

THE OPERATOR FORM OF THE EFFECTIVE POTENTIAL GOVERNING THE TIME EVOLUTION IN n -DIMENSIONAL SUBSPACE OF STATES

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This paper presents the operator form of the effective potential V governing the time evolution in 2 and 3 and n dimensional subspace of states. The general formula for the n dimensional case is considered the starting point for the calculation of the explicit formulae for 2 and 3 dimensional degenerate and non-degenerate cases. We relate the 2 and 3 dimensional cases to some physical systems which are currently investigated.

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1. Introduction

The time evolution of physical systems in the Hilbert space is described by the Shrödinger equation:

$$i\frac{\partial}{\partial t}|\psi; t\rangle = H|\psi; t\rangle, \quad (1)$$

where $\hbar = c = 1$. If we choose the initial conditions:

$$|\psi; t = 0\rangle \equiv |\psi\rangle, \quad (2)$$

then the time evolution is described by a unitary operator $U(t)|\psi\rangle = |\psi; t\rangle$ ($|\psi\rangle, |\psi; t\rangle \in \mathcal{H}$, $U(t) = e^{-itH}$).

Vector $|\psi; t\rangle \in \mathcal{H}$ carries complete information about the physical system considered. In particular, the properties of the system which are described by vectors belonging to a closed subspace $\mathcal{H}_{||}$, of $\mathcal{H}$ can be extracted from $|\psi; t\rangle$. In such a case it is sufficient to know the component $|\psi; t\rangle_{||} \in \mathcal{H}_{||}$ of $|\psi; t\rangle$. The subspace $\mathcal{H}_{||}$ is defined by a projector P : $\mathcal{H}_{||} = \mathcal{P}\mathcal{H}$, which simply means that $|\psi; t\rangle_{||} = P|\psi; t\rangle$.

Alternatively, the same result can be obtained by studying the time evolution not in the total space of states $\mathcal{H}$ but in a closed subspace $\mathcal{H}_{||}$. In this way the total state space is split into two orthogonal subspaces $\mathcal{H}_{||}$ and $\mathcal{H}_{\perp} = \mathcal{H} \ominus \mathcal{H}_{||}$, and the Shrödinger equation can be replaced by equations describing each of the subspaces respectively. The equation for $\mathcal{H}_{||}$ has the following form [1–3]:

$$\left(i\frac{\partial}{\partial t} - P H P\right) |\psi; t\rangle_{||} = |\chi; t\rangle - i \int_0^{\infty} K(t - \tau) |\psi; \tau\rangle_{||} d\tau, \quad (3)$$

$$Q = I - P, \quad (4)$$

$$K(t) = \Theta(t) P H Q e^{-i Q H Q} Q H P, \quad (5)$$

$$|\chi; t\rangle = P H Q e^{-i Q H Q} |\psi\rangle_{\perp}, \quad (6)$$

where

$$\Theta(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}.$$

Of course, $K(t) \neq 0$ only if $[P, H] \neq 0$. Condition (2) can now be rewritten as

$$|\psi; t = 0\rangle_{||} \equiv |\psi\rangle_{||}, |\psi; t = 0\rangle_{\perp} \equiv |\psi\rangle_{\perp}, \quad (7)$$

where $|\psi\rangle_{\perp} \equiv Q|\psi\rangle$. If we now assume that at the initial moment no states from $\mathcal{H}_{\perp}$ are occupied, $|\psi\rangle_{\perp} = 0$, (that is $|\chi; t\rangle \equiv 0$, $|\psi\rangle \equiv |\psi\rangle_{||}$) and define the evolution operator for the subspace $\mathcal{H}_{||}$:

$$|\psi; t\rangle_{||} \equiv P|\psi; t\rangle \equiv P U(t) |\psi\rangle \equiv P U(t) P |\psi\rangle_{||}, \quad (8)$$

so

$$U_{||}(t) |\psi\rangle_{||} \stackrel{\text{def}}{=} P U(t) P |\psi\rangle_{||}, \quad (9)$$

we can transform (3) into

$$\left(i\frac{\partial}{\partial t} - P H P\right) U_{||} |\psi\rangle_{||} = -i \int_0^{\infty} K(t - \tau) U_{||}(\tau) |\psi\rangle_{||} d\tau. \quad (10)$$

An equivalent differential form of (10) has been found by Królikowski and Rzewuski [1, 2]:

$$\left(i\frac{\partial}{\partial t} - H_{||}(t)\right) U_{||}(t) |\psi\rangle_{||} = 0, \quad t \geq 0, \quad U_{||}(0) = P, \quad (11)$$

where the $H_{||}(t)$ denotes the effective Hamiltonian:

$$H_{||}(t) \equiv PHP + V_{||}(t). \quad (12)$$

For every effective Hamiltonian $H_{||}$ governing the time evolution in $\mathcal{H}_{||} \equiv P\mathcal{H}$, which in general can depend on time t [1–3], the following identity holds [4–7]:

$$H_{||}(t) \equiv i \frac{\partial U_{||}(t)}{\partial t} [U_{||}(t)]^{-1} P, \quad (13)$$

where $[U_{||}(t)]^{-1}$, is defined as follows

$$[U_{||}(t)]^{-1} U_{||}(t) = U_{||}(t) [U_{||}(t)]^{-1} \equiv P. \quad (14)$$

In the nontrivial case

$$[P, H] \neq 0, \quad (15)$$

from (13), using (12) and (9) we find

$$H_{||}(t) \equiv PHU(t)P[U_{||}(t)]^{-1}P \quad (16)$$

$$\stackrel{\text{def}}{=} PHP + V_{||}(t). \quad (17)$$

and thus

$$V_{||}(t) \equiv PHQU(t)[U_{||}(t)]^{-1}P.$$

Assumption (15) means that transitions of states from $\mathcal{H}_{||}$ into $\mathcal{H}_{\perp}$ and from $\mathcal{H}_{\perp}$ into $\mathcal{H}_{||}$, *i.e.*, the decay and regeneration processes, are allowed. Thus [4–6],

$$H_{||}(0) \equiv PHP, \quad V_{||}(0) = 0, \quad V_{||}(t \rightarrow 0) \simeq -itPHQHP, \quad (18)$$

so, in general $H_{||}(0) \neq H_{||}(t \gg t_0 = 0)$ [4–7] and $V_{||}(t \neq 0) \neq V_{||}^+(t \neq 0)$, $H_{||}(t \neq 0) \neq H_{||}^+(t \neq 0)$. According to the ideas of the standard scattering theory, it can be stated that operator $H_{||}(t \rightarrow \infty) \equiv H_{||}(\infty) \stackrel{\text{def}}{=} H_{||}$ describes the bounded or quasistationary states of the subsystem considered (and in this sense it is similar to *e.g.* the LOY-effective Hamiltonian [8]).

From (10) and (11), (12) it follows that the action of $V_{||}(t)$ on $U_{||}(t)$ has the following form:

$$V_{||}(t)U_{||}(t) = -i \int_0^{\infty} K(t - \tau)U_{||}(\tau)d\tau. \quad (19)$$

The approximate form of $V_{||}$ can be obtained from (10) and (19) with the use of the retarded solution of:

$$\left(i\frac{\partial}{\partial t} - P H P\right) G(t) = P\delta(t), \quad (20)$$

where $G(t)$ is the retarded Green operator:

$$G \equiv G(t) = -i\Theta(t)e^{-itPH P}P. \quad (21)$$

Then, using the iteration procedure for the equation (10) for $U_{||}$ [2,5-7] we get:

$$U_{||} = U_{||}^0(t) + \sum_{n=1}^{\infty} (-i)^n \underbrace{L \circ L \circ L \circ \dots \circ L}_{n \text{ times}} \circ U_{||}^0(t). \quad (22)$$

$U_{||}^0(t)$ is the solution of the following "free" equation [5-7]:

$$\left(i\frac{\partial}{\partial t} - P H P\right) U_{||}^0(t) = 0, \quad U_{||}^0(0) = P. \quad (23)$$

$\circ$ stands for the convolution

$$f \circ g(t) = \int_0^{\infty} f(t-\tau)g(\tau),$$

and

$$L \equiv L(t) = G \circ K(t).$$

Equations (19) and (22) yield:

$$V_{||}(t)U_{||}(t) = -iK \circ U_{||}^0(t) + \sum_{n=1}^{\infty} (-i)^n K \circ L \circ L \circ \dots \circ L \circ U_{||}^0(t). \quad (24)$$

If $\|L(t)\| < 1$ then the series (24) is convergent. It is worth noticing that, unlike in the standard perturbation series, it is not necessary for the perturbation H^1 to be small in relation to H^0 (the full Hamiltonian $H = H^0 + H^1$) if $\|L(t)\| < 1$. This is considered one of the advantages of this approach over the standard ones as it can describe both weak and strong interactions [6]. If for every $t \geq 0$ $\|L(t)\| < 1$ then to the lowest order of $L(t)$, $V_{||}(t)$ is expressed by [6]:

$$V_{||}(t) \simeq V_{||}^1(t) = -i \int_0^{\infty} K(t-\tau)e^{i(t-\tau)PH P} P d\tau. \quad (25)$$

This formula was used to compute $V_{||}(t)$ for one-dimensional subspace $\mathcal{H}_{||}$ and to find the matrix elements of $V_{||}(t)$ acting in a two-state subspace $\mathcal{H}_{||}$ in [7]. In some problems it is more useful and more convenient to use the operator form of $V_{||}(t)$ rather than the the matrix elements of $V_{||}(t)$ only. Searching for the global transformation properties of $V_{||}(t)$ under some operators expressing symetries of the system is as an example of such problems.

Result (25) will be the starting point for the following considerations concerning the explicit operator form of $V_{||}(t)$ in n , 2 and 3 dimensional cases.

2. Effective potential $V_{||}$ in n -dimensional subspace of states

Let us consider a general case of effective potential $V_{||,n}$, acting in an n -dimensional subspace of states. Formally, the equation corresponding to Eq. (12) has the following form:

$$H_{||,n}(t) \stackrel{\text{def}}{=} PHP + V_{||,n}(t). \quad (26)$$

The projector P is defined in the following way:

$$P = \sum_{j=1}^n |e_j\rangle\langle e_j| \equiv I_{||}, \quad (27)$$

where $I_{||}$ is the unit operator in $\mathcal{H}_{||}$, $\{|e_j\rangle\}_{j \in \mathcal{A}}$ and $\{|e_j\rangle\}_{j=1,2,\dots,n} \subset \{|e_j\rangle\}_{j \in \mathcal{A}}$ are complete sets of orthonormal vectors $\langle e_j | e_k \rangle = \delta_{jk}$ in $\mathcal{H}$ and $\mathcal{H}_{||} \subset |\mathcal{H}$, respectively. Consequently, if the state space for the problem is $\mathcal{H}$ then $\mathcal{H}_{||} = \mathcal{PH}$ and P is the unity in $\mathcal{H}_{||}$, $P = I_{||}$ [7].

The subspace $\mathcal{H}_{||}$ can also be spanned by the eigenvectors of the Hermitian matrix PHP :

$$PHP|\lambda_j\rangle = \lambda_j|\lambda_j\rangle, \quad (j = 1, 2, \dots, n). \quad (28)$$

Using $|\lambda_j\rangle$ we define projectors P_j [9], where for simplicity the non-degenerate case of λ_j is assumed:

$$P_j \stackrel{\text{def}}{=} \frac{1}{\langle \lambda_j | \lambda_j \rangle} |\lambda_j\rangle\langle \lambda_j|, \quad (j = 1, 2, 3). \quad (29)$$

Of course, these projectors fulfill the following completeness condition:

$$\sum_{j=1}^n P_j = P. \quad (30)$$

The operator PHP can now be written as follows:

$$PHP = \sum_{j=1}^n \lambda_j P_j, \quad (31)$$

and following:

$$Pe^{\pm itPHP} = P \sum_{j=1}^n e^{\pm it\lambda_j} P_j. \quad (32)$$

This result can be directly applied to equation (25)

$$V_{\parallel,n}(t) = -i \sum_{j=1}^n \int_0^t PHQ e^{-i(t-\tau)(QH Q - \lambda_j)} QHP d\tau P_j. \quad (33)$$

The integration can be easily performed, with the result:

$$V_{\parallel,n}(t) = -i \sum_{j=1}^n \left\{ PHQ \frac{e^{-it(QH Q - \lambda_j)} - 1}{QH Q - \lambda_j} QHP \right\} P_j = \sum_{j=1}^n \Xi(\lambda_j, t) P_j, \quad (34)$$

where

$$\Xi(\lambda, t) \stackrel{\text{def}}{=} PHQ \frac{e^{-it(QH Q - \lambda)} - 1}{QH Q - \lambda} QHP. \quad (35)$$

Knowing that [10]

$$\lim_{t \rightarrow \infty} \Xi(\lambda, t) = PHQ \frac{1}{QH Q - \lambda + i0} QHP, \quad (36)$$

and defining

$$\Sigma(\lambda) \stackrel{\text{def}}{=} PHQ \frac{1}{QH Q - \lambda + i0} QHP,$$

we finally get:

$$V_{\parallel,n} \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} V_{\parallel,n}(t) = -i \lim_{t \rightarrow \infty} \sum_{j=1}^n \Xi(\lambda_j, t) P_j = -i \sum_{j=1}^n \Sigma(\lambda_j) P_j. \quad (37)$$

3. $V_{\parallel}$ in a two dimensional subspace

In this section we find the explicit formula for $V_{\parallel}$ in a two-dimensional subspace of states using the framework presented above. In this case PHP , being a $[2 \times 2]$ Hermitian matrix, has the following form:

$$PHP = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad (38)$$

$$H_{ij} = H_{ji}^*,$$

where $H_{j,k} = \langle \mathbf{e}_j | H | \mathbf{e}_k \rangle$. The eigenvalues of PHP are easy to calculate:

$$PHP|\lambda\rangle = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \lambda \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad (39)$$

$$\lambda^{1,2} = \frac{1}{2}(H_{11} + H_{22}) \pm \sqrt{|H_{12}|^2 + \frac{1}{4}(H_{11} - H_{22})^2}, \quad (40)$$

and if we adopt the symbols used in [4–7]:

$$\lambda^{1,2} \stackrel{\text{def}}{=} H_0 \pm \kappa, \quad (41)$$

where

$$H_0 = \frac{1}{2}(H_{11} + H_{22}), \quad \kappa = \sqrt{|H_{12}|^2 + \frac{1}{4}(H_{11} - H_{22})^2}. \quad (42)$$

Following, the eigenvector $|\lambda^1\rangle$ can be chosen as follows:

$$|\lambda^1\rangle = \begin{pmatrix} \frac{H_{12}}{H_0 + \kappa - H_{11}} \\ 1 \end{pmatrix}, \quad (43)$$

and the projector P_1 :

$$\begin{aligned} P_1 &= \frac{1}{\langle \lambda_1 | \lambda_1 \rangle} |\lambda_1\rangle \langle \lambda_1| \\ &= \frac{1}{\frac{|H_{12}|^2}{(H_0 + \kappa - H_{11})^2} + 1} \begin{pmatrix} \frac{H_{12}}{H_0 + \kappa - H_{11}} \\ 1 \end{pmatrix} \begin{pmatrix} \frac{H_{21}}{H_0 + \kappa - H_{11}}, 1 \end{pmatrix}, \end{aligned} \quad (44)$$

so, explicitly

$$P_1 = \begin{bmatrix} \frac{(H_0 + \kappa - H_{11}) \cdot |H_{12}|^2}{|H_{12}|^2 + (H_0 + \kappa - H_{11})^2} & \frac{(H_0 + \kappa - H_{11}) \cdot H_{12}}{|H_{12}|^2 + (H_0 + \kappa - H_{11})^2} \\ \frac{(H_0 + \kappa - H_{11}) \cdot H_{21}}{|H_{12}|^2 + (H_0 + \kappa - H_{11})^2} & \frac{(H_0 + \kappa - H_{11})^2}{|H_{12}|^2 + (H_0 + \kappa - H_{11})^2} \end{bmatrix}. \quad (45)$$

For clarity let us define:

$$P_j \stackrel{\text{def}}{=} \begin{bmatrix} p_{11}^j & p_{12}^j \\ p_{21}^j & p_{22}^j \end{bmatrix} \quad (j = 1, 2).$$

Both P_1 and PHP can be represented by Pauli matrices:

$$P_1 = p_0^1 \sigma_0 + p_x^1 \sigma_x + p_y^1 \sigma_y + p_z^1 \sigma_z,$$

$$PHP = H_0 \sigma_0 + H_x \sigma_x + H_y \sigma_y + H_z \sigma_z,$$

and the calculation of the coefficients p^j yields:

$$\begin{aligned} p_0 &= \frac{1}{2}(p_{11} + p_{22}) = \frac{1}{2}, \\ p_x &= \frac{1}{2}(p_{12} + p_{21}) = \frac{H_x}{2\kappa}, \\ p_y &= \frac{1}{2}i(p_{12} - p_{21}) = \frac{H_y}{2\kappa}, \\ p_z &= \frac{1}{2}(p_{11} - p_{22}) = \frac{H_z}{2\kappa}. \end{aligned} \quad (46)$$

We can see from the above that p_ν , ($\nu = 0, x, y, z$) can be expressed by H_ν , ($\nu = 0, x, y, z$), so finally we get the following expression for P_1

$$P_1 = \frac{1}{2} \left(\left(1 - \frac{H_0}{\kappa} \right) \sigma_0 + \frac{1}{\kappa} PHP \right). \quad (47)$$

Keeping in mind the fact that in $\mathcal{H}_\parallel$ we have $\sigma_0 = I_\parallel = P$, we obtain:

$$P_1 = \frac{1}{2} \left(\left(1 - \frac{H_0}{\kappa} \right) P + \frac{1}{\kappa} PHP \right), \quad (48)$$

and after performing the same calculation for P_2 :

$$P_2 = \frac{1}{2} \left(\left(1 + \frac{H_0}{\kappa} \right) P - \frac{1}{\kappa} PHP \right). \quad (49)$$

It is easy to verify that the completeness condition (30) is fulfilled:

$$P_1 + P_2 = P.$$

If we now come back to Eq. (34) and use the results obtained in this section, the effective potential $V_\parallel$ will have the following form:

$$\begin{aligned} V_\parallel(t) &= -\frac{1}{2}\Xi(H_0 + \kappa, t) \left[\left(1 - \frac{H_0}{\kappa} \right) P + \frac{1}{\kappa} PHP \right] \\ &\quad - \frac{1}{2}\Xi(H_0 - \kappa, t) \left[\left(1 + \frac{H_0}{\kappa} \right) P - \frac{1}{\kappa} PHP \right]. \end{aligned} \quad (50)$$

Matrix elements of this $V_{||}(t)$ are exactly the same as those obtained in [7].

As noted in [7] this result is significant. For example in the case of neutral K mesons the assumption of CPT invariance and CP noninvariance in the quantum theory, that is $[CPT, H] = 0$ and $[CP, H] \neq 0$, yields:

$$h_{11} - h_{22} \neq 0, \quad (51)$$

where $h_{ij} = \langle \mathbf{e}_i | H_{||} | \mathbf{e}_j \rangle$ are the matrix elements of $H_{||} \equiv PHP + V_{||}$, $V_{||} \stackrel{\text{def}}{=} V_{||}(\infty)$, which runs counter to the usual assumption. More remarks on this problem can be found in the conclusions.

The case of both eigenvalues of PHP equal can easily be obtained from the general case described above. The assumption of both eigenvalues equal for a Hermitian $[2 \times 2]$ matrix yields $H_{11} = H_{22}$ and $H_{12} = H_{21} = 0$. It is easy to verify that $\lambda^1 = \lambda^2 \iff \kappa = 0$. Then:

$$\begin{aligned} \lambda^1 &= \lambda^2 = H_0, \\ PHP &= H_0 P, \end{aligned} \quad (52)$$

and

$$P e^{itPHP} \equiv e^{itH_0} P. \quad (53)$$

Thus, from equations (25) and (33):

$$V_{||}(t) \simeq V_{||}^1(t) = -\Xi(H_0, t)P. \quad (54)$$

Furthermore, if apart from assuming the degenerate case of PHP we take $t \rightarrow \infty$ we will get the same result as obtained from the Wigner-Weisskopf approximation by *e.g.* Lee, Oehme and Yang [8]. It is interesting to notice that in this case $h_{11} = h_{22}$ (where $h_{jj} = \langle j | H_{||} | j \rangle$) with $[CPT, H] = 0$, whereas in the case of $\lambda^1 \neq \lambda^2$ under the same conditions we have (51).

4. $V_{||}$ in a three dimensional subspace

This section describes the explicit formula for $V_{||}$ in a three dimensional subspace of states in a very similar way as it was done for the two dimensional case.

In this case the PHP matrix is a $[3 \times 3]$ matrix, for example

$$PHP = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}, \quad (55)$$

$$H_{ij} = H_{ji}^*,$$

and has the following characteristic equation:

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0, \quad (56)$$

$$\begin{aligned} A &= -(H_{11} + H_{22} + H_{33}), \\ B &= H_{11}H_{22} + H_{11}H_{33} + H_{22}H_{33} - |H_{13}|^2 - |H_{23}|^2 - |H_{12}|^2, \\ C &= -(H_{11}H_{22}H_{33} + 2\operatorname{Re}(H_{12}H_{23}H_{31}) \\ &\quad - H_{11}|H_{23}|^2 - H_{22}|H_{13}|^2 - H_{33}|H_{12}|^2). \end{aligned}$$

It is easy to notice that $A, B, C \in \Re$ so, given the fact that PHP is a Hermitian matrix, equation (56) is a third order equation with real coefficients and real solutions. To find the solutions we will use the Cardano formulae. Bearing in mind that the solutions are real we get the following three cases $\lambda_1 \neq \lambda_2 \neq \lambda_3$, $\lambda_1 = \lambda_2 = \lambda \neq \lambda_3$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$. Let us find the eigenvectors, projectors and the quasipotential for each of the above cases.

4.1. $\lambda_1 \neq \lambda_2 \neq \lambda_3$

In this case the three solutions of the characteristic equation (56) are given by the following formulae:

$$\begin{aligned} \lambda_1 &= -2\left(\frac{A^2-3B}{3}\right)^{\frac{1}{2}} \cos \frac{1}{3}\alpha - \frac{A}{3}, \\ \lambda_2 &= -2\left(\frac{A^2-3B}{3}\right)^{\frac{1}{2}} \cos \frac{1}{3}(\alpha + 2\pi) - \frac{A}{3}, \\ \lambda_3 &= -2\left(\frac{A^2-3B}{3}\right)^{\frac{1}{2}} \cos \frac{1}{3}(\alpha + 4\pi) - \frac{A}{3}, \end{aligned} \quad (57)$$

where $\cos \alpha = \frac{\frac{3}{2}(\frac{2A^3}{27} - \frac{B}{3} + C)}{\frac{2}{3}(\frac{A^2-3B}{3})^{\frac{3}{2}}}$. The following basis of orthogonal eigenvectors can be chosen:

$$|\lambda_j\rangle = \begin{pmatrix} H_{13}(H_{22} - \lambda_j) - H_{23}H_{12} \\ H_{23}(H_{11} - \lambda_j) - H_{13}H_{21} \\ |H_{12}|^2 - (H_{11} - \lambda_j)(H_{22} - \lambda_j) \end{pmatrix}, \quad (58)$$

where $j = 1, 2, 3$. Using these eigenvectors we create projectors P in the way given in Sec. 2:

$$\begin{aligned} P_j &= \frac{1}{\langle \lambda_j | \lambda_j \rangle} |\lambda_j\rangle \langle \lambda_j| \\ &= \{ |H_{13}(H_{22} - \lambda_j) - H_{23}H_{12}|^2 \\ &\quad + |H_{23}(H_{11} - \lambda_j) - H_{13}H_{21}|^2 \\ &\quad + [|H_{12}|^2 - (H_{11} - \lambda_j)(H_{22} - \lambda_j)]^2 \}^{-1} \\ &\quad \times \begin{bmatrix} p_{11}^j & p_{12}^j & p_{13}^j \\ p_{21}^j & p_{22}^j & p_{23}^j \\ p_{31}^j & p_{32}^j & p_{33}^j \end{bmatrix}, \quad (j = 1, 2, 3), \end{aligned} \quad (59)$$

where

$$\begin{aligned}
 p_{11}^j &= |H_{13}(H_{22} - \lambda_j) - H_{23}H_{12}|^2, \\
 p_{12}^j &= (H_{13}(H_{22} - \lambda_j) - H_{23}H_{12})(H_{32}(H_{11} - \lambda_j) - H_{31}H_{12}), \\
 p_{13}^j &= (H_{13}(H_{22} - \lambda_j) - H_{23}H_{12})(|H_{12}|^2 - (H_{11} - \lambda_j)(H_{22} - \lambda_j)), \\
 p_{21}^j &= (H_{23}(H_{11} - \lambda_j) - H_{13}H_{21})(H_{31}(H_{22} - \lambda_j) - H_{32}H_{21}), \\
 p_{22}^j &= |H_{23}(H_{11} - \lambda_j) - H_{13}H_{21}|^2, \\
 p_{23}^j &= (H_{23}(H_{11} - \lambda_j) - H_{13}H_{21})(|H_{12}|^2 - (H_{11} - \lambda_j)(H_{22} - \lambda_j)), \\
 p_{31}^j &= (|H_{12}|^2 - (H_{11} - \lambda_j)(H_{22} - \lambda_j))(H_{31}(H_{22} - \lambda_j) - H_{32}H_{21}), \\
 p_{32}^j &= (|H_{12}|^2 - (H_{11} - \lambda_j)(H_{22} - \lambda_j))(H_{32}(H_{11} - \lambda_j) - H_{31}H_{12}), \\
 p_{33}^j &= (|H_{12}|^2 - (H_{11} - \lambda_j)(H_{22} - \lambda_j))^2,
 \end{aligned}$$

(where $j=1,2,3$). And consequently the quasipotential

$$V_{\parallel,3}(t) = -i \sum_{j=1}^3 \Xi(\lambda_j, t) P_j, \quad (60)$$

$$4.2. \lambda_1 = \lambda_2 = \lambda \neq \lambda_3$$

In this case we have the following expressions for the solutions of the characteristic equation (56):

$$\begin{aligned}
 \lambda &= \left(\frac{2A^3}{27} - \frac{B}{3} + C\right)^{\frac{1}{3}} - \frac{A}{3}, \\
 \lambda_3 &= -2\left(\frac{2A^3}{27} - \frac{B}{3} + C\right)^{\frac{1}{3}} - \frac{A}{3}.
 \end{aligned} \quad (61)$$

In this case to define one of the projectors, say P_3 we can use the result presented above, so the projector will be given by formula (59). We do not actually need to know the remaining two projectors explicitly as

$$V_{\parallel,3}(t) = -i\Xi(\lambda, t)(P_1 + P_2) + \Xi(\lambda_3, t)P_3, \quad (62)$$

and $P_1 + P_2 = P - P_3$, P is the unity in the considered space so:

$$V_{\parallel,3}(t) = -i\Xi(\lambda, t)(P - P_3) + \Xi(\lambda_3, t)P_3. \quad (63)$$

$$4.3. \lambda_1 = \lambda_2 = \lambda_3 = \lambda$$

This case is the simplest one, and the solutions are:

$$\lambda = -\frac{A}{3} = H_{11} = H_{22} = H_{33}. \quad (64)$$

In this case PHP is a diagonal matrix in any basis. In fact, this is true for any n -dimensional Hermitian matrix with all eigenvalues equal, so we get a form of quasipotential which is identical to the two dimensional degenerate case (54).

$$V_{||} = -\Xi(\lambda, t)P. \quad (65)$$

Again, if apart from assuming the three-fold degenerate case of PHP we take $t \rightarrow \infty$ we will get a result which is analogous to the one obtained from the Wigner–Weisskopf approximation by *e.g.* Lee, Oehme and Yang [8].

5. Equation for ρ matrix in $\mathcal{H}_{||}$

This section contains one possible application of the results obtained above, which is the equation for the density matrix ρ in $\mathcal{H}_{||}$.

Very often systems of the type described in Section 1. are considered as open systems interacting with an unknown rest, *i.e.*, with the reservoir [11, 12]. Then, for the description of the time evolution in subspace $\mathcal{H}_{||}$, instead of the state vector $|\psi; t\rangle_{||}$ solving equations (3), (11), density matrix ρ is used. The ρ -matrix in quantum mechanics fulfills the following equation:

$$\frac{\partial}{\partial t}\rho = i[\rho, H], \quad (66)$$

where H is the total Hamiltonian of the system under consideration acting in the Hilbert state space $\mathcal{H}$. H and ρ are Hermitian. The consideration of such systems sometimes begins with a phenomenological Hamiltonian $H_{\text{eff}} \equiv H_{||}$, acting in an n -dimensional subspace $\mathcal{H}_{||}$. Such Hamiltonians are of the LOY type or the type used in the master equation approaches [11, 12]. These Hamiltonians are not Hermitian, therefore the time evolution of the reduced ρ -matrix, *i.e.*, $\rho_{||}$ (where $\rho_{||}$ denotes the part of ρ -matrix acting in $\mathcal{H}_{||}$), is given by [11]

$$\frac{\partial}{\partial t}\rho_{||} = -i(H_{||}\rho_{||} - \rho_{||}H_{||}^+), \quad (67)$$

where

$$\rho_{||}(t) \equiv \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) & \dots & \rho_{1n}(t) & 0 & \dots \\ \rho_{12}^*(t) & \rho_{22}(t) & \dots & \rho_{2n}(t) & 0 & \dots \\ \dots & \dots & \dots & \dots & 0 & \dots \\ \rho_{1n}^*(t) & \rho_{2n}^*(t) & \dots & \rho_{nn}(t) & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \quad (68)$$

At this point one remark concerning the above should be made: all properties of $\rho_{||}(t > 0)$ solving this evolution equation are determined by the form and properties of $H_{||}$ so for the same initial conditions ρ_0 but different $H_{||}$ a different $\rho_{||}(t)$ can be obtained.

Let us notice that the solution of Eq. (66) has the following form

$$\rho(t) \equiv U(t)\rho_0 U^\dagger(t), \quad (69)$$

where $\rho_0 \equiv \rho(0)$ and $U(t)$ is the total unitary evolution operator for the system considered. From this we conclude that the exact reduced ρ -matrix for a given complete and closed subspace $\mathcal{H}_\parallel$ of the total state space $\mathcal{H}$ is

$$\rho_\parallel(t) \equiv P\rho(t)P. \quad (70)$$

If the subsystem described by $\rho_\parallel(t)$ is an open system, *i.e.*, if transitions from subspace $\mathcal{H}_\parallel$ into $\mathcal{H} \ominus \mathcal{H}_\parallel$ (and *vice versa*) occur, then P cannot commute with the total Hamiltonian H .

Now, in order to describe an n state system of the considered type, ρ_0 of the form (68) and a projector defining the subspace of the form (27), or another unitary one equivalent to it, should be chosen. It is easy to verify that for this P we have

$$\rho_0 \equiv P\rho_0 P, \quad (71)$$

so, in this case (see (70) and (69))

$$\rho_\parallel(t) \equiv P\rho(t)P \equiv PU(t)P\rho_0 PU^\dagger(t)P. \quad (72)$$

Using the identity (9) we have

$$\rho_\parallel(t) \equiv U_\parallel(t)\rho_0 U_\parallel^\dagger(t). \quad (73)$$

It can be easily verified that $\rho_\parallel(t)$ fulfills the following equation,

$$i\frac{\partial}{\partial t}\rho_\parallel(t) = \left(i\frac{\partial U_\parallel(t)}{\partial t}\right)\rho_\parallel(t) + \rho_\parallel(t)\left(i\frac{\partial U_\parallel^\dagger(t)}{\partial t}\right), \quad (74)$$

or, equivalently

$$i\frac{\partial}{\partial t}\rho_\parallel(t) \equiv H_\parallel(t)\rho_\parallel(t) - \rho_\parallel(t)H_\parallel^\dagger(t), \quad (75)$$

(where $H_\parallel(t)$ is given by the identity (13)), which is analogous to (67).

6. Conclusions

This paper deals with the operator form of the effective potential governing the time evolution in n -dimensional subspace of states. The general expression for such an effective potential has been found in Section 2. Sections 3 and 4 dealt with the explicit form of such an operator for 2 and 3

dimensional cases. In Section 5 an application of the formalism developed in the previous sections to the density matrix has been suggested.

The approach presented in this paper can be considered a natural extension of the Wigner–Weisskopf approach to the single line width to more level subsystems which interact with the rest of the physical system. It has been shown that in the case of n level systems the WW approach may only be suitable if the PHP is n -fold degenerate, which of course is not always the case.

The physical problem which is currently investigated with the use of similar methods is the neutral kaon complex and the possible violation of the CPT symmetry. This problem is obviously a 2 dimensional problem and can be researched with the use of the formalism developed in Section 3. The standard approach to the problem developed in [8] uses the WW approximation to describe the time evolution of the $K_0, \bar{K}_0$ complex and proves to be quite a successful approximation of the physical reality. As noted at the end of Section 3, one of the conclusions which can be drawn here is that $h_{11} = h_{22}$. This can be measured, and the parameter $\delta_{CPT} \sim h_{11} - h_{22}$ is used in tests of CPT conservation. However, if we want to retain the geometry of the problem (*i.e.* we do not want to reduce the problem to a one dimensional problem by assuming PHP degeneration) we will find that $\delta_{CPT} \neq 0$ even under CPT conserved. For a more extensive discussion of this problem see [4, 5, 7].

The three dimensional case has not as yet been applied to describe an actual physical system and the possibilities of doing so will be investigated in future papers. One possibility is to use the density matrix approach which has been proposed in Section 5, to the description of multi-level atomic transitions. Experiments designed to demonstrate the Quantum Zeno effect provide an example of such multi-level systems. For example Cook suggested an experiment which should demonstrate this effect on an induced transition in a single, trapped ion [13]. This experiment assumes the ion to have a 3 - level structure, and to describe it the density matrix approach is usually used (see for example [14]). This gives us a possibility to use the results obtained in Section 4.1 ($\lambda_1 \neq \lambda_2 \neq \lambda_3$) and Section 5 to construct a suitable equation for the reduced three dimensional density matrix. This, however, is beyond the scope of this paper and, as noted earlier, will be researched in future papers.

REFERENCES

- [1] W. Królikowski, J. Rzewuski, *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. & Phys.* **4**, 19 (1956).
- [2] W. Królikowski, J. Rzewuski, *Nuovo Cimento* **B25**, 739 (1975), and references therein.
- [3] K. Urbanowski, *Acta. Phys. Pol.* **B14**, 485 (1983).
- [4] K. Urbanowski, *Phys. Lett.* **A171**, 151 (1992) .
- [5] K. Urbanowski, *Int. J. Mod. Phys..* **A10**, 1151 (1995).
- [6] K. Urbanowski, *Phys. Rev.* **A50**, 2847 (1994).
- [7] K. Urbanowski, *Int. J. Mod. Phys..* **A8**, 3721 (1993).
- [8] T.D. Lee, R. Oehme, C.N. Yang, *Phys. Rev.* **106**, 340 (1957).
- [9] K. Urbanowski, J. Piskorski, Improved Lee, Oehme and Yang Approximation, preprint No. phys/9803030; K. Urbanowski, J. Piskorski, On Departures from the Lee, Oehme and Yang Approximation, January 2000, in press.
- [10] F.W. Byron, R.W. Fuller, *Mathematics of Classical and Quantum Physics*, Vol.2., Addison and Wesley Publishing Company, Inc., Reading Massachusetts, USA, 1970.
- [11] J. Ellis, J.S. Hagelin, D.V. Nanopoulos, M. Srednicki, *Nucl. Phys.* **B241**, 381 (1984). J. Ellis, N.E. Mavromatos, D.V. Nanopoulos, *Phys. Lett.* **B293**, 142 (1992).
- [12] P. Huet, M.E. Peskin, *Nucl. Phys.* **B434**, 3 (1995).
- [13] R. J. Cook, *Phys. Scr.* **T21**, 49 (1988).
- [14] S.M. Chumakov, K.E. Hellwing, A.L. Rivera, *Phys. Lett.* **A197**, 73 (1995).