DYNAMICS OF A PLANAR DOMAIN WALL WITH OSCILLATING THICKNESS IN $\lambda\Phi^4$ MODEL

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Domain wall-type solution with oscillating thickness in a real, scalar field model is investigated with the help of a polynomial approximation. We propose a simple extension of the polynomial approximation method. In this approach we calculate higher order corrections to the planar domain wall solution and find that the domain wall with oscillating thickness radiates.

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1. Introduction

Topological defects constitutes an important class of solutions in field-theoretical models with degenerate vacua. They play very important role in several branches of physics. Let us mention here field-theoretical cosmology and the cosmic strings hypothesis (see [1–3]), dynamics of superconductors, superfluids and liquid crystals in condensed matter physics (see [3,7]) as well as a flux tube in QCD.

This short and incomplete list shows the necessity of having effective computing methods to study the dynamics of topological defects. In spite of the increasing development of mathematical techniques to solve nonlinear equations, exact solutions seem not to be the rule and numerical methods have been the most common approach to study properties of topological defects. Therefore analytical and perturbative methods are of great interest and importance.

In this paper we study domain wall type solution in the $\lambda\Phi^4$ model. In paper [8] excited domain wall of this kind was considered and the radiation emitted from the domain wall was found. In our approach we further develop the method of a so-called polynomial approximation (see [9,10]), which is

used to construct the domain wall type solution with time-dependent thickness. We show how to compute corrections to the polynomial solution. The corrections consist of two parts: the static one and the time-dependent one. The time-dependent part of the correction contains radiation emitted by domain wall with oscillating thickness.

The plan of our paper is the following. In the next section we present the model. In Section 3 we derive the time-dependent planar domain wall solution in the polynomial approximation. Section 4 is devoted to the detailed analysis of this solution. We present the method of finding correction to the polynomial solution. In Sections 5 and 6 we calculate static and time-dependent part of this correction. In Section 7 we shortly summarize the main points of our work.

2. The model

We consider the model with single, scalar, real-valued field Φ, defined by the action

$$\mathcal{S} = \int d^4x \left[-\frac{1}{2}\eta_{\mu\nu}\partial^\mu\Phi\partial^\nu\Phi - \frac{\lambda}{2}(\Phi^2 - v^2)^2 \right], \tag{1}$$

where $\eta_{\mu\nu} = \mathrm{diag}\,(-1,1,1,1)$ and λ, v are positive constants. The corresponding equation of motion for the field Φ has the form

$$\partial_\mu\partial^\mu\Phi - 2\lambda\Phi(\Phi^2 - v^2) = 0. \tag{2}$$

The energy functional for the model is given by the formula

$$E[\Phi] = \frac{1}{2}\int d^3\vec{x}\left[(\partial_t\Phi)^2 + (\partial_{\vec{x}}\Phi)^2 + \lambda(\Phi^2 - v^2)^2\right]. \tag{3}$$

It is convenient to rescale the space-time coordinates and the scalar field as follows

$$\begin{aligned} \phi &= \frac{\Phi}{v}, \\ t &= \alpha x^0, \\ \xi &= \alpha x^3, \\ \tilde{x}^1 &= \alpha x^1, \quad \tilde{x}^2 = \alpha x^2, \end{aligned} \tag{4}$$

where $\alpha = \sqrt{\lambda v^2}$. The new variables are dimensionless. The vacuum values of ϕ in the considered model are equal to ± 1. Configuration of the field which smoothly interpolates between these two vacua is called the domain wall. Our goal is to construct the domain wall configuration, localized on the $\tilde{x}^1 - \tilde{x}^2$ plane, with the time-dependent width. The planar domain

wall distinguishes the direction perpendicular to the wall plane, given in our case by the coordinate lines of ξ. As we consider the configuration with ϕ independent of the coordinates $\tilde{x}^1$ and $\tilde{x}^2$, we can restrict our approach to the $1+1$ dimensional model. Equation (2) then takes the form

$$-\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2\phi}{\partial\xi^2}-2\phi(\phi^2-1)=0. \tag{5}$$

Static solutions of this equation are well-known. They have the form

$$\phi(\xi)=\pm\tanh\xi\,. \tag{6}$$

3. Domain wall solution in the polynomial approximation

In this section we construct approximate domain wall type solution of Eq. (5) with time-dependent width. In order to realize this we use the method of a polynomial approximation, whose detailed description is given in [9]. The basic idea of this approach is to approximate the scalar field inside the domain wall by the polynomial in the variable ξ with time-dependent coefficients. Thus, inside an interval $[-\xi_1,\xi_0]$ (ξ_0 and ξ_1 are positive) we have

$$\phi(t,\xi)=a(t)\xi+\frac{1}{2!}b(t)\xi^2+\frac{1}{3!}c(t)\xi^3. \tag{7}$$

The domain wall solution is characterized by the fact, that for sufficiently large $\mid\xi\mid$ the field approaches its vacuum values ±1

$$\begin{aligned}\phi(t,\xi) &= +1 \quad \text{for} \quad \xi\geq\xi_0,\\ \phi(t,\xi) &= -1 \quad \text{for} \quad \xi\leq-\xi_1.\end{aligned} \tag{8}$$

It is possible (see [9]) to adopt more accurate asymptotics for large $\mid\xi\mid$, with exponential correction $\exp(-2\alpha\xi)$ to the vacuum values ±1. For simplicity of further calculations we use expressions (8).

We can tune accuracy of our approximation changing degree of the polynomial (7). One can easily notice, that the cubic Ansatz presented above is the simplest, nontrivial choice. The Ansatz (7) should be smoothly matched with the vacuum solutions at $\xi=\xi_0$ and $\xi=-\xi_1$. The matching conditions follow in a standard manner from Eq. (5). One integrates Eq. (5) over ξ in arbitrary small intervals $[-\xi_1-\varepsilon,-\xi_1+\varepsilon]$; $[\xi_0-\varepsilon,\xi_0+\varepsilon]$ and lets $\varepsilon\to0$. This implies

$$\begin{aligned}\partial_\xi\phi(t,\xi)\mid_{\xi=\xi_0} &= 0,\\ \partial_\xi\phi(t,\xi)\mid_{\xi=-\xi_1} &= 0.\end{aligned} \tag{9}$$

From Eq. (8) one gets

$$\begin{aligned} \phi(t,\xi_0) &= +1, \\ \phi(t,-\xi_1) &= -1. \end{aligned} \tag{10}$$

The Ansatz (7) with matching conditions (9)–(10) allows us to find solution of Eq. (5) in the proposed form. Inserting the expansion (7) into the matching conditions we get, analogously to the case of cylindrical domain wall discussed in [9], the following conditions for functions $a(t)$, $b(t)$, $c(t)$ and parameters ξ_0, ξ_1

$$\xi_0 = \xi_1, \quad a(t) = \frac{3}{2\xi_0}, \quad b(t) = 0, \quad c(t) = -\frac{8a^3}{9}. \tag{11}$$

Then we insert the expansion (7) into (5) and equate to zero coefficients in front of successive powers of ξ. After some easy algebra, considering conditions (11) we get the following equation

$$\ddot{a} + \frac{8}{9}a^3 - 2a = 0. \tag{12}$$

Function $c(t)$ is related to $a(t)$ (see formula (11)) and the last step in our approach is to solve Eq. (12). We construct an approximate solution of Eq. (12), which is convenient for the further analysis. With variables redefined as follows

$$a(t) = \frac{3}{2}A(t), \quad \tau = 2t, \tag{13}$$

Eq. (12) takes the form

$$\ddot{A} = A - A^3, \tag{14}$$

where now dot denotes the derivative with respect to τ. It is easy to notice that there exist periodic solutions. Eq. (14) can be regarded as equation of motion for a particle moving in one dimensional potential field (see Fig. 1)

$$U(A) = -\frac{A^2}{2} + \frac{A^4}{4}. \tag{15}$$

In the further part of our discussion we consider only the periodic solutions oscillating around $A = 1$, because of the connection between A and the domain wall thickness parameter ξ_0. When $A(t) > 0$ for all τ we have also $\xi_0, \xi 1 > 0$, which is consistent with the interpretation of these parameters. The periodic solutions oscillating around $A = -1$ are symmetric (in the sense that when one changes ξ_0 and ξ_1 by each other it has no influence on the dynamics of our system), so we don't discuss this situation. Considering

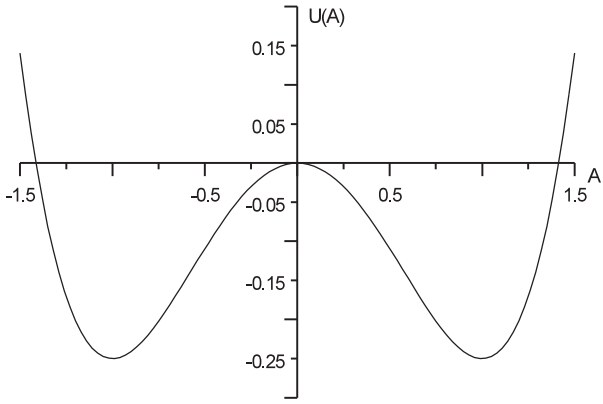

Fig. 1. The potential field $U(A) = -\frac{A^2}{2} + \frac{A^4}{4}$.

all these remarks we restrict our investigation to solutions oscillating with small amplitude around $A = 1$. We can write

$$A(\tau) = 1 + \Psi(\tau), \tag{16}$$

where $\Psi(\tau)$ is small, periodic function. Inserting (16) into Eq. (14) one gets

$$\ddot{\Psi} + \Psi = -\frac{1}{2}\Psi^3 - \frac{3}{2}\Psi^2. \tag{17}$$

Expression (17) has the form of oscillator equation with nonlinear terms. Our goal is to find its perturbative solution with initial conditions: $\Psi(0) = \Psi_0, \dot{\Psi}(0) = 0$, which describes initially static, sqeezed wall. One can solve the nonlinear oscillator equation of type (17) using Krylov–Bogolubov method (see [19]).The general solution has the form

$$\Psi(\tau) = \omega \cos\varphi(\tau) + \frac{1}{2}\alpha_0(\omega) + \sum_{k\geq 1} \alpha_k(\omega)\cos k\varphi(\tau) + \beta_k(\omega)\sin k\varphi(\tau), \tag{18}$$

where functions $\omega,\ \varphi,\ \alpha,\ \beta$ can be calculated in a standard manner (see [19]) with initial condition $\omega(0) = \delta$. In our case we find

$$\Psi(\tau) = \delta\cos\left[\left(1 + \frac{3\delta^2}{16}\right)\tau\right] - \frac{3\delta^2}{4} + \frac{\delta^2}{4}\cos\left[2\left(1 + \frac{2\delta^2}{16}\right)\tau\right] + O(\delta^3), \tag{19}$$

where δ is a small parameter. Thus we get solution $a(t)$

$$a(t) = \frac{3}{2}\left(1 + \delta\cos\left[2\left(1 + \frac{3\delta^2}{16}\right)t\right] - \frac{3\delta^2}{4} + \frac{\delta^2}{4}\cos\left[4\left(1 + \frac{3\delta^2}{16}\right)t\right]\right) + O(\delta^3). \tag{20}$$

Formula (20) agrees well with the numerical solutions of Eq. (12). In the last step we have to insert the solution (20) into the Ansatz (7) considering conditions (11). The final expression for the field of the oscillating domain wall in the polynomial approximation has the form

$$\begin{aligned}\phi^{(0)}(t,\xi) \;=\; & \left[\frac{3}{2}\left(\xi-\frac{1}{3}\xi^3\right)+\frac{3}{2}\delta\cos\Omega t\left(\xi-\xi^3\right)+\frac{3}{8}\delta^2\cos 2\Omega t\left(\xi-3\xi^3\right)\right.\\ & \left.-\frac{9}{8}\delta^2\left(\xi-\frac{1}{3}\xi^3\right)\right]\Theta(\xi_0(t)-\mid\xi\mid)\\ & +\Theta(\xi-\xi_0(t))-\Theta(-\xi_0(t)-\xi)+O(\delta^3), \qquad (21)\end{aligned}$$

where

$$\Omega = 2+\frac{3}{8}\delta^2+O(\delta^3). \qquad (22)$$

The function $\xi_0(t)$ can be regarded as the half-width of the domain wall in the ξ coordinate. It is given by the formula

$$\xi_0(t) = 1-\delta\cos\Omega t+\frac{3}{4}\delta^2+\frac{1}{4}\delta^2\cos 2\Omega t+O(\delta^3). \qquad (23)$$

From Eq. (22) one can get period of oscillation of the domain wall $T = 2\pi/\Omega$: $T = \pi - O(\delta^2)$. In the case of linear oscillations around static solution (one can regard this situation neglecting nonlinear terms in (17) — it is admissible because Ψ is small by definition) one gets $\Omega = 2$ and $T = 2$. The formula (21) can be adopted to the special, static case of domain wall solution, taking $\delta = 0$. It is equivalent to the $(1,0)$ central, critical point of Eq. (17). The general solution $\phi^{(0)}(t,\xi)$ can be then treated as a small oscillation (with amplitude given by the parameter δ) around the static solution $\phi_s^{(0)}(\xi)$

$$\phi_s^{(0)}(\xi) = \frac{3}{2}\left(\xi-\frac{1}{3}\xi^3\right)\Theta(1-\mid\xi\mid). \qquad (24)$$

Accordingly to Eq. (21) initially the domain wall is a bit sqeezed. Then it oscillates with small amplitude around the static solution.

4. Correction to the polynomial solution

In this section we propose a simple extension of the pure polynomial solution obtained above. Formula (21) gives us the simplest, approximate domain wall-type solution in our model. One can study wider range of phenomena (*e.g.* nontrivial asymptotics of the domain wall solution as well as

radiation, which can be emitted by the oscillating domain wall) considering higher-order corrections to this solution. Our goal is to calculate perturbatively corrections to the zeroth order polynomial solution. Let us denote it by $\phi^{(1)}(t,\xi)$. Thus, we can write the field satisfying Eq. (2) as

$$\phi(t,\xi) = \phi^{(0)}(t,\xi) + \phi^{(1)}(t,\xi). \tag{25}$$

In order to find the correction $\phi^{(1)}(t,\xi)$ let us insert expression (25) into the field equation (5). Neglecting nonlinear terms in the field $\phi^{(1)}$ we get

$$\partial_\mu\partial^\mu\phi^{(1)} - 2\left[3\phi^{(0)2} - 1\right]\phi^{(1)} = -\partial_\mu\partial^\mu\phi^{(0)} + 2\left[\phi^{(0)2} - 1\right]\phi^{(0)}. \tag{26}$$

We expect that global character of the domain wall solution during time evolution won't change because of topological stability (in the other words: the solution remains the planar domain wall type all the time, only small corrections can occur). That's why the linearization in (26) may be done. Eq. (26) is a linear one with source term $j(t,\xi)$ given by the formula

$$j(t,\xi) = \frac{\partial^2\phi^{(0)}}{\partial t^2} - \frac{\partial^2\phi^{(0)}}{\partial\xi^2} + 2\Big[\phi^{(0)2} - 1\Big]\phi^0. \tag{27}$$

As it was done for the polynomial solution $\phi^{(0)}$ we can split $j(t,\xi)$ into static and time-dependent part

$$j(t,\xi) = j_s(\xi) + j_d(t,\xi). \tag{28}$$

From Eq. (21) one gets

$$j_s(\xi) = \left[\frac{27}{4}\left(\xi - \frac{1}{3}\xi^3\right)^3 + \xi^3\right]\Theta(1 - |\,\xi\,|) - \left(\delta'(\xi-1) + \delta'(\xi+1)\right), \tag{29}$$

where prim denotes derivative with respect to ξ, and

$$\begin{aligned} j_d(t,\xi) = \Bigg[& -\frac{3}{2}\delta\Omega^2(\xi - \xi^3)\cos\Omega t - 3\delta(4\xi - \xi^3)\cos\Omega t \\ & + \frac{81}{4}\delta\left(\xi - \frac{1}{3}\xi^3\right)^2(\xi - \xi^3)\cos\Omega t\Bigg]\Theta(\xi_0(t) - |\,\xi\,|) \\ & + \left[\frac{27}{4}\left(\xi - \frac{1}{3}\xi^3\right) + \xi^3\right]\left[\Theta(\xi_0(t) - |\,\xi\,|) - \Theta(1 - |\,\xi\,|)\right] \\ & + \left[\frac{3}{2}\delta\Omega^2\left(\xi - \frac{1}{3}\xi^3\right)\cos\Omega t - 3\delta\, sgn\,\xi(1 - 3\xi^2)\cos\Omega t\right]\delta(\xi_0(t) - |\,\xi\,|) \end{aligned}$$

$$
\begin{aligned}
&+\frac{3}{2} sgn\,\xi(1-\xi^2)\left(\delta(\xi_0(t)-\mid\xi\mid)-\delta(1-\mid\xi\mid)\right)\\
&+\frac{3}{2}\delta(\xi-\xi^3)\cos\Omega t\;\delta'(\xi_0(t)-\mid\xi\mid)\\
&-\frac{3}{2}\left(\xi-\frac{1}{3}\xi^3\right)\left[\delta'(\xi_0(t)-\mid\xi\mid)-\delta'(1-\mid\xi\mid)\right]\\
&+\delta\Omega^2\cos\Omega t\left[\delta(\xi+\xi_0(t))-\delta(\xi-\xi_0(t))\right]\\
&-\left[\delta'(\xi-\xi_0(t))-\delta'(\xi-1)\right]-\left[\delta'(\xi+\xi_0(t))-\delta'(\xi+1)\right]. \qquad (30)
\end{aligned}
$$

Eq. (26) is complicated due to unpleasant form of the polynomial solution given by Eq. (21). We can simplify the operator on the left-hand side of Eq. (26) noticing, that $\phi^{(0)}$ can be split into static and time-dependent part, and predominant component $\tanh\xi$ (which is exact, static solution of the initial field equation) can be extracted from the static part as follows

$$\phi^{(0)}=\tanh\xi+\delta\phi_s^{(0)}(\xi)+\delta\phi_d^{(0)}(t,\xi), \qquad (31)$$

Inserting expression (31) into the evolution equation and rewriting some terms on the right-hand side we get finally

$$
\begin{aligned}
\partial_\mu\partial^\mu\phi^{(1)}-2\left[3\tanh^2\xi-1\right]\phi^{(1)}&=j_s(\xi)+j_d(t,\xi)\\
+6\left(\phi_s^{(0)2}-\tanh^2\xi\right)\phi^{(1)}&\\
+6\,\delta\phi_d^{(0)}\left(\delta\phi_d^{(0)}+2\,\delta\phi_s^{(0)}+2\tanh\xi\right)\phi^{(1)}.& \qquad (32)
\end{aligned}
$$

The component in front of $\phi^{(1)}$ in the third term on the right-hand side is nonzero only on the border of the domain wall and quickly tends to zero when $\mid\xi\mid\to\infty$. Analogously, the component in front of $\phi^{(1)}$ in the fourth term is of order of the small parameter δ. Thus, we have the following, strongly suggested method of solving Eq. (32): in the first step we put $\phi^{(1)}=0$ on the right-hand side of Eq. (32) and solve linear equation for $\phi^{(1)}$ with source term $j_s(\xi)+j_d(t,\xi)$. The solution obtained in this way can be inserted to the right-hand side of Eq. (32), so in the second step we have to solve equation with the same linear operator but a new source term. We find full solution as a result of such as iterating procedure. Nevertheless if components in front of $\phi^{(1)}$ on the right-hand side are small (and that is our case), we can find predominant part of full solution in the first step.

Due to the form of source term (which consist of static and time -dependent part) it seems natural to split $\phi^{(1)}$ into two components: the static one and the time-dependent one. They are denoted respectively by $\phi_s^{(1)}$ and $\phi_d^{(1)}$. From Eq. (32) we get

$$\phi_s^{(1)\,\prime\prime}-2\left(3\tanh^2\xi-1\right)\phi_s^{(1)}=j_s(\xi)+6\left(\phi_s^{(0)2}-\tanh^2\xi\right)\phi_s^{(1)}, \qquad (33)$$

for the static part of solution, and

$$\begin{aligned}\Box\phi_d^{(1)} - 2\left[3\tanh^2\xi - 1\right]\phi_d^{(1)} &= j_d(t,\xi)\\ +6\,\delta\phi_d^{(0)}\left(\delta\phi_d^{(0)} + 2\,\delta\phi_s^{(0)} + 2\tanh\xi\right)&\left(\phi_d^{(1)} + \phi_s^{(1)}\right)\\ +6\left(\phi_s^{(0)2} - \tanh^2\xi\right)&\phi_d^{(1)},\end{aligned} \tag{34}$$

for the time-dependent part. In the last equation we use the standard notation

$$\Box = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial\xi^2}. \tag{35}$$

5. Static correction to the polynomial solution

Our goal in this section is to solve Eq. (33) and find static part of the function $\phi^{(1)}(t,\xi)$. We adopt the method of solving, which was discussed in the previous section. In the first step we put $\phi_s^{(1)} = 0$ on the right-hand side of Eq. (33) and solve it in a standard way as a linear, inhomogeneous equation. Then solution $\phi_s^{(1)}$ found in this way is inserted to the r.h.s. of Eq. (33) and in the next step equation with a new source term should be solved. Nevertheless, if expression which occur in a new source term as a result of this procedure is small we get pretty good approximation in the first step. The term in front of $\phi_s^{(1)}$ on the r.h.s. of Eq. (33) is nonzero only on the border of the domain wall and quickly tends to zero when $|\,\xi\,| \to \infty$. Eventually we expect to have a good approximation in the region outside the wall. This result seems to be acceptable — inside the domain wall the accuracy can be improved by higher degree of the polynomial in Ansatz (7). Finally we are going to solve the linear equation

$$\phi_s^{(1)\,\prime\prime} - 2\left[3\tanh^2\xi - 1\right]\phi_s^{(1)} = j_s(\xi). \tag{36}$$

The solution can be found by the standard Green's function technique. There exist two linearly independent solutions of the homogenous part of Eq. (36)

$$\begin{aligned} f_1(x) &= \frac{1}{\cosh^2 x},\\ f_2(x) &= \frac{1}{8}\sinh 2x + \frac{3}{8}\tanh x + \frac{3}{8}\frac{x}{\cosh^2 x}. \end{aligned} \tag{37}$$

As the Green's function we take

$$\begin{aligned} G(\xi,x) &= f_1(x)\,f_2(\xi)\Theta(\xi - x)\\ &\quad - f_1(\xi)\,f_2(x)[\Theta(\xi - x) - \Theta(-x)]. \end{aligned} \tag{38}$$

The Green's function $G(\xi, x)$ obeys the condition

$$G(\xi = 0, x) = 0 \quad \text{for} \quad x \in (-\infty, \infty). \tag{39}$$

The general solution of Eq. (36) has then the form

$$\phi_s^{(1)}(\xi) = A\, f_1(\xi) + B\, f_2(\xi) + \int\limits_R G(\xi, x) j_s(x) dx, \tag{40}$$

where A and B are arbitrary constants. Inserting the Green's function (38) into the formula (40) we get

$$\phi_s^{(1)}(\xi) = \left(A - \int\limits_0^{\xi} f_2(x) j_s(x) dx \right) f_1(\xi) + \left(B + \int\limits_{-\infty}^{\xi} f_1(x) j_s(x) dx \right) f_2(\xi). \tag{41}$$

We have to put $A = 0$ and $B = 0$ because we are looking for a solution generated by source term, not for the homogeneous equation solution. Another reason for keeping $B = 0$ is quick (exponential) growth of the function $f_2(x)$ for $\mid \xi \mid \to \infty$. In the case $B \neq 0$ the solution $\phi_s^{(1)}(\xi)$ grows exponentially and doesn't meet requirements of the perturbative calculus. Combining expressions (29), (38) and (41) we get finally

$$\phi_s^{(1)}(\xi > 1) = -\left[\int\limits_0^1 f_2(x) \left(\frac{27}{4} (x - \frac{1}{3} x^3)^3 + x^3 \right) dx \right] f_1(\xi), \tag{42}$$

and

$$\phi_s^{(1)}(0 \leq \xi \leq 1) = \left[\int\limits_0^{\xi} f_2(x) \left(\frac{27}{4} \left(x - \frac{1}{3} x^3 \right)^3 + x^3 \right) dx \right] f_1(\xi)$$
$$+ \left[\int\limits_{-1}^{\xi} f_1(x) \left(\frac{27}{4} \left(x - \frac{1}{3} x^3 \right)^3 + x^3 \right) dx \right] f_2(\xi). \tag{43}$$

Solution for $\xi < 0$ has opposite sign.

Integration in the formula (42) can be done numerically, yielding

$$\phi_s^{(1)}(\xi > 1) = \frac{C}{\cosh^2(\xi)}, \tag{44}$$

where $C \simeq -0.731$. As one can expect $\phi_s^{(1)}(\xi)$ tends to zero when $\xi \to \infty$. The static solution with the correction $\phi_s^{(0)} + \phi_s^{(1)}$ is presented in Fig. 2, where one can compare it with the pure polynomial solution $\phi_s^{(0)}$ and the strict, static solution $\tanh \xi$. As it is easy to notice the correction obtained above forces the polynomial solution to the well-known strict, static solution. It gives us a strong argument, that splitting the static part of polynomial solution into two parts (one of them is $\tanh \xi$) is acceptable. This fact will be used in the next section.

Eq. (33) can be also solved in a straightforward way using numerical methods. Numerical analysis confirms the results obtained above.

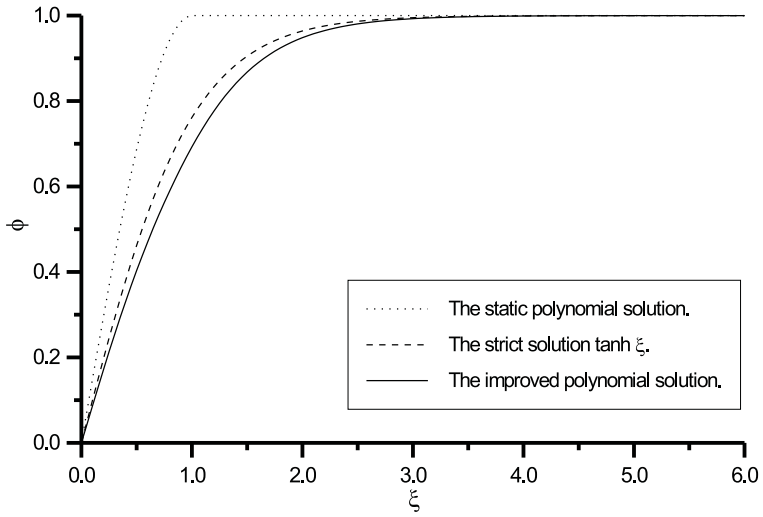

Fig. 2. The static polynomial solution, the improved polynomial solution and the strict solution $\tanh \xi$.

6. Time-dependent correction to the polynomial solution

In this section we find time-dependent correction to the polynomial approximation. We have to solve Eq. (34) using the iterative method proposed in Section 4. The second term on the r.h.s. of the Eq. (34) is of the order of the small parameter δ and can be neglected in the first approximation. The coefficient in front of $\phi_d^{(1)}$ in the third term on the r.h.s. is the same as in Eq. (33) discussed in the previous section. Now it is necessary to specify more precisely the scenario of evolution of the discussed domain wall. For $t < 0$ the domain wall is static and, as was discussed in the Section 6 has the form of the strict, static solution *tanh* ξ (see (6)). In $t = 0$ the domain wall

is squeezed by 2δ as a result of action of an external force. The evolution at later times is governed by the field equation (34).

This scenario helps us to understand the role of the last term on the r.h.s. of Eq. (34). In the simplest case we can state that it is small for $\xi \to \pm\infty$, and by neglecting it we obtain good approximation in the first step in this region. Nevertheless from the discussion presented above we can conclude that we get quite good solution in the whole range of ξ. To prove this we rewrite the third term on the l.h.s. Thus we have

$$\Box\phi_d^{(1)} - 2\left[3\phi_s^{(0)\,2}(\xi) - 1\right]\phi_d^{(1)} = j_d(t,\xi) \\ +6\,\delta\phi_d^{(0)}\left(\delta\phi_d^{(0)} + 2\,\delta\phi_s^{(0)} + 2\tanh\xi\right)\left(\phi_d^{(1)} + \phi_s^{(1)}\right). \tag{45}$$

Due to the results of Section 6 the static part of the polynomial solution is corrected by the function $\phi_s^{(1)}$, which has been calculated above. When we take some terms of higher order (which can be obtained in the next steps of calculation) we can change $\phi_s^{(0)\,2}$ on the r.h.s. of Eq. (34) by "improved" expression $(\phi_s^{(0)}+\phi_s^{(1)})^2$. From the discussion presented above and illustrated in Fig. 2 we have

$$\phi_s^{(0)} + \phi_s^{(1)} \approx \tanh\xi. \tag{46}$$

In the first step we are then going to solve the linear, inhomogeneous equation

$$\Box\phi_d^{(1)} - 2\left[3\tanh^2\xi - 1\right]\phi_d^{(1)} = j_d(t,\xi). \tag{47}$$

This can be rewritten as a wave equation

$$\left[\frac{\partial^2}{\partial t^2} + D^2\right]\phi_d^{(1)}(t,\xi) = -j_d(t,\xi), \tag{48}$$

where operator D^2 has the form

$$D^2 = -\left[\frac{\partial^2}{\partial\xi^2} - 2(3\tanh^2\xi - 1)\right]. \tag{49}$$

Eq. (48) can be solved by the standard Green's function technique. We calculate Green's function for operator D^2 using expansion in eigenfunctions. We are looking for the Green's function $G(\xi,t;\xi',t')$, which fulfil the equation

$$\left[\frac{\partial^2}{\partial t^2} + D^2\right]G(\xi,t;\xi',t') = \delta(t-t')\delta(\xi-\xi'), \tag{50}$$

and obeys the condition

$$G(\xi,t;\xi',t') = 0 \quad \text{for} \quad t < t'. \tag{51}$$

If the set of eigenfunctions $\{\psi_n\}$ is given, we can construct Green's function in a standard manner. Detailed description of this procedure is given in [20]. The Green's function can be written as

$$G(\xi,t;\xi',t') = K(\xi,t;\xi',t')\Theta(t-t'), \tag{52}$$

where propagator $K(\xi,t;\xi',t)$ has the form

$$K(\xi,t;\xi',t') = \psi_0(\xi)\psi_0^*(\xi')(t-t') + \sum_{n\neq 0}\psi_n(\xi)\psi_n^*(\xi')\frac{\sin\left[\sqrt{\lambda_n}(t-t')\right]}{\sqrt{\lambda_n}}. \tag{53}$$

The solution of the inhomogeneous equation is given by the formula

$$\phi_d^{(1)}(t,\xi) = -\int\limits_R d\xi' \int\limits_{-\infty}^{t} dt' K(\xi,t;\xi',t')\, j_d(t',\xi'). \tag{54}$$

Because $j_d(t<0,\xi)=0$ we can simplify (54) as follows

$$\phi_d^{(1)}(t,\xi) = -\int\limits_R d\xi' \int\limits_{0}^{t} dt' K(\xi,t;\xi',t')\, j_d(t',\xi'). \tag{55}$$

The problem of calculating time-dependent correction to the polynomial solution is then reduced to finding the system of eigenfunctions of operator D^2. This system is well-known (see *e.g.* [8])and consists of two discrete eigenvalues and continuous spectrum with corresponding eigenfunctions

$$\psi_0(x) = \frac{\sqrt{3}}{2}\frac{1}{\cosh^2 x} \quad \text{for} \quad \lambda_0 = 0, \tag{56}$$

$$\psi_1(x) = \sqrt{\frac{3}{2}}\frac{\sinh x}{\cosh^2 x} \quad \text{for} \quad \lambda_1 = 3, \tag{57}$$

$$\psi_k(x) = \frac{1}{\sqrt{2\pi(k^2+1)(k^2+4)}}\, e^{ikx}\left[2 - \frac{3}{\cosh^2 x} - 3ik\tanh x - k^2\right], \tag{58}$$

where

$$\lambda_k = k^2+4, \quad k\in R^+.$$

The eigenfunctions from the continuous part of the spectrum correspond to the real eigenvalues and can be split into two sets of eigenfunctions, orthogonal to each other

$$\psi_k^{(1)}(x) = \frac{1}{\sqrt{2\pi(k^2+1)(k^2+4)}} \times\left[\left(2-k^2-\frac{3}{\cosh^2 x}\right)\cos kx + 3k\sin kx\tanh x\right], \tag{59}$$

and

$$\psi_k^{(2)}(x) = \frac{1}{\sqrt{2\pi(k^2+1)(k^2+4)}} \times \left[\left(2-k^2-\frac{3}{\cosh^2 x}\right)\sin kx - 3k\cos kx\tanh x\right]. \quad (60)$$

Inserting formulae (56)–(60) into (52)–(55) we get the expression for the retarded Green's function

$$G_r(\xi,t;\xi',t') = \Theta(t-t')\Bigg((t-t')\psi_0(\xi)\psi_0^*(\xi') + \frac{1}{\sqrt{3}}\sin\left[\sqrt{3}(t-t')\right]\psi_1(\xi)\psi_1^*(\xi') + \sum_{i=1,2}\int_0^\infty dk \frac{\sin\left[\sqrt{k^2+4}(t-t')\right]}{\sqrt{k^2+4}}\psi_k^{(i)}(\xi)\psi_k^{(i)*}(\xi')\Bigg). \quad (61)$$

We get the solution $\phi_d^{(1)}(t,\xi)$ integrating $G_r(\xi,t;\xi',t')$ with the source term $j_d(t',\xi')$ in the whole range of ξ'. As $j_d(t',\xi')$ is odd in variable ξ', part of G_r even in this variable does not give any contribution to the final solution. The Green's function can be rewritten as

$$G_r(\xi,t;\xi',t') = \Theta(t-t')\left(\frac{1}{\sqrt{3}}\sin\left[\sqrt{3}(t-t')\right]\psi_1(\xi)\psi_1^*(\xi') + \int_0^\infty dk \frac{\sin\left[\sqrt{k^2+4}(t-t')\right]}{\sqrt{k^2+4}}\psi_k^{(1)}(\xi)\psi_k^{(1)*}(\xi')\right). \quad (62)$$

Finally, inserting Eq. (62) into (55) we can write solution $\phi_d^{(1)}(t,\xi)$. For convenience we split it into two parts due to the parts of Green's function (62). It reads

$$\phi_d^{(1)}(t,\xi) = \phi_{d\,(\sim)}^{(1)}(t,\xi) + \phi_{d\,(-)}^{(1)}(t,\xi), \quad (63)$$

where $\phi_{d\,(\sim)}^{(1)}$ and $\phi_{d\,(-)}^{(1)}$ are given by formulae

$$\phi_{d\,(\sim)}^{(1)}(t,\xi) = -\frac{1}{2\pi}\int_R d\xi' \int_0^t dt' \int_0^\infty dk \frac{\sin\left[\sqrt{k^2+4}(t-t')\right]}{(k^2+1)(k^2+4)^{3/2}} j_d(t',\xi')$$

$$\times \left((2-k^2)\sin k\xi - 3k\cos k\xi \tanh\xi - \frac{3\cos k\xi}{\cosh^2\xi} \right)$$
$$\times \left((2-k^2)\sin k\xi' - 3k\cos k\xi' \tanh\xi' - \frac{3\cos k\xi'}{\cosh^2\xi'} \right), \tag{64}$$

$$\phi^{(1)}_{d\,(-)}(t,\xi) = -\frac{\sqrt{3}}{2}\frac{\sinh\xi}{\cosh^2\xi}\int\limits_R d\xi' \int\limits_0^t dt' j_d(t',\xi')\sin\left[\sqrt{3}(t-t')\right]\frac{\sinh\xi'}{\cosh^2\xi'}. \tag{65}$$

Let us give an interpretation of these two parts of the final solution. $\phi^{(1)}_{d\,(-)}$ is connected with the ψ_1 mode in the system of eigenfunctions of the operator D^2. The interaction with ψ_1 generates in the time-dependent correction $\phi^{(1)}_d$ the component, which exponentially vanishes for large ξ and oscillates. It can be treated as a form of excitation of the domain wall given by ψ_1. The crucial role which this component plays in dynamics of our system is more transparently visible in the process of collision of kinks (see [16] and [17]).

The component $\phi^{(1)}_{d\,(\sim)}$ has the form of a wavepacket — the oscillations of the domain wall generate radiation. Thus we have the following scenario of the discussed phenomena: in $t=0$ the external force squeezed the domain wall by 2δ. The wall oscillates around the static configuration. The oscillations generate radiation as in the formula (64). It is natural to restrict $\phi^{(1)}_{d\,(\sim)}$ to waves going out from the wall

$$\phi^{(1)}_{d\,(\sim)} \to \phi^{(1)}_{d\,(\to)}(t,\xi>0) = -\frac{1}{4\pi}\int\limits_R d\xi' \int\limits_0^t dt' \frac{1}{(k^2+1)(k^2+4)^{3/2}} j_d(t',\xi')$$
$$\times\cos\left[k(\xi-\xi') - \sqrt{k^2+4}(t-t')\right]\left[3k\tanh\xi\left(2-k^2-\frac{3}{\cosh^2\xi'}\right)\right.$$
$$\left.-3k\tanh\xi'\left(2-k^2-\frac{3}{\cosh^2\xi}\right)\right]$$
$$+\sin\left[k(\xi-\xi') - \sqrt{k^2+4}(t-t')\right]\left[-2\left(2-k^2-\frac{3}{\cosh^2\xi}\right)\right.$$
$$\left.\times\left(2-k^2-\frac{3}{\cosh^2\xi'}\right) - 9k^2\tanh\xi\tanh\xi'\right]. \tag{66}$$

For $\xi<0$ the result is analogous.

The solution $\phi^{(1)}_{d\,(\rightarrow)}$ has the form of a wavepacket

$$\phi^{(1)}_{d\,(\rightarrow)}(t,\xi) = \int_0^\infty dk F(k,t,\xi) e^{i(k\xi - \sqrt{k^2+4}t)}. \tag{67}$$

The waves have the frequency $\omega(k) = \sqrt{k^2+4}$ so they satisfy the dispersion relation

$$\omega^2(k) - k^2 = 4. \tag{68}$$

It agrees well with our expectations, because for large $\mid \xi \mid$ Eq. (47) reads

$$\Box \phi^{(1)}_d - 4\phi^{(1)}_d = 0. \tag{69}$$

To plot the function $\phi^{(1)}_{d\,(\rightarrow)}$ given by the formula (66) we calculate it numerically for some fixed values of t and $\xi \in (0, 20)$, with the step $\Delta\xi = 0.1$. The final result is presented in Fig. 3–7. One can easily see the causal character of obtained solution. In the region outside the light-cone the field $\phi^{(1)}_{d\,(\rightarrow)}$ is nearly equal to 0. The fluctuations are caused mainly by computation errors. This fact was checked by decreasing the step of integration — the fluctuations decrease then too. The field $\phi^{(1)}_{d\,(\rightarrow)}$ is small — on the figures it is multiplied by factor 10^2. This once again confirms the approximation used in our approach.

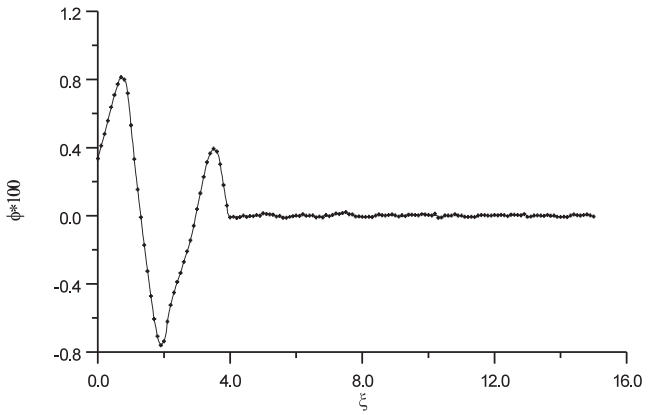

Fig. 3. The radiation emitted by the oscillating domain wall for $t = 3$.

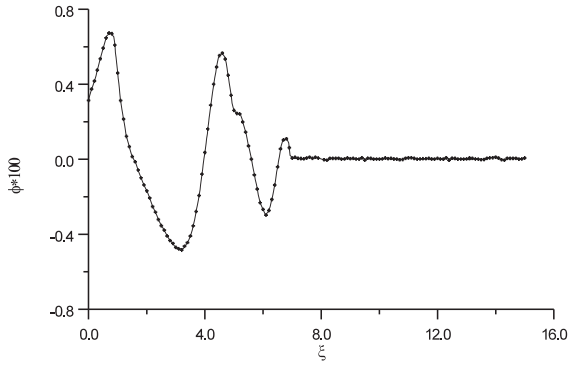

Fig. 4. The radiation emitted by the oscillating domain wall for $t = 6$.

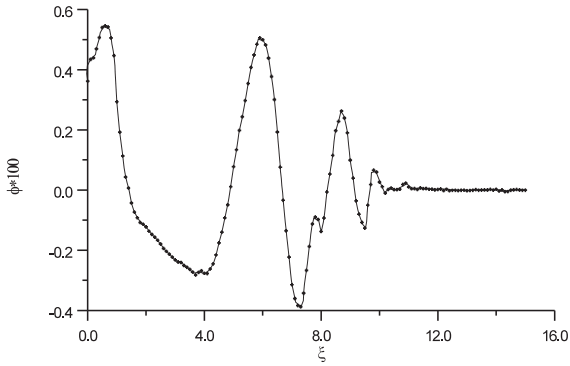

Fig. 5. The radiation emitted by the oscillating domain wall for $t = 9$.

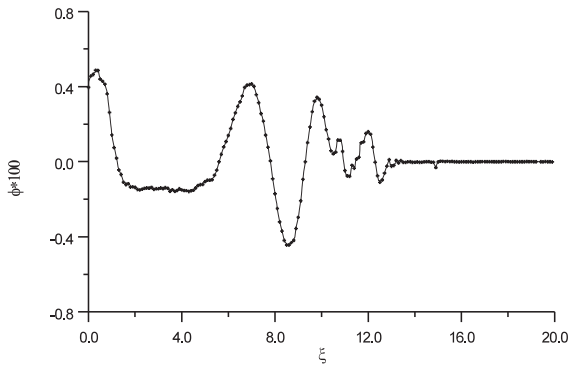

Fig. 6. The radiation emitted by the oscillating domain wall for $t = 12$.

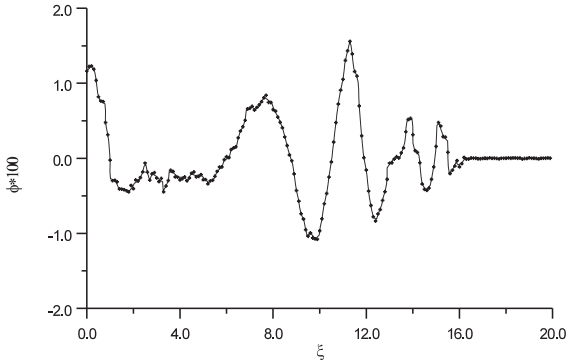

Fig. 7. The radiation emitted by the oscillating domain wall for $t = 15$.

7. Remarks

Let us briefly summarize the main points of our work. We have derived the planar domain wall solution with the time-dependent thickness in the polynomial approximation. The planar domain wall oscillates with small amplitude. We have then proposed an approach enabling us to calculate the correction to the pure polynomial solution. We have split this correction into two parts. The time-dependent part contains the component which is interpreted as the radiation from the oscillating domain wall.

There are several possible extensions and generalizations of our approach. It can be applied without much trouble to the other field-theoretical models. In our paper we discuss only the case of small oscillations around the static solution. As one can easily see in Fig. 1 there exist also solutions oscillating with amplitude which is not small. This case should be discussed separately. Another problem is the backreaction of the radiation in the considered, oscillating domain wall. As we have the energy flowing out from the domain wall, one can expect dumping of oscillations. We have checked that the energy per unit area of the domain wall is conserved up to terms neglected. The problem of the backreaction of the domain wall deserve a separate investigation. It will be discussed in the next paper.

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