{f} = \sum_{k=-\infty}^{\infty} f(k\hbar) \mid k\rangle\langle k \mid = f(\widehat{p}) \tag{3.25}$$

if $g = g(\exp(i\theta), \exp(-i\theta))$ then by (3.23)

$$\widehat{g} = \int_{-\pi}^{+\pi} g(\exp(i\theta), \exp(-i\theta)) \mid \theta\rangle d\theta \langle\theta \mid = g(\exp(i\widehat{\theta}), \exp(-i\widehat{\theta})) . \tag{3.26}$$

We now prove a simple theorem that shows how restrictive is the condition (3.25).

Theorem 1 *There doesn't exist a family of operators in* $L^2(S^1)$. $\left\{\widehat{\Phi}(p,\theta) : p \in \mathbb{R}, -\pi \le \theta < \pi\right\}$ *such that i)* $\mathrm{Tr}\left\{\widehat{\Phi}(p,\theta)\right\} = a \in \mathbb{R}$ *for every* p *and* θ *ii)* $\int_{-\infty}^{\infty} dp \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} f(p)\widehat{\Phi}(p,\theta) = f(\widehat{p})$ *for every* $f = f(p)$.

Proof. *Assume that ii) holds true. Then,*

$$\int_{-\infty}^{\infty} dp f(p) \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} \langle k \mid \widehat{\Phi}(p,\theta) \mid k\rangle = f(k\hbar) \tag{3.27}$$

for every $|\, k\rangle$ *and every function* $f = f(p)$. *Hence*

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} d\theta \langle k \mid \widehat{\Phi}(p,\theta) \mid k\rangle = \delta\left(p - k\hbar\right) . \tag{3.28}$$

Summing up both sides of (3.28) with respect to k *and using i) one gets*

$$a = \sum_{k=-\infty}^{\infty} \delta\left(p - k\hbar\right) \tag{3.29}$$

which is, of course a contradiction. ■

This theorem leads to an important conclusion. If one wants the quantization of the particle on the circle S^1 to satisfy a simple and natural condition (3.25), then the corresponding Stratonovich-Weyl quantizer must be defined on $\mathbb{J}\times S^1$, where $\mathbb{J}$ is some discrete subset of $\mathbb{R}$. Moreover, it is an easy matter to prove that $\mathbb{J} = \mathbb{Z}$. Indeed we have

Theorem 2 *Let* $\left\{\widehat{\Phi}(\beta,\theta) : \beta \in \mathbb{J} \subset \mathbb{R}, \quad -\pi \leq \theta < \pi\right\}$ *be a family of operators in* $L^2(S^1)$ *such that i)* $\mathrm{Tr}\left\{\widehat{\Phi}(\beta,\theta)\right\} = a \in \mathbb{R}$ *for every* $\beta \in \mathbb{J}$ *and* $\theta \in [-\pi,\pi)$ *ii)* $\sum\limits_{\beta\in\mathbb{J}} \int\limits_{-\pi}^{\pi} \frac{d\theta}{2\pi} f(\beta\hbar)\widehat{\Phi}(\beta,\theta) = f(\widehat{p})$ *for every function* $f = f(p)$ *on* $\mathbb{R}\times S^1$, *then* $\mathbb{J} = \mathbb{Z}$ *and* $a = 1$.

Proof. *Let* $f = f(p)$ *be of the form*

$$f = f(p) = \begin{cases} 1 \text{ for } p = m\hbar \\ 0 \text{ for } p \neq m\hbar \end{cases}$$

for some $m \in \mathbb{Z}$. *Then from i) and ii) one has*

$$\sum_{\beta\in\mathbb{J}} f(\beta\hbar) \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} \mathrm{Tr}\left\{\widehat{\Phi}(\beta,\theta)\right\} = \sum_{\beta\in\mathbb{J}} a f(\beta\hbar) = \mathrm{Tr}\left\{f\left(\widehat{p}\right)\right\} = 1 . \tag{3.30}$$

Hence $m \in \mathbb{J}$ *and* $a = 1$. *As* m *is an arbitrary element of* $\mathbb{Z}$, $\mathbb{Z} \subset \mathbb{J}$. *Let now* $\beta_0 \in \mathbb{J}$ *be any element of* $\mathbb{J}$ *and assume the function* $f = f(p)$ *to be of the following form*

$$f = f(p) = \begin{cases} 1 \text{ for } p = \beta_0\hbar \\ 0 \text{ for } p \neq \beta_0\hbar \end{cases} .$$

Thus we have (rememeber that $a = 1$)

$$\sum_{\beta\in\mathbb{J}} f(\beta\hbar) \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} \mathrm{Tr}\left\{\widehat{\Phi}(\beta,\theta)\right\} = f(\beta_0\hbar) = 1 = \mathrm{Tr}\left\{f\left(\widehat{p}\right)\right\}$$
$$= \sum_{k\in\mathbb{Z}} \langle k \mid f\left(\widehat{p}\right) \mid k\rangle = \sum_{k\in\mathbb{Z}} f(k\hbar)\,. \tag{3.31}$$

Hence $\beta_0 \in \mathbb{Z} \Longrightarrow \mathbb{J} \subset \mathbb{Z}$. Consequently, $\mathbb{J} = \mathbb{Z}$. The proof is complete. ■

This result can be interpreted as the quantization (discretization) of the classical phase space. In another word one can assume that the deformation quantization on the cylinder consists not only in the Moyal $*$-deformation of the usual product algebra $C^\infty(\mathbb{R}\times S^1)$ but also in reducing the classical phase space $\mathbb{R}\times S^1$ to its subset $\hbar\mathbb{Z}\times S^1$. We suppose that similar discretization of the phase space should be observed for other coordinate spaces of non-trivial topology.

Of course this changes the original idea of deformation quantization and also the idea of an axiomatic approach to the Stratonovich-Weyl correspondence between functions on the phase space and the quantum operators [14,15,18,19].

Now we study the rule of transformation for $\widehat{\Omega}(n,\theta)$ under the following group of transformations acting on $\mathbb{Z}\times S^1$

$$\begin{array}{ll} n' = n + n_0, & n', n, n_0 \in \mathbb{Z}; \\ \theta' = [\theta + \theta_0], & -\pi \le \theta', \theta, \theta_0 < \pi\,. \end{array} \tag{3.32}$$

Substituting (3.32) into (3.22) after straightforward calculations one gets

$$\widehat{\Omega}(n + n_0, [\theta + \theta_0]) = \widehat{U}(n_0, -\theta_0)\widehat{\Omega}(n,\theta)\widehat{U}^+(n_0, -\theta_0)\,, \tag{3.33}$$

where, according to (3.6),

$$\widehat{U}(n_0, -\theta_0) = \exp\left\{i\left(-\frac{\theta_0}{\hbar}\widehat{p} + n_0\widehat{\theta}\right)\right\}. \tag{3.34}$$

(Compare (3.33) with (2.42)). Therefore

$$\widehat{\Omega}(n,\theta) = \widehat{U}(n,-\theta)\widehat{\Omega}(0,0)\widehat{U}^+(n,-\theta)\,. \tag{3.35}$$

Consequently, having $\widehat{\Omega}(0,0)$ and the group property (3.33) we can find $\widehat{\Omega}(n,\theta)$ for every $n \in \mathbb{Z}$ and every $\theta \in [-\pi,\pi)$.

An interesting question arises. Assume that we don't know the family of operators $\left\{\widehat{\Omega}(n,\theta)\right\}$ which enter into the formula (3.17), but we assume that

the group property (3.33) and the natural properties (3.25) and (3.26) are satisfied for this family. The question is what information one can extract from these assumptions. Simple calculations show that with (3.33) assumed we get

$$(3.25) \Longleftrightarrow \langle k \mid \widehat{\Omega}(0,0) \mid k\rangle = \delta_{k,0} \qquad \forall k \in \mathbb{Z} \tag{3.36}$$

and

$$(3.26) \Longleftrightarrow \sum_{n=-\infty}^{\infty} \langle n \mid \widehat{\Omega}(0,0) \mid n+k\rangle = 1 \qquad \forall k \in \mathbb{Z}\,. \tag{3.37}$$

Of course $\widehat{\Omega}(n,\theta)$ considered by Mukunda [11] and given here by (3.18) satisfies (3.36) and (3.37). Note, that taking in (3.37) $k=0$ we obtain

$$\mathrm{Tr}\left\{\widehat{\Omega}(0,0)\right\} = 1\,. \tag{3.38}$$

The same condition follows from (3.36) when the sum of both sides of the relation $\langle k \mid \widehat{\Omega}(0,0) \mid k\rangle = \delta_{k,0}$ is considered.

By (3.35) $\mathrm{Tr}\left\{\widehat{\Omega}(n,\theta)\right\} = \mathrm{Tr}\left\{\widehat{\Omega}(0,0)\right\}$. Thus one arrives at the conclusion that the group property (3.33) and the conditions (3.25) or (3.26) yield the trace condition (3.20).

Now we examine the relation between $\widehat{\Omega}(n,\theta)$ given by (3.18) and the parity operator $\widehat{P}$ which is defined as follows

$$\widehat{P} := \sum_{k=-\infty}^{\infty} \mid k\rangle\langle -k \mid = \int_{-\pi}^{+\pi} \mid \theta\rangle d\theta \langle -\theta \mid \tag{3.39}$$

(compare with (2.21)). Then analogously as in the $\mathbb{R}^2$ case we define

$$\widehat{I}(n,\theta) := \widehat{U}(n,\theta)\widehat{P}\,. \tag{3.40}$$

Inserting (3.7) or (3.9) and (3.39) into (3.40) one has

$$\widehat{I}(n,\theta) = \sum_{k=-\infty}^{\infty} \exp\left\{i\theta\left(k+\frac{n}{2}\right)\right\} \mid k+n\rangle\langle -k \mid \tag{3.41}$$

or

$$\widehat{I}(n,\theta) = \int_{-\pi}^{+\pi} \exp\left(in\tau\right) \left|\left[\tau - \frac{\theta}{2}\right]\right\rangle d\tau \left\langle\left[\tau + \frac{\theta}{2}\right]\right|, \tag{3.42}$$

respectively.

It is an easy matter to prove the following properties of $\widehat{I}(n,\theta)$

$$\widehat{I}^{+}(n,\theta) = \widehat{I}(n,\theta) \tag{3.43}$$

$$\mathrm{Tr}\left\{\widehat{I}(n,\theta)\right\} = \frac{1}{2}\left\{1+(-1)^{n}\right\} \tag{3.44}$$

$$\mathrm{Tr}\left\{\widehat{I}(n,\theta)\widehat{I}(n',\theta')\right\} = 2\pi\delta_{n,n'}\delta^{(S)}(\theta-\theta')\,. \tag{3.45}$$

Then one has also the group property similar to (3.33) but only for $\widehat{I}(2n,\theta)$ *i.e.*,

$$\widehat{I}(2(n+n_0),[\theta+\theta_0]) = \widehat{U}\left(n_0,\frac{\theta_0}{2}\right)\widehat{I}(2n,\theta)\widehat{U}^{+}\left(n_0,\frac{\theta_0}{2}\right) \tag{3.46}$$
$$n,n_0\in\mathbb{Z}, \qquad \theta,\theta_0\in[-\pi,\pi)\,.$$

From (3.22), (3.39) and (3.41) employing also (3.21) and (3.45) we get

$$\widehat{\Omega}(0,0) = \widehat{P} + \tfrac{2}{\pi}\sum_{k,l=-\infty}^{\infty}\tfrac{(-1)^{l}}{(2l+1)}\mid 2l-k+1\rangle\langle k\mid, \tag{3.47}$$

$$\widehat{\Omega}(n,\theta) = \widehat{I}(2n,[-2\theta]) + \tfrac{2}{\pi}\sum_{k=-\infty}^{\infty}\exp(2ik\theta)$$
$$\times\left\{\sum_{l=-\infty}^{\infty}\tfrac{(-1)^{l}}{(2l+1)}\exp\left\{-i(2l+1)\theta\right\}\mid 2(n+l)-k+1\rangle\langle k\mid\right\}, \tag{3.48}$$

$$\widehat{\Omega}(n,\theta) = \sum_{n'=-\infty}^{\infty}\int_{-\pi}^{+\pi}\mathrm{Tr}\left\{\widehat{I}(n',\theta')\widehat{\Omega}(n,\theta)\right\}\widehat{I}(n',\theta')\tfrac{d\theta'}{2\pi}\,, \tag{3.49}$$

$$\widehat{I}(n,\theta) = \sum_{n'=-\infty}^{\infty}\int_{-\pi}^{+\pi}\mathrm{Tr}\left\{\widehat{\Omega}(n',\theta')\widehat{I}(n,\theta)\right\}\widehat{\Omega}(n',\theta')\tfrac{d\theta'}{2\pi}\,, \tag{3.50}$$

where

$$\mathrm{Tr}\left\{\widehat{I}(n',\theta')\widehat{\Omega}(n,\theta)\right\} = 2\pi\sum_{k=-\infty}^{\infty}\left\{\delta_{n',2n}\delta^{(S)}(\theta'+2\theta)\right.$$
$$\left.+\tfrac{1}{\pi}\left[1-(-1)^{n'}\right]\tfrac{(-1)^{\frac{n'-1}{2}-n+k}}{n'-2n}\exp(-in'\theta)\delta^{(S)}(\theta')\right\}. \tag{3.51}$$

Now it is evident that one can use the family $\left\{\widehat{I}(n,\theta): n\in\mathbb{Z},\theta\in[-\pi,\pi)\right\}$ as the basis of operators acting in the Hilbert space $L^{2}(S^{1})$. This basis has been used in the paper by Kasperkovitz and Peev [13].

Thus we can define the correspondence between functions on the phase space $\mathbb{R}\times S^1$ and the operators in the following form

$$\widehat{\mathcal{F}} := c \sum_{n=-\infty}^{\infty} \int_{-\pi}^{+\pi} f(n\hbar,\theta)\widehat{I}(n,\theta)\frac{d\theta}{2\pi}\,, \tag{3.52}$$

where $c \neq 0$ is some real number. Using (3.45) one gets

$$f(n\hbar,\theta) = \frac{1}{c}\mathrm{Tr}\,\left\{\widehat{I}(n,\theta)\widehat{\mathcal{F}}\right\}\,. \tag{3.53}$$

However, we can easily show that the correspondence given by (3.52) and (3.53) doesn't satisfy both (3.25) and (3.26) conditions. Indeed, for $f = 1$ one has

$$\begin{aligned} & c \sum_{n=-\infty}^{\infty} \int_{-\pi}^{+\pi} 1\widehat{I}(n,\theta)\tfrac{d\theta}{2\pi} \\ & = c\left(\widehat{1} + \tfrac{2}{\pi}\sum_{k,l=-\infty}^{\infty} \tfrac{(-1)^l}{2l+1} \mid 2l-k+1\rangle\langle -k \mid\right) \neq \widehat{1} \qquad \forall c \in \mathbb{R} - \{0\}\,. \end{aligned} \tag{3.54}$$

Moreover, as we know from (3.46), only the family $\left\{\widehat{I}(2n,\theta)\right\}$ has the group property similar to that given by (3.33). Consequently, the *operator basis* $\left\{\widehat{\Omega}(n,\theta)\right\}$ *seems to be more justified from the physical point of view than* $\left\{\widehat{I}(n,\theta)\right\}$.

From the considerations of the present section it follows that the basis $\left\{\widehat{\Omega}(n,\theta)\right\}$ defines a one to one correspondence between functions on the "quantized" phase space $\hbar\mathbb{Z}\times S^1$ and the operators acting in $L^2(S^1)$ (or more precisely in the Gelfand triplet $C^\infty(S^1) \subset L^2(S^1) \subset (C^\infty(S^1))'$).

Now we are in a position to find the Moyal $*$-product for the quantization on the cylinder.

Let $f_1 = f_1(n\hbar,\theta)$ and $f_2 = f_2(n\hbar,\theta)$ be arbitrary functions on $\hbar\mathbb{Z}\times S^1$ and let $\widehat{f_1}$ and $\widehat{f_2}$ be the corresponding operators. As in the $\mathbb{R}^2$ case we denote by $f_1 * f_2 = (f_1 * f_2)(n\hbar,\theta)$ the function which corresponds to the product $\widehat{f_1}\widehat{f_2}$.

From (3.24) and (3.17) one gets

$$(f_1 * f_2)(n\hbar,\theta) = \mathrm{Tr}\left\{\widehat{\Omega}(n,\theta)\widehat{f}_1\widehat{f}_2\right\} = \tfrac{1}{4\pi^2}\sum_{n',n''=-\infty}^{\infty}\int_{-\pi}^{+\pi} d\theta' \tag{3.55}$$
$$\times \int_{-\pi}^{+\pi} d\theta''\left\{f_1(n'\hbar,\theta')\mathrm{Tr}\left\{\widehat{\Omega}(n,\theta)\widehat{\Omega}(n',\theta')\widehat{\Omega}(n'',\theta'')\right\} f_2(n''\hbar,\theta'')\right\}.$$

Employing (3.22) and the formula

$$\tfrac{2}{\pi}\sum_{k=-\infty}^{\infty}\tfrac{(-1)^k}{(2k+1)}\exp\{i(2k+1)\theta\} = \tfrac{4}{\pi}\sum_{k=0}^{\infty}\tfrac{(-1)^k}{(2k+1)}\cos\{(2k+1)\theta\}$$
$$= sgn\{\cos(\theta)\}\,, \tag{3.56}$$

where, as usual, the function $sgn(x)$ is defined by

$$sgn(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases} \tag{3.57}$$

we find the fundamental result

$$\begin{aligned}&\mathrm{Tr}\left\{\widehat{\Omega}(n,\theta)\widehat{\Omega}(n',\theta')\widehat{\Omega}(n'',\theta'')\right\}\\ &= \exp\{2i[(n''-n)(\theta'-\theta)-(n'-n)(\theta''-\theta)]\}\\ &\times\{1+sgn(\cos(\theta''-\theta))\,sgn(\cos(\theta'-\theta))\\ &+sgn(\cos(\theta'-\theta))\,sgn(\cos(\theta''-\theta'))\\ &+sgn(\cos(\theta''-\theta'))\,sgn(\cos(\theta''-\theta))\}\,.\end{aligned} \tag{3.58}$$

Inserting (3.58) into (3.55) and changing the variables, $\varphi' := \theta'-\theta$, $\varphi'' := \theta''-\theta$, $m' := n'-n$ and $m'' := n''-n$ one obtains

$$\begin{aligned}&(f_1 * f_2)(n\hbar,\theta) = \tfrac{1}{4\pi^2}\sum_{m',m''=-\infty}^{\infty}\int_{-\pi}^{+\pi} d\varphi'\\ &\times\int_{-\pi}^{+\pi} d\varphi'' f_1((n+m')\hbar,\theta+\varphi')\\ &\times\exp\{2i(m''\varphi'-m'\varphi'')\}\times\{1+sgn(\cos\varphi'')\,sgn(\cos\varphi')\\ &+sgn(\cos\varphi')\,sgn(\cos(\varphi''-\varphi'))\\ &+sgn(\cos(\varphi''-\varphi'))\,sgn(\cos\varphi'')\}f_2((n+m'')\hbar,\theta+\varphi'')\}\,.\end{aligned} \tag{3.59}$$

Now we show that using the formal expansions of $f_1((n+m')\hbar, \theta+\varphi')$ and $f_2((n+m'')\hbar, \theta+\varphi'')$ one can bring (3.59) to the form similar to (2.30).

To this end we insert into (3.59)

$$f_1((n+m')\hbar, \theta+\varphi') = \sum_{k=0}^{\infty} \tfrac{1}{k!} \tfrac{\partial^k f_1(p,\theta+\varphi')}{\partial p^k}|_{p=n\hbar} (m'\hbar)^k$$
$$f_2((n+m'')\hbar, \theta+\varphi'') = \sum_{k'=0}^{\infty} \tfrac{1}{k'!} \tfrac{\partial^{k'} f_2(p,\theta+\varphi'')}{\partial p^{k'}}|_{p=n\hbar} (m''\hbar)^{k'} . \qquad (3.60)$$

Thus

$$\begin{aligned}
&(f_1 * f_2)(n\hbar, \theta) = \tfrac{1}{4\pi^2} \sum_{m',m''=-\infty}^{\infty} \sum_{k,k'=0}^{\infty} \int_{-\pi}^{+\pi} d\varphi' \int_{-\pi}^{+\pi} d\varphi'' \\
&\times \tfrac{1}{k!k'!} \tfrac{\partial^k f_1(p,\theta+\varphi')}{\partial p^k}|_{p=n\hbar} (m'\hbar)^k (m''\hbar)^{k'} \exp\{2i\,(m''\varphi' - m'\varphi'')\} \\
&\times \big\{1 + sgn\,(\cos\varphi'')\, sgn\,(\cos\varphi') + sgn\,(\cos\varphi')\, sgn\,(\cos\,(\varphi''-\varphi')) \\
&+ sgn\,(\cos\,(\varphi''-\varphi'))\, sgn\,(\cos\varphi'')\big\} \tfrac{\partial^{k'} f_2(p,\theta+\varphi'')}{\partial p^{k'}}|_{p=n\hbar} \\
&= \tfrac{1}{4\pi^2} \sum_{m',m''=-\infty}^{\infty} \sum_{k,k'=0}^{\infty} \int_{-\pi}^{+\pi} d\varphi' \int_{-\pi}^{+\pi} d\varphi'' \\
&\times \tfrac{1}{k!k'!} \tfrac{\partial^k f_1(p,\theta+\varphi')}{\partial p^k}|_{p=n\hbar} \left(-\tfrac{\hbar}{2i}\right)^k \left[\tfrac{\partial^k}{\partial\varphi''^k} \exp(-2im'\varphi'')\right] \\
&\times \left(\tfrac{\hbar}{2i}\right)^{k'} \left[\tfrac{\partial^{k'}}{\partial\varphi'^{k'}} \exp(2im''\varphi')\right] \{1 + sgn\,(\cos\varphi'')\, sgn\,(\cos\varphi') \\
&+ sgn\,(\cos\varphi')\, sgn\,(\cos\,(\varphi''-\varphi')) \\
&+ sgn\,(\cos\,(\varphi''-\varphi'))\, sgn\,(\cos\varphi'')\big\} \tfrac{\partial^{k'} f_2(p,\theta+\varphi'')}{\partial p^{k'}}|_{p=n\hbar} . \qquad (3.61)
\end{aligned}$$

Employing the formula

$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \exp(\pm 2im\varphi) = \frac{1}{2} \sum_{l=-\infty}^{\infty} \delta(\varphi + \pi l) \qquad (3.62)$$

and integrating by parts one gets

$$
\begin{aligned}
&(f_1 * f_2)(n\hbar,\theta) = \tfrac{1}{4} \sum_{k,k'=0}^{\infty} \int_{-\pi}^{+\pi} d\varphi' \int_{-\pi}^{+\pi} d\varphi'' \\
&\times \tfrac{1}{k!k'!} \tfrac{\partial^{k'+k} f_1(p,\theta+\varphi')}{\partial\theta^{k'}\partial p^k}|_{p=n\hbar} \left(\tfrac{\hbar}{2i}\right)^k \left(-\tfrac{\hbar}{2i}\right)^{k'} \\
&\times \{\delta(\varphi') + \delta(\varphi'+\pi) + \delta(\varphi'-\pi)\} \\
&\times \{\delta(\varphi'') + \delta(\varphi''+\pi) + \delta(\varphi''-\pi)\} \\
&\times \big\{1 + sgn\,(\cos\varphi'')\, sgn\,(\cos\varphi') \\
&+ sgn\,(\cos\varphi')\, sgn\,(\cos(\varphi''-\varphi')) \\
&+ sgn\,(\cos(\varphi''-\varphi'))\, sgn\,(\cos\varphi'')\big\} \tfrac{\partial^{k+k'} f_2(p,\theta+\varphi'')}{\partial\theta^k \partial p^{k'}}|_{p=n\hbar} \\
&= \sum_{k,k'=0}^{\infty} \int_{-\pi}^{+\pi} d\varphi' \int_{-\pi}^{+\pi} d\varphi'' \\
&\times \tfrac{1}{k!k'!} \left(\tfrac{\hbar}{2i}\right)^k \left(-\tfrac{\hbar}{2i}\right)^{k'} \tfrac{\partial^{k'+k} f_1(p,\theta+\varphi')}{\partial\theta^{k'}\partial p^k}|_{p=n\hbar} \\
&\times \delta(\varphi')\delta(\varphi'') \tfrac{\partial^{k+k'} f_2(p,\theta+\varphi'')}{\partial\theta^k \partial p^{k'}}|_{p=n\hbar} \\
&= \left\{ \sum_{k,k'=0}^{\infty} \tfrac{1}{k!k'!} \left(-\tfrac{i\hbar}{2}\right)^k \left(\tfrac{i\hbar}{2}\right)^{k'} \tfrac{\partial^{k'+k} f_1(p,\theta)}{\partial\theta^{k'}\partial p^k} \tfrac{\partial^{k+k'} f_2(p,\theta)}{\partial\theta^k \partial p^{k'}} \right\}|_{p=n\hbar} . \qquad (3.63)
\end{aligned}
$$

Finally, we find the fundamental result

$$
\begin{aligned}
&(f_1 * f_2)(n\hbar,\theta) = \left\{ f_1(p,\theta) \exp\left(\tfrac{i\hbar}{2} \overleftrightarrow{\mathcal{P}}\right) f_2(p,\theta) \right\}|_{p=n\hbar} \\
&\overleftrightarrow{\mathcal{P}} := \tfrac{\overleftarrow{\partial}}{\partial\theta}\tfrac{\overrightarrow{\partial}}{\partial p} - \tfrac{\overleftarrow{\partial}}{\partial p}\tfrac{\overrightarrow{\partial}}{\partial\theta} \\
&\exp\left(\tfrac{i\hbar}{2}\overleftrightarrow{\mathcal{P}}\right) = \left\{ \sum_{k'=0}^{\infty} \tfrac{1}{k'!} \left(\tfrac{i\hbar}{2}\right)^{k'} \tfrac{\overleftarrow{\partial}^{k'}}{\partial\theta^{k'}} \tfrac{\overrightarrow{\partial}^{k'}}{\partial p^{k'}} \right\} \left\{ \sum_{k=0}^{\infty} \tfrac{1}{k!} \left(-\tfrac{i\hbar}{2}\right)^k \tfrac{\overleftarrow{\partial}^{k}}{\partial p^k} \tfrac{\overrightarrow{\partial}^{k}}{\partial\theta^k} \right\} .
\end{aligned} \qquad (3.64)
$$

Concluding, one obtains the *Moyal $*$-product of the form given for the $\mathbb{R}^2$ case (see*(2.30)*) and the only difference is that in the case of cylinder, after performing calculations, one must put $p = n\hbar$, $n \in \mathbb{Z}$.*

We suppose that analoguos facts will be observed in more general cases. In other words if we want the general theory of deformation quantization [1,9,10] to give the "physical" results then we should expect that, in general, the non-trivial topology of the coordinate space will cause the quantization of the classical phase space in the form similar to the case of the cylinder. This very interesting problem is now under consideration and we hope to present some results soon.

Here we consider also the Wigner function for the cylinder (compare with [11,12]). If $\widehat{\rho} = \widehat{\rho}^{+}$ is the density operator of the system then analogously as in the $\mathbb{R}^2$ case (see (2.43) and (2.44)) we define the corresponding Wigner function by

$$w = w(n\hbar, \theta) := \frac{1}{2\pi} \mathrm{Tr} \left\{ \widehat{\Omega}(n, \theta) \widehat{\rho} \right\} . \tag{3.65}$$

If

$$\widehat{\rho} = \sum_{j=0}^{\infty} \rho_j \mid \Psi_j \rangle \langle \Psi_j \mid , \qquad \langle \Psi_j \mid \Psi_{j'} \rangle = \delta_{jj'} \tag{3.66}$$

is the spectral representation of $\widehat{\rho}$ then inserting (3.22) and (3.66) into (3.65) one gets

$$\begin{aligned} w = w(n\hbar, \theta) = & \tfrac{1}{2\pi} \exp(-2in\theta) \sum_{j=0}^{\infty} \rho_j \sum_{k=-\infty}^{\infty} \{ \exp(2ik\theta) \\ & \times \left[\langle \Psi_j \mid 2n - k \rangle \langle k \mid \Psi_j \rangle \right. \\ & \left. + \tfrac{2}{\pi} \sum_{l=-\infty}^{\infty} \tfrac{(-1)^l}{(2l+1)} \exp\{-i(2l+1)\theta\} \langle \Psi_j \mid 2(n+l) - k + 1 \rangle \langle k \mid \Psi_j \rangle \right] \} . \end{aligned} \tag{3.67}$$

One quickly shows that

$$\widehat{\rho} = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{+\pi} w(n\hbar, \theta) \widehat{\Omega}(n, \theta) d\theta \tag{3.68}$$

and

$$\widehat{\rho} = \widehat{\rho}^{+} \iff w = \overline{w} , \tag{3.69}$$

$$\mathrm{Tr}\,(\widehat{\rho}) = 1 \iff \sum_{n=-\infty}^{\infty} \int_{-\pi}^{+\pi} w(n\hbar, \theta) d\theta = 1 , \tag{3.70}$$

$$\mathrm{Tr}\,\left(\widehat{\rho} \widehat{g}^{+} \widehat{g}\right) \geq 0 \iff \sum_{n=-\infty}^{\infty} \int_{-\pi}^{+\pi} (\overline{g} * g)\,(n\hbar, \theta) w(n\hbar, \theta) d\theta \geq 0 . \tag{3.71}$$

Then, the expected value of an observable $\widehat{f}$ is given by

$$\left\langle \widehat{f} \right\rangle = \frac{\sum_{n=-\infty}^{\infty} \int_{-\pi}^{+\pi} f(n\hbar, \theta) w(n\hbar, \theta) d\theta}{\sum_{n=-\infty}^{\infty} \int_{-\pi}^{+\pi} w(n\hbar, \theta) d\theta} . \tag{3.72}$$

Finally, the Liouville-von Neumann equation reads

$$\frac{dw}{dt} = \{H, w\}_M \tag{3.73}$$

and the Heisenberg equation is given by

$$\frac{df}{dt} = \{f, H\}_M \ , \tag{3.74}$$

where the Moyal bracket $\{\cdot, \cdot\}_M$ is defined as follows

$$\{f_1, f_2\}_M = \frac{1}{i\hbar}(f_1 * f_2 - f_2 * f_1)(n\hbar, \theta) \tag{3.75}$$

and H stands for the Hamiltonian.

4. Conclusions

As it has been shown, in order to obtain physically correct results for deformation quantization on the cylinder it is necessary to reduce (quantize) the classical phase space $\mathbb{R}\times S^1$ to its subset $\hbar\mathbb{Z}\times S^1$ like in [11,13]. Consequently, the Stratonovich-Weyl quantizer is also defined on this quantized phase space. It is expected that the same will be observed in the case of any phase space with non-trivial topology (see [24]).

Similar results concerning quantization of the classical phase space $\mathbb{R}\times S^1$ were recently obtained by del Olmo and González [25,26].

We must note another promising aproach to the quantization on the cylinder, namely the method of coherent states [25,27,28]. The relation between deformation quantization and the coherent states method for the cylinder has been considered in [25]. Nevertheless this relation and also the relation between deformation quantization and quantization on the "quantum cylinder" [29] are not clear.

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