

REPRESENTATIONS OF THE HEISENBERG
ALGEBRA BY DIFFERENCE OPERATORS

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We construct a class of representations of the Heisenberg algebra in terms of the complex shift operators subject to the proper continuous limit imposed by the correspondence principle. We find a suitable Hilbert space formulation of our construction for: (1) real shifts, (2) purely imaginary shifts. The representations involving imaginary shifts are free of spectrum doubling. We determine the corresponding coordinate and momentum operators satisfying the canonical commutation relations. The eigenvalues of the coordinate operator are in both cases discrete.

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1. Introduction

The aim of this paper is to find a possibly wide class of difference operator representations for the coordinate and momentum operators leading to a generalization of the standard Quantum Mechanics (QM) in a way compatible with the discreteness of space–time. It is important to stress that at this stage we are at the level of *kinematics*. Thus, we are looking for proper definitions of the coordinate and momentum operators, their domains, the “classical limit” (to recover the standard QM in the continuous limit), the Hilbert space *etc.* We assume that the Heisenberg Commutation Relations (CR) hold on some dense set of states. A deeper understanding of, seemingly, exotic realizations of CR for which one of the operators is bounded

might be needed to advance this program. In other words, the lack of such understanding was perhaps responsible for the relatively unsuccessful, albeit very interesting, results of the early attempts in this direction [1–4].

Our research, though not directly related to, is motivated by an old problem of the nature of the space–time geometry below the Planck scale. There have been many speculations and indications that at this level new discrete structures are likely to emerge (see the arguments put forward in string theory [5], in various approaches to quantum gravity [6–10], in ’t Hooft’s approach [11] *etc.*) and that, as a consequence, some “unusual” representations of the Heisenberg algebra maybe appropriate [12]. Finally, discrete models are widely used because of technical reasons, perhaps the most obvious examples being lattice gauge theories [13] and Regge calculus approach to general relativity [14] to name only a few.

The first step to realize this program has been taken in [15], where representations of the Heisenberg CR in terms of real shift operators have been analyzed. We have discussed the Bargmann–Fock (annihilation–creation operators) as well as the Schrödinger (position and momentum operators) representations. We have found that our coordinate operator has a discrete spectrum. In this paper we generalize our previous results by including the complex shift operators as well. In particular, we find that the operators discussed in [9, 16, 17] are included in our scheme. Those operators correspond to purely imaginary shifts and in the last two references they are used in a physically interesting context of the black hole’s energy quantization.

We work directly with the coordinate (Schrödinger) representation, as opposed to the annihilation–creation type, for the former seems to be physically more fundamental; the sentiment captured so nicely in the following quote “*In physics the only observations we must consider are position observations...*” [18].

2. Momentum and coordinate operators

We start with the following *Ansatz* for the generalized momentum operator P and the corresponding “discrete” derivative D

$$P \equiv -i\hbar D = -i\hbar \sum_{k=-N}^N \frac{1}{\Delta x} \alpha_k E_{\Delta x}^k, \quad (1)$$

where the complex shift operator is defined as

$$E_{\Delta x}^k f(x) = f(x + k\Delta x), \quad (2)$$

and the shift Δx can be complex. We keep the notation the same as in [15] with the exception of the overall $1/\Delta x$ factor in the formula (1) that defines

the constants α_k 's. With this definition our formal equations for α_k become identical to those of [15]. The coordinate operator X is assumed to have the form

$$X = \sum_k \frac{1}{2} \beta_k \left[\hat{x} E_{\Delta x}^k + E_{\Delta x}^k \hat{x} \right], \quad (3)$$

with unspecified constants β_k .

We would like to have the standard coordinate ($\hat{x}$) and momentum ($\hat{p} = -i\hbar\partial/\partial x$) in the continuum limit. Hence, we impose the *classical limit* conditions

$$\lim_{\Delta x \rightarrow 0} X = \hat{x}, \quad \lim_{\Delta x \rightarrow 0} P = \hat{p}. \quad (4)$$

In addition, we demand the symmetry:

$$\alpha_{-k} = \alpha_k \quad (5)$$

and that D be the best fit to $\partial/\partial x$, *i.e.* we assume the operator D to be *optimal* in the sense of [15]. This determines the coefficients α_k in a unique way, independent of Δx

$$\alpha_0 = 0, \quad \alpha_k = (-1)^{k+1} \frac{(N!)^2}{k(N+k)!(N-k)!}. \quad (6)$$

Finally, the operator X will be determined from the Heisenberg CR

$$[X, P] = i\hbar \mathbb{I}. \quad (7)$$

At the formal level (7) imply the same solutions for β_k 's as in the case of the real shifts [15]. However, to determine Hermiticity we have to specify the scalar product and the corresponding Hilbert space. This is an especially pressing issue for the imaginary shifts where the standard setup involving functions in $L^2(\mathbb{R})$ is not suitable. For the real shifts the analysis of (7) has been done in [15]. For general complex shifts Δx we were unable to construct a Hilbert space on which both X and P would be Hermitean. We are therefore concentrating on two types of shifts: (a) real shifts, (b) purely imaginary shifts.

By abuse of notation we will write Δx for real shifts, $i\Delta x$ for imaginary shifts, respectively. Thus Δx is always real. The following section is devoted to a construction of a suitable Hilbert space supporting imaginary shift representations.

3. Analytic setup for imaginary shifts

We choose a particular model of a Hilbert space. Suppose $\Lambda > 0$ and consider a linear space

$$C_\Lambda = \{f \in L^2(\mathbb{R}), f(p) = 0 \text{ for } |p| > \Lambda\} . \quad (8)$$

One might think of Λ as a momentum cut-off and C_Λ is a space of wave functions compactly supported in the momentum space (*e.g.* we could fix Λ to be of the order of $1/l_P$, where l_P is the Plank length). In fact, we assume that $\Lambda \sim 1/\Delta x$. It is easy to see that C_Λ is a closed subspace of $L^2(\mathbb{R})$. Using the Fourier transform we obtain the coordinate representation of C_Λ which will be denoted by H_Λ . Following [19] we adopt the following

Definition 1 *An entire function $F(x)$ is said to be of exponential type at most Λ if $\limsup_{|x| \rightarrow \infty} \frac{\ln |F(x)|}{|x|} \leq \Lambda$.*

Now, one can give the following description of H_Λ ([19], Theorems 16 and 17)

Theorem 1 (Paley–Wiener) $H_\Lambda = \{F \in L^2(\mathbb{R}) : F \text{ is entire of exponential type at most } \Lambda\}$

The shift operator $E_{\Delta z}$ acts on H_Λ and, consequently, it acts on C_Λ . Using the convention

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ipx} f(p) dp , \quad (9)$$

we get that

$$E_{\Delta z} : f(p) \mapsto e^{ip\Delta z} f(p) , \quad f \in C_\Lambda . \quad (10)$$

Since f vanishes outside of a compact set, $E_{\Delta z}$ is unitary if $\Delta z = \Delta x \in \mathbb{R}$, Hermitean if $\Delta z = i\Delta x$ respectively. The former case is analyzed in [15] by a slightly different method. In this paper we are interested in the latter case. $D_{i\Delta x}$ acts on C_Λ as a multiplication operator by the function

$$D_N(p) = \frac{1}{i\Delta x} \sum_{k=-N}^N \alpha_k z^k , \quad (11)$$

where $z = e^{\Delta xp}$. For real α_k , D_N is skew-Hermitean. In particular, this is true for the optimal discretization.

Now, we turn to the study of the operator of multiplication by x . We first introduce a dense subspace of C_Λ defined as

$$C_\Lambda^0 = \left\{ f \in C_\Lambda : f \text{ is a. c. , } f(\Lambda) = f(-\Lambda) = 0 \right\} .$$

The corresponding subspace of H_A will be denoted by H_A^0 . We note that C_A^0 is invariant under $E_{\Delta z}$. On H_A^0 we can integrate by parts obtaining

$$xF(x) = \frac{1}{\sqrt{2\pi}} \int_{-A}^A e^{-ipx} \frac{1}{i} \frac{d}{dp} f(p) dp .$$

We are interested in finding pairs X, D_N of operators satisfying Heisenberg CR in the sense of definition given in Sec. 6 of [15]. We look for X of the form

$$X = \frac{1}{2i} \sum_m \left\{ \beta_m z^m \frac{d}{dp} + \frac{d}{dp} \beta_m z^m \right\} . \quad (12)$$

Then the Heisenberg CR (7) reads $[D_N(p), X] = 1$ and it implies

$$-\frac{\Delta x}{i} z \left(\sum_m \beta_m z^m \right) \frac{d}{dz} (D_N(p)) = 1 . \quad (13)$$

Example For the optimal discretization scheme and $N = 1$ we have found in [15] that $\alpha_1 = 1/2 = \alpha_{-1}$. Then (11) implies

$$D_1(p) = \frac{1}{i\Delta x} \sinh(p\Delta x) , \quad -A \leq p \leq A ,$$

and (13) yields

$$\sum_m \beta_m z^m = \frac{1}{(\cosh p\Delta x)} , \quad -A \leq p \leq A .$$

In summary

$$D_1(p) = \frac{1}{i\Delta x} \sinh(p\Delta x) ,$$

$$X(p) = \frac{1}{2i} \left[\frac{1}{\cosh(p\Delta x)} \frac{d}{dp} + \frac{d}{dp} \frac{1}{\cosh(p\Delta x)} \right] ,$$

is a conjugate pair in the sense of [20].

Below we will show that the pair $X(p), D_1(p)$ is unitary equivalent to the canonical conjugate pair $\frac{1}{i} \frac{d}{dy}, \frac{1}{i} y$ on $L^2 \left(\left[-\frac{\sinh A\Delta x}{\Delta x}, \frac{\sinh A\Delta x}{\Delta x} \right] \right)$. This, in particular, implies that the spectrum of a self-adjoint extension of $X(p)$ is a 1-dimensional lattice. It is perhaps worth mentioning that we do not have in this case a doubling of canonical conjugate pairs, a phenomenon occurring for real shifts [15]. Finally, we would like to point out that the above pair $X(p), D_1(p)$ appears in [16]. We believe that the present work provides a proper analytic setup for that work. For example, to define the position operator, the authors of [16] had to confine their attention to the zero momentum sector of the Hilbert space (formula (7) in [16]). There is no need for such an assumption in our approach.

4. Optimal discretization for imaginary shifts

In this section we consider the case of the optimal discretization scheme. We limit ourselves to the most important aspects as the details are analogous to the real case studied in Sec. VI of [15]. Our goal is to understand (13), in particular, the structure of zeros of $\frac{d}{dz}D_N(p) \equiv D'_N$. We immediately have

Lemma 1 *Let $D_N(p)$ be optimal and N be odd. Then $D'_N(p)$ has no roots for $p \in [-\Lambda, \Lambda]$.*

Proof: The proof is very similar to the proof of Lemma 2 in [15]. We compute $i\Delta x z \frac{d}{dz}D_N$ to obtain

$$i\Delta x z \frac{d}{dz}D_N = (-1)^{N+1} \frac{(N!)^2}{(2N)!} z^{-N} \left[(1-z)^{2N} - (-1)^N \binom{2N}{N} z^N \right].$$

Thus z_0 is a zero of D'_N iff it satisfies

$$\frac{(1-z)^{2N}}{z^N} = (-1)^N \binom{2N}{N}, \quad z = e^{p\Delta x}, \quad -\Lambda \leq p \leq \Lambda.$$

It is clear that for odd N this equation has no real roots ($z > 0$). This completes the proof.

Lemma 2 *If N is even and $\Lambda = 1/\Delta x$, then $D'_N(p)$ has no roots for $p \in [-\Lambda, \Lambda]$.*

Proof: Now, the equation determining the zeros of D'_N reads

$$\frac{(1-z)^{2N}}{z^N} = \binom{2N}{N}, \quad z = e^{p\Delta x}, \quad -\Lambda \leq p \leq \Lambda.$$

In other words

$$\sinh^2 \frac{p\Delta x}{2} = a_N, \quad a_N = \frac{1}{4} \sqrt{\binom{2N}{N}}, \quad -\Lambda \leq p \leq \Lambda.$$

In [15] we have proved that $\{a_N\}$ is increasing and $\lim_{N \rightarrow \infty} a_N = 1$. Note that

$$0 \leq \sinh^2 \frac{p\Delta x}{2} \leq \sinh^2 \frac{\Lambda\Delta x}{2}.$$

Thus we have no roots iff $\sinh^2 \frac{\Lambda\Delta x}{2} < a_N$ for every even N .

Since $\{a_N\}$ is increasing the latter inequality is equivalent to $\sinh^2(\Lambda\Delta x/2) < a_2 = \sqrt{6}/4$ which implies that

$$\Lambda\Delta x < 2\operatorname{arcsch}\left(\frac{{}^4\sqrt{6}}{2}\right) \simeq 1.4379\dots,$$

which holds if $\Lambda = 1/\Delta x$.

Remark: It is clear from the proof of *Lemma 2* that if $\Lambda\Delta x$ exceeds the bound 1.4379... then D'_2 has exactly two roots. This case is then very similar to the case of real shifts ([15], Sec. VI). From now on we assume that D'_N has no real roots on the interval of interest. One easily checks if $-\Lambda \leq p \leq \Lambda$, then $i\Delta x D'_N > 0$ if N is odd and $i\Delta x D'_N < 0$ if N is even. Hence, the map

$$\omega : p \longrightarrow iD_N(p), \quad [-\Lambda, \Lambda] \longrightarrow [-|D_N(\Lambda)|, |D_N(\Lambda)|]$$

is bijective. Let us point out that $iD_N(p)$ is real for $p \in [-\Lambda, \Lambda]$. We call the new variable $y = iD_N(p)$. The corresponding Hilbert space is $L^2([-|D_N(\Lambda)|, |D_N(\Lambda)|])$. The map ω induces an isomorphism of $L^2([- \Lambda, \Lambda])$ and $L^2([-|D_N(-\Lambda)|, |D_N(\Lambda)|])$. In particular, ω maps

$$X \longrightarrow \frac{1}{i} \frac{d}{dy}, \quad D_N \longrightarrow \frac{1}{i} y. \quad (14)$$

This situation should be contrasted with the results we have obtained in [15] for real shifts, for which the pairs X, D_N are, after a suitable rescaling, unitarily equivalent to *two* copies of the canonical conjugate pairs $\frac{1}{i} \frac{d}{dy}, \frac{1}{i} y$ defined on $L^2([-1, 1])$.

5. Imaginary vs real shifts

The aim of this section is to clarify the differences between the real and imaginary shift representations of the Heisenberg CR. In doing so we will come across an interesting connection to problems in signal theory, centered around the so-called Shannon theorem [21]. Let us consider $f \in C_A$. Every such function can be written in terms of its Fourier coefficients. We define

$$\hat{f}(n) = \frac{1}{2\Lambda} \int_{-\Lambda}^{\Lambda} e^{-i\frac{n\pi}{\Lambda}p} f(p) dp. \quad (15)$$

In view of (9) we obtain

$$F\left(\frac{n\pi}{\Lambda}\right) = \frac{2\Lambda}{\sqrt{2\pi}} \hat{f}(n). \quad (16)$$

Since $f(p) = \sum_n \hat{f}(n) e^{i \frac{n\pi}{\Lambda} p}$, we conclude that

$$f(p) = \frac{\sqrt{2\pi}}{2\Lambda} \sum_n F\left(\frac{n\pi}{\Lambda}\right) e^{i \frac{n\pi}{\Lambda} p}. \quad (17)$$

Using the definition of $F(x)$ we arrive at *Shannon's theorem*

$$F(x) = \sum_n F\left(\frac{n\pi}{\Lambda}\right) \frac{\sin(n\pi - \Lambda x)}{n\pi - \Lambda x}, \quad F \in H_\Lambda. \quad (18)$$

Thus $F(x)$ can be recovered from its sample values on the lattice $\frac{\pi}{\Lambda}\mathbb{Z}$.

Our objective now is to understand how the real and imaginary shifts act on those sample values. It is clear that the real shift $E_{\Delta x}$, for $\Delta x = \pi/\Lambda$, gives $(E_{\Delta x} F)(n\pi/\Lambda) = F(\frac{n+1}{\Lambda}\pi)$. The question that presents itself at this point is whether there exists a simple formula for the action of the imaginary shift. We know that the action of $E_{i\Delta x}$ is most conveniently expressed on C_Λ by $f(p) \mapsto e^{p\Delta x} f(p)$. Thus, in view of (16) we obtain

$$(E_{i\Delta x} F)\left(\frac{n\pi}{\Lambda}\right) = \frac{2\Lambda}{\sqrt{2\pi}} (\widehat{e^{p\Delta x} f})(n),$$

and with the help of the convolution theorem and (16) we get

$$(E_{i\Delta x} F)\left(\frac{n\pi}{\Lambda}\right) = (-1)^n \sinh(\Lambda\Delta x) \sum_{m \in \mathbb{Z}} (-1)^m \frac{F\left(\frac{m\pi}{\Lambda}\right)}{\Lambda\Delta x - i(n-m)\pi}. \quad (19)$$

We conclude that the imaginary shift $E_{i\Delta x}$ involves *infinite number of real shifts* by π/Λ . One can easily extend these computations to the “discrete” derivative $D_{i\Delta x}$. For simplicity, we only consider here the case $N = 1$ for which we obtain

$$\begin{aligned} & (D_{i\Delta x}^1 F)\left(\frac{n\pi}{\Lambda}\right) \\ &= (-1)^n \frac{\sinh(\Lambda\Delta x)}{\Delta x} \sum_{m \in \mathbb{Z}} (-1)^m \frac{(m-n)\pi}{(\Lambda\Delta x)^2 + (m-n)^2\pi^2} F\left(\frac{m\pi}{\Lambda}\right). \end{aligned} \quad (20)$$

We remark that the main contribution to the sum (20) comes from the region $m \sim n \pm \frac{\Lambda\Delta x}{\pi}$. Since $\Lambda \sim 1/\Delta x$, we conclude that (20) can be interpreted as a smeared central difference scheme. However, due to the slow decay one cannot ignore the tail of the expansion (20). It is perhaps fair to say that the utility of difference schemes involving infinitely many points is questionable from a computational point of view. If, however, one is willing to accept that in principle our goal is not to find useful computational methods of the existing quantum theory but explore the foundations of quantum kinematics, then the imaginary shifts merit a careful study. In the next section we comment on the problem of fermion doubling viewed from the perspective of different choices of the discrete derivative.

6. Remarks on fermion doubling

In this section we briefly comment on the problem of the spectrum doubling for the lattice Dirac operator. For an elementary introduction to the problem one can consult [13]. Let us consider the 2-D lattice Dirac equation

$$i\hbar \frac{\partial}{\partial t} \Psi = \gamma_5 P \Psi, \quad (21)$$

where $\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, P is a momentum operator. If one uses, say, the central difference scheme

$$\frac{\partial}{\partial x} \Psi(x) \mapsto \frac{\Psi(n\Delta x + \Delta x) - \Psi(n\Delta x - \Delta x)}{2\Delta x},$$

followed by the plane wave substitution $\Psi(n\Delta x) = e^{-i\omega t + in\Delta x p} \Phi(n)$ one obtains the dispersion relation

$$\omega^2 = \frac{\sin^2(p\Delta x)}{(\Delta x)^2}.$$

Spectrum doubling arises because the right hand side has two zeros in the Brillouin zone. We have shown in [15] that the spectrum doubling is tantamount to the presence of two copies of the canonical conjugate pairs. This effect occurs for any optimal discretization and real Δx . In particular, the central difference scheme is included in that scheme and corresponds to $N = 1$. The situation for the imaginary shift is different! In the simplest case of $N = 1$ we have $P = \sinh(p\Delta x)/\Delta x$ ($-\Lambda \leq p \leq \Lambda$) and the dispersion relation reads

$$\omega^2 = \frac{\sinh^2(p\Delta x)}{\Delta x^2}, \quad -\Lambda \leq p \leq \Lambda.$$

There is no spectrum doubling; only a single copy of the canonical conjugate pair occurs. One can directly interpret (21) as a lattice equation by using the interpretation of the imaginary shift in terms of infinitely many real shifts presented in Sec. 5.

7. Summary and conclusions

This paper concludes the project of looking for Heisenberg algebra representations in terms of the complex shift operators of the form (1) and (3). In the present paper we have limited our attention to the Schrödinger picture, hence to the Hermitean X, P pairs. For the case of purely imaginary shifts we have formulated a Hilbert space approach using the space of entire functions. Assuming furthermore the *optimal* form of the momentum

operator introduced in [15] we have found that (i) the spectrum of the self-adjoint extension of the coordinate operator X comprises a lattice, (ii) all the conjugate pairs are, after a suitable rescaling, unitary equivalent to the canonical pair $\frac{1}{i}y, \frac{1}{i}\frac{d}{dy}$ on $L^2([-1, 1])$ (iii) the pairs X, D do not exhibit the doubling which occurs for real shifts.

As the next step within this approach one could analyze evolution (wave) equations for state vectors and operators corresponding to observables other than X, P . Some work in this direction has been done a long time ago ([2–4]). We, nevertheless, expect that the most interesting feature arising from our work, which is the lattice structure of the spectrum of X , has not been sufficiently elucidated. In particular, one could pursue a potentially interesting analogy with the solid state physics. But this will be a subject of the future research.

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