RUNNING COUPLINGS IN HAMILTONIANS*

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We describe key elements of the perturbative similarity renormalization group procedure for Hamiltonians using two, third-order examples: ϕ^3 interaction term in the Hamiltonian of scalar field theory in 6 dimensions and triple-gluon vertex counterterm in the Hamiltonian of QCD in 4 dimensions. These examples provide insight into asymptotic freedom in Hamiltonian approach to quantum field theory. The renormalization group procedure also suggests how one may obtain ultraviolet-finite effective Schrödinger equations that correspond to the asymptotically free theories, including transition from quark and gluon to hadronic degrees of freedom in case of strong interactions. The dynamics is invariant under boosts and allows simultaneous analysis of bound state structure in the rest and infinite momentum frames.

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1. Introduction

Canonical Hamiltonians of quantum field theories can be written in terms of creation and annihilation operators. Let us denote those operators by q. In this notation, for example, a triple-gluon vertex in the light-front QCD Hamiltonian has the structure

$$H_Y = \sum_{123} \int [123]\, \delta(1+2-3)\, g\, Y_{123}\; q_1^\dagger\, q_2^\dagger\, q_3 + \text{ h.c.}\,, \tag{1.1}$$

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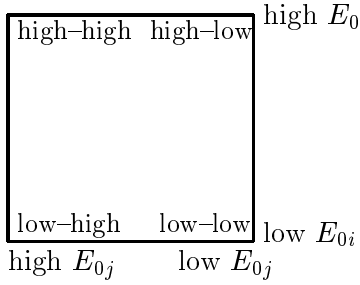
where symbols 1, 2 and 3 denote colors, spins or momenta of gluons, [123] is a shorthand notation for the integration measure over the gluon three-momenta, $\delta(1+2-3)$ is the Dirac δ-function of three-momentum conservation, and Y_{123} is a function of the gluon quantum numbers, implied by QCD. g denotes a bare canonical coupling constant, which is expected from Lagrangian approach to require ultraviolet renormalization. However, the Hamiltonian lacks many of the Lagrangian density symmetries that are employed in the perturbative Lagrangian renormalization procedure and, in all light-front Hamiltonians, transverse and longitudinal directions are treated differently. For that reason, and to introduce a method for renormalization of Hamiltonians such as (1.1), we need to review the origin of the ultraviolet divergences.

One can evaluate matrix elements of H_Y between states of the form $|ijk\ldots\rangle = q_i^\dagger q_j^\dagger q_k^\dagger \ldots |0\rangle$. Consider $\langle 12|H_Y|3\rangle$, which is proportional to Y_{123}. The trouble is that Y_{123} does not vanish when the relative motion of particles 1 and 2 becomes very energetic. In other words, the interaction Hamiltonian directly couples states of small kinetic energy to states of arbitrarily high kinetic energy. For example, when we square H_Y and evaluate $\sum_i H_Y|i\rangle\langle i|H_Y$, the range of $|i\rangle$s to sum over, on the energy scale, is infinite. The sum diverges, and the square of H_Y does not exist. If one tries to find eigenstates of the Hamiltonian, the divergence will dominate finite terms. Also, $\exp -iHt$ diverges and no conclusions can be drawn from knowing H as it stands. The problem is worse than the infinite energy range of interaction implies by itself: the function Y_{123} in QCD grows when the energy difference between kinetic energies of particles 1 and 2, and particle 3 grows. The larger are the kinetic energies of the intermediate particles, the more important become the interactions, and we are sent into an abyss of Fock space states without bounds. All physically relevant local quantum field theories have this trouble.

To see the essence of the problem, imagine a matrix of the Hamiltonian matrix elements in the basis of eigenstates of certain H_0, $H_0|i\rangle = E_{0i}|i\rangle$, $H_{ij} \equiv \langle i|H|j\rangle$. Our problem is that the corners marked low–high and

$$H_{ij} = H_{0ij} + H_{Yij} =$$

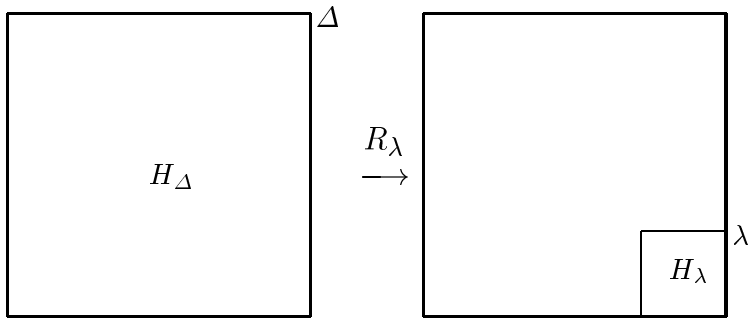


high–low on the above figure, contain too large matrix elements for the

Hamiltonian matrix to have eigenvalues that are independent of the matrix boundaries. In order to understand the boundary-dependence problem of the eigenvalues, we put an upper bound, denoted by Δ, on the basis states energies and we work out what happens with eigenstates of Hamiltonian matrices whose size is limited by the conditions $E_{0i} \leq \Delta$ and $E_{0j} \leq \Delta$, when we change Δ. In particular, we ask what properties must H_Y have for the spectrum of H_Δ to have a limit when $\Delta \to \infty$.

Wilson asked this question in case of a model Hamiltonian with big energy gaps between successive energy scales that were included in his calculation, and he studied the influence of coupling between small and large energy states on the lowest eigenvalues [1]. His method for dealing with the Δ-dependence of the spectrum in the multiscale eigenvalue problem (in general, the problem of dependence on regularization, of any kind), was based on an iterative procedure. Initially, one solves the highest energy part of dynamics and focuses on its lowest eigenvalue levels whose dynamics, in turn, is dominated by states with energies lower by one energy gap. Then, one solves this next lower energy scale dynamical problem, and one repeats the process many times. Starting from the energy scale Δ, one eventually arrives at a finite scale λ. This is schematically indicated on the following figure.

Wilson's approach



In this figure, we see a new small matrix of size λ, denoted by H_λ. This matrix is calculated using an operation R_λ, which is constructed in the sequence of steps lowering the cutoff from Δ to λ. The construction is based on the principle that the smallest eigenvalues of the small matrix, H_λ, should be the same as the smallest eigenvalues of the big matrix H_Δ. The algebraic derivation of the small Hamiltonian H_λ is designed to guarantee the equality of the smallest eigenvalues. The crux is that if all matrix elements of H_λ are independent of Δ (in general, independent of the regularization

one uses to define H_Δ) then, the eigenvalues of H_λ must be independent of the regularization. Therefore, if we know what to do with the regularization dependence of matrix elements of H_λ, then we know how to go about regularization dependence of the spectrum of H_Δ.

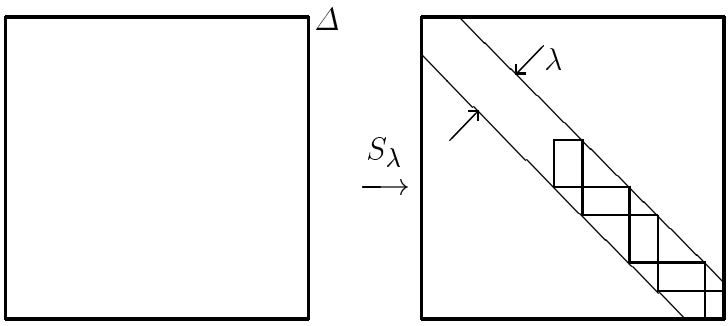
Note, that λ is finite and can be chosen arbitrarily, as long as $\Delta \to \infty$. The set of transformations that connect Hamiltonians with different values of λ is called renormalization group. [2] Physical results should be independent of λ, by construction. It is clear that H_λ cannot be equal to merely $P_\lambda H_\Delta P_\lambda$, where P_λ denotes a projection operator that projects on the space of states with energies $E_{0i} < \lambda$. Some additional terms must be included, which reproduce dynamical effects from above λ. Similarly, we do not expect that H_Δ is merely $P_\Delta H_{\rm canonical} P_\Delta$, where $H_{\rm canonical}$ is built from H_0 and terms such as (1.1). By the same argument as for H_λ, some additional terms are required in H_Δ, to include effects from above the cutoff Δ. We will call those terms *counterterms*. The problem is how to find them. According to Wilson, they are found from the condition that all matrix elements of H_λ become independent of the regularization in the limit $\Delta \to \infty$, for all finite values of λ. One must take care not only of the diverging (*i.e.* Δ-dependent) regularization dependence, but also of the finite regularization effects. The coupling constant g in Eq. (1.1) is changed in the renormalization process as a result of introducing counterterms.

The problem with the transformation R_λ is that the large energy gaps are absent in physically relevant cases, and perturbation theory based on energy scales alone fails. One can try to take advantage of a small coupling constant but we know that naive perturbative expansion does not work in degenerate cases. For example, one may think of effects familiar from elementary degenerate perturbation theory for eigenvalues of a Hamiltonian matrix with a few rows and columns. We know that perturbation theory cannot work unless one properly chooses the initial basis states in the degenerate subspace — eigenstates of the interaction matrix (if it is a few rows and columns). Only in that basis the perturbative limit $g \to 0$ exists. Otherwise perturbation theory produces vanishing energy denominators that lead to diverging terms and the calculation is misleading. In case of quantum field theories of interest, the situation is much more involved than in simple matrix case due to multiply degenerated continuous spectra of H_0. Moreover, in asymptotically free theories, we expect that the interaction strength grows when we go from H_Δ to H_λ and small energy denominators are certainly expected to produce large effects. Thus, the degeneracy of spectra and strength of couplings do not allow us to do a precise analysis of R_λ and H_λ. So, the operation R_λ is of limited applicability in the canonical approach to quantum field theory.

2. Similarity for Hamiltonians

There is an alternative approach [3, 4] (see the figure below). Instead of calculating a small Hamiltonian matrix, we can also calculate a narrow matrix. Namely, a similar (in the sense of algebraic similarity) matrix that has the same eigenvalues but whose matrix elements $H_{\lambda ij}$ vanish if $|E_{0i} - E_{0j}| > \lambda$ (or another condition of "narrowness" is satisfied — hermitian matrices can be diagonalized and, therefore, partial diagonalization to a narrow matrix should be possible). The choice of near-diagonal form is motivated by the following property of near-diagonal matrices: when we act with them on a state of some finite energy, a single action of the matrix can rise the energy by at most λ, *i.e.* by its width on the energy scale. In perturbation theory for eigenvalues of a near-diagonal matrix, corrections will not be sensitive to the cutoff Δ up to the order $n \sim \Delta/(2\lambda)$, since one has to go up in energies and come back to the initial energy range through action of interactions, and at least n are needed to reach the boundary starting from λ. In similarity, the crux is that if matrix elements of the narrow matrix of finite width λ are independent of regularization when $\Delta \to \infty$, then the spectrum of H_λ will be independent of regularization to all orders of perturbation theory.

Similarity approach



The transformation S_λ is called similarity transformation. The effective Hamiltonian matrices with various widths λ are connected by transformations that are called similarity renormalization group transformations for Hamiltonians. We limit our discussion to unitary transformations S_λ. The key feature of the similarity approach is that perturbative construction of S_λ avoids small energy denominators entirely — they are limited from below by the width λ. In turn, the perturbatively calculated narrow Hamiltonians can be diagonalized numerically, which is the ultimate way to find solutions

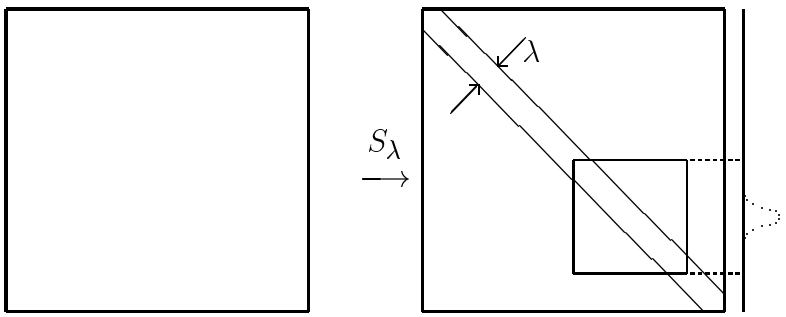
to complex non-perturbative problems of the original theory. One could ask, why don't we go all the way to $\lambda = 0$, which would mean complete diagonalization through S_λ with $\lambda = 0$? This is impossible in perturbation theory. We can trust perturbation theory for calculating H_λ only for not too small values of λ.

In other words, the perturbative similarity transformation S_λ involves energy changes that are at least as large as λ and the problem with large effects in perturbative evaluation of effective Hamiltonians is overcome. But that does not mean we eliminated any of the nonperturbative effects. They are still hidden in the narrow effective Hamiltonian, as much as they were in the initial one. The only thing we accomplish through similarity, is the elimination of direct couplings between states of interest to us and very high energy states. This is a prerequisite that we need to define an ultraviolet finite, nonperturbative Hamiltonian eigenvalue problem in quantum field theory.

One can apply various perturbative procedures for calculating S_λ and H_λ. Wegner invented a beautifully simple scheme for evaluating near diagonal Hamiltonians in solid state physics [5]. It was shown [6] that Wegner's equation can be employed in the renormalization group scheme. A number of variations reported in the literature in different areas, is growing [7].

The bottom line is that when solving for the spectrum of a narrow Hamiltonian, we do not have to diagonalize the whole matrix of size $\Delta \to \infty$. We can select a window, as is illustrated in the next figure.

Window Hamiltonians



Diagonalization of that small window is much simpler than diagonalization of the whole matrix and the window eigenvalues match the whole matrix eigenvalues in the middle range of window energies [8]. The reason is that the wave functions have width comparable to λ on the energy scale, which is indicated on the figure (see also Appendix B in Ref. [3]). One cannot expect

wave functions of eigenstates of the initial Hamiltonian matrix, H_Δ, to be dominated by some small range of energies. In contrast, the eigenstates of matrix H_λ are expected to have wave functions that have this property. Recent calculations of quarkonium and glueball spectra in light-front QCD exploit this feature [9, 10]

Ref. [8] outlines the similarity procedure starting from an initial Hamiltonian, through evaluation of the narrow Hamiltonians (having found the necessary counterterms in the initial Hamiltonian) in perturbation theory, to diagonalization of a small window to get the bound state energy, using a matrix example. The matrix model is asymptotically free and has a bound state. The coupling constant is a function of the effective Hamiltonian width λ, we say it "runs". For example, it may equal about 0.06 at $\lambda = 65$ TeV and about 1 at 1 GeV. Still, the window Hamiltonian of a few GeV size can be calculated in second order perturbation theory and the window bound state eigenvalue deviates from exact solution by only 10%. Once we understand that example, we can return to quantum field theories with interactions of the form (1.1) and attempt a calculation of the corresponding "window" Hamiltonians, with running couplings. Calculations can be carried out using the notion of effective particles.

3. Similarity for particles

When we deal with huge Hamiltonian matrices of quantum field theory the number of states is as big as we let it be and the number of matrix elements we have to think about becomes very quickly incomprehensible. We have to reduce the amount of information that we need to know at the beginning. Imagine we would know matrix elements of the interaction α/r in atomic basis functions, numerically, but we would not know that they all correspond to the Coulomb force. It would be very hard to relate what happens in one atomic system to what happens in another one. Therefore, when we aim at universal calculations of effective Hamiltonians in theories that contain interactions such as (1.1), we may proceed to a new version of similarity transformation, which avoids dealing directly with Hamiltonian matrix elements in a particular basis and, instead, operates at the level of field operators [11].

Let us introduce a unitary transformation $\mathcal{U}_\lambda$ that transforms field operators (denoted here by ϕ, independently of their spin or other quantum numbers they carry),

$$\phi_\lambda(x) = \mathcal{U}_\lambda \; \phi_\infty(x) \; \mathcal{U}_\lambda^\dagger \,. \tag{3.1}$$

$\phi_\infty(x)$ denotes a bare quantum field operator that, at any prescribed time, can be expanded into creation and annihilation operators for bare particles in a canonical fashion that we do not need to define here very precisely.

$\phi_\lambda(x)$ denotes an operator that is built in exactly the same way from creation and annihilation operators for effective (dressed) particles. This kind of transformation is motivated by physics of hadrons, whose structure can be explained in a constituent quark model. Dressed particles in a given theory interact differently than the bare ones. Namely, bare ones have interactions like (1.1), while the effective ones can only exchange momentum transfers that are limited by λ. This is secured by the construction of $\mathcal{U}_\lambda$, to be explained below. Therefore, *the effective particle wave functions of eigenstates of the Hamiltonian may quickly fall off when momenta or number of the effective particles deviate from the physically dominant values.* This is why one can hope to obtain a constituent picture of hadrons in QCD using similarity for particles. More generally, the expected convergence of eigenstate expansion in effective particle basis in Fock space opens a door to studies of few-body systems in quantum field theory.

In order to set up equations that will allow us to calculate Hamiltonians for effective particles, let's rewrite Eq. (3.1) in terms of the creation and annihilation operators,

$$q_\lambda = \mathcal{U}_\lambda \; q_\infty \; \mathcal{U}_\lambda^\dagger \,. \tag{3.2}$$

All we need to do next is: take the bare Hamiltonian of our theory, as it is given initially in terms of q_∞, calculate counterterms it needs to contain in addition to the canonical terms, obtain this way our initial $H_\infty(q_\infty)$, and rewrite it in terms of q_λ. $\mathcal{U}_\lambda$ is secured to be unitary by construction. The whole point of the construction is that the resulting $H_\lambda(q_\lambda)$ is to contain only such interaction terms that, when we evaluate their matrix elements between Fock basis states of effective particles, the resulting effective Hamiltonian matrix is narrow, of width λ, as in the similarity procedure for Hamiltonian matrices we discussed in previous Section. It will not be necessary to go into details here. Only a brief outline of the scheme follows.

Since rewriting the Hamiltonian in different degrees of freedom does not change the operator itself, we have $H_\lambda(q_\lambda) = H_\infty(q_\infty)$. One may think about $H_\lambda(q_\lambda)$ as a QCD Hamiltonian written in terms of constituent quarks and gluons, and about $H_\infty(q_\infty)$ as the same QCD Hamiltonian written in terms of canonical quarks and gluons, associated with partons, or current quarks (to make the connection between hadronic rest frame constituents and partons in the infinite momentum frame, we have to use the light-front form of Hamiltonian dynamics, see [12] for an outline of light-front QCD in the context of renormalization group procedure for Hamiltonians).

Applying the transformation $\mathcal{U}_\lambda$, one obtains

$$\mathcal{H}_\lambda \equiv H_\lambda(q_\infty) = \mathcal{U}_\lambda^\dagger H_\infty(q_\infty) \mathcal{U}_\lambda \,.$$

This relation means that the operator $\mathcal{H}_\lambda$ has the same coefficient functions in front of products of q_∞ as the effective Hamiltonian H_λ has in front of the unitarily equivalent products of q_λ. Differentiating $\mathcal{H}_\lambda$ one obtains

$$\frac{d}{d\lambda}\,\mathcal{H}_\lambda = -[\mathcal{T}_\lambda,\, \mathcal{H}_\lambda]\,, \tag{3.3}$$

where the generator $\mathcal{T}_\lambda$ is related to $\mathcal{U}_\lambda$ by

$$\mathcal{T}_\lambda = \mathcal{U}_\lambda^\dagger \frac{d}{d\lambda}\mathcal{U}_\lambda\,. \tag{3.4}$$

The script letters are introduced to indicate that the operators can be conveniently thought about as expanded into sums of products of operators q_∞. The latter are independent of λ and are not differentiated in Eqs. (3.3) and (3.4). In other words, Eqs. (3.3) and (3.4) describe only the flow of coefficients in front of the creation and annihilation operators. Effective Hamiltonians are obtained from $\mathcal{H}_\lambda$ using $H_\lambda(q_\lambda) = \mathcal{U}_\lambda \mathcal{H}_\lambda \mathcal{U}_\lambda^\dagger$.

The key element now is how one defines $\mathcal{T}_\lambda$. This is the domain of similarity for effective particles. In its essence [3,4], one studies what one has to do to get the narrow Hamiltonian matrices as a result of the procedure, and these studies tell us what to put for $\mathcal{T}_\lambda$. There exist infinitely many choices. The one that the present author used to get results described in the next Sections is of the following form [11, 13]

$$[\mathcal{T}_\lambda,\, \mathcal{H}_{0\lambda}] = \frac{d}{d\lambda}\,(1 - F_\lambda)[\mathcal{G}_\lambda]\,. \tag{3.5}$$

The symbols $\mathcal{G}$ and F require explanation. The effective Hamiltonian H_λ contains form factors of width λ in all its vertices. If we denote an operator without the form factors by G_λ, our Hamiltonian takes the form $H_\lambda = F_\lambda[G_\lambda]$, where the operator F_λ inserts the form factors. With these form factors, momentum transfers in interactions between effective particles are guaranteed to be at most of the order of λ. $\mathcal{G}_\lambda = \mathcal{U}_\lambda^\dagger G_\lambda \mathcal{U}_\lambda$. We divide $\mathcal{G}_\lambda$ into two parts, a part that is bilinear in q_∞, and an interaction part that would vanish if the coupling constant were equal 0, so that $\mathcal{G}_\lambda = \mathcal{G}_0 + \mathcal{G}_{I\lambda}$. The operator $\mathcal{G}_{I\lambda}$ satisfies the following differential equation as a consequence of Eqs. (3.2)–(3.5),

$$\frac{d}{d\lambda}\mathcal{G}_{I\lambda} = \left[f\mathcal{G}_I,\, \left\{ \frac{d}{d\lambda}(1-f)\mathcal{G}_I \right\}_{\mathcal{G}_0} \right]\,. \tag{3.6}$$

We dropped the subscript λ on the right-hand side for clarity. f denotes the similarity form factor introduced by F_λ and the curly bracket with the subscript $\mathcal{G}_0$ denotes a solution for $\mathcal{T}_\lambda$ resulting from Eq. (3.5).

4. Asymptotic freedom in scalar theory

Since the interaction term (1.1) is only a part of the QCD Hamiltonian and the function Y_{123} depends on spins and momenta of gluons, let us first discuss the case of scalar field with classical Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2) - \frac{g}{3!}\phi^3\,. \tag{4.1}$$

In this case, the interaction term in the corresponding Hamiltonian is of the form (1.1), but $Y_{123} = 1/2$ and calculations are much simpler than in QCD. Our goal is to describe results for the light-front Hamiltonian for effective bosons calculated in perturbation theory up to third power in g. Although our presentation is based on Ref. [13] that uses plain expansion in powers of g, the reader may also wish to compare our results with Ref. [14], where a different scheme is used, including transverse locality and coupling coherence [9,10].

The light-front Hamiltonian corresponding to the Lagrangian density (4.1) reads

$$\begin{aligned} H_\infty \;=\; & \int [k] \frac{k^{\perp\,2} + \mu^2}{k^+} a^\dagger_{\infty k} a_{\infty k} \\ & + \frac{g}{2} \int [k_1 k_2 k_3] 2(2\pi)^5 \delta^5(k_1 + k_2 - k_3) \\ & \times (a^\dagger_{\infty k_1} a^\dagger_{\infty k_2} a_{\infty k_3} + a^\dagger_{\infty k_3} a_{\infty k_2} a_{\infty k_1}) r_\Delta + X_\Delta, \end{aligned} \tag{4.2}$$

where r_Δ is a smooth regularization factor and X_Δ denotes counterterms (derivable in perturbation theory). In n dimensions, $[k]$ means

$$\frac{\theta(k^+) dk^+ d^{n-2}k^\perp}{2k^+(2\pi)^{n-1}}\,.$$

We choose

$$r_\Delta = \exp \frac{-(\eta_1 + \eta_2)\,\kappa_{12}^{\perp\,2}}{\Delta^2}\,, \tag{4.3}$$

where $x_1 = k_1^+/k_3^+$ and $\kappa_{12}^\perp = k_1^\perp - x_1 k_3^\perp$, $\eta_i = \eta(x_i)$, and η is a useful function of its argument. A natural choice is $\eta(x) = 1$, for it is simple. Leaving η unspecified will help us identify finite regularization effects.

The similarity form factor for an operator containing u creation operators and v annihilation operators is defined by

$$f_\lambda(u,v) = \exp\left[-\frac{(\mathcal{M}_u^2 - \mathcal{M}_v^2)^2}{\lambda^4}\right]. \tag{4.4}$$

The script notation for invariant masses means $\mathcal{M}_u^2 = (k_1 + \ldots + k_u)^2$, where the minus components of the momentum four-vectors are given by $k_i^- = (k_i^{\perp\,2} + \mu^2)/k_i^+$ for $i = 1, \ldots, u$, and similarly for v.

Equation (3.6) can now be solved order by order using expansion in powers of g. Firstly, one obtains the counterterms X_Δ as the initial conditions at $\lambda = \infty$ that render regularization independent finite λ Hamiltonians. To order g^3, the regularization dependence of H_λ lets us identify two counterterms: the mass counterterm

$$\beta_{\infty 11} = \int [k] \, \frac{\delta\mu_\infty^2}{k^+} a_{\infty k}^\dagger a_{\infty k} \, , \tag{4.5}$$

and the vertex counterterm

$$\gamma_{\infty 21} = \int [k_1 k_2 k_3] \, 2(2\pi)^5 \delta^5(k_1 + k_2 - k_3) \, \gamma_\infty(k_1, k_2, k_3) \, a_{\infty k_1}^\dagger a_{\infty k_2}^\dagger a_{\infty k_3} \, r_\Delta \, . \tag{4.6}$$

Without loss of generality, we assume that some gedanken experimental data require the mass squared parameter in effective Hamiltonian with $\lambda = \lambda_0$ to be equal $\mu^2 + \delta\mu_0^2$. This means that when one calculates observables using the effective Hamiltonian, $\mu_{\lambda_0}^2$ must equal $\mu^2 + \delta\mu_0^2$ to fit the data. This condition, by tracing the renormalization group equation for H_λ back to $\lambda = \infty$, tells us that

$$\delta\mu_\infty^2 = \delta\mu_0^2 - \left(\frac{g}{2}\right)^2 \frac{1}{2(2\pi)^5} \int_0^1 \frac{dx}{x(1-x)} \int d^4\kappa^\perp \frac{2}{\mathcal{M}^2 - \mu^2} \left[f_{\lambda_0}^2(\mathcal{M}^2, \mu^2) - 1 \right] r_{\Delta\beta} . \tag{4.7}$$

The script $\mathcal{M}$ denotes invariant mass, $\mathcal{M}^2 = (\kappa^{\perp\,2} + \mu^2)/x(1-x)$, and the regularization factor is

$$r_{\Delta\beta} = \exp\left\{ \frac{-2[\eta(x) + \eta(1-x)]\kappa^{\perp\,2}}{\Delta^2} \right\} . \tag{4.8}$$

Integration gives two diverging terms, one proportional to Δ^2 and another one proportional to $\log \Delta$. The remaining finite part depends on our choice of the function η. For example, evaluating the integral for $\eta(x) = 1/x$ one obtains

$$\delta\mu_\infty^2 = g^2 \frac{1}{(4\pi)^3} \left[\frac{1}{24} \Delta^2 \, - \, \mu^2 \frac{5}{6} \log \frac{\Delta}{\mu} \, + \, \mu_\eta^2 \right] , \tag{4.9}$$

where μ_η has a finite limit when $\Delta \to \infty$. The logarithmically divergent part is independent of the function η and agrees with results for the Lagrangian mass squared counterterm obtained using Feynman diagrams and dimensional regularization [15, 16] in the following sense: when one changes

Δ to Δ' the logarithmic part of the counterterm changes with Δ as the mass squared changes as a function of the renormalization scale in Eq. (7.1.22) in [16].

The vertex counterterm is defined by the requirement that the effective vertex in the Hamiltonian H_{λ_0} is free from regularization dependence for arbitrary finite values of λ_0. The one loop regularization sensitive contributions to the effective vertex function are given by

$$\gamma_\infty(k_1,k_2,k_3) + \left(\frac{g}{2}\right)^3 \frac{\pi^2}{2(2\pi)^5}$$
$$\times \left[\frac{1}{2}\left[\int_{x_1}^1 \frac{dx}{x(1-x)(x-x_1)} \int_0^\infty \kappa^2 d\kappa^2 \; 8\frac{x-x_1}{x x_2 \mathcal{M}^4} \exp\left(\frac{-c_\eta \kappa^2}{\Delta^2}\right) + (x_1 \leftrightarrow x_2)\right]\right.$$
$$\left. + \int_0^1 \frac{dx}{x(1-x)} \int_0^\infty \kappa^2 d\kappa^2 \frac{-3}{\mathcal{M}^4} \exp\left(\frac{-d_\eta \kappa^2}{\Delta^2}\right)\right], \tag{4.10}$$

where

$$\begin{aligned} c_\eta \;=\; & \eta(x) + \eta(1-x) + \left\{\eta\left(\frac{x_1}{x}\right) + \eta\left(\frac{x-x_1}{x}\right)\right\}\left(\frac{x_1}{x}\right)^2 \\ & + \eta\left(\frac{x-x_1}{x_2}\right) + \eta\left(\frac{(1-x)}{x_2}\right) \end{aligned} \tag{4.11}$$

and

$$d_\eta = 2[\eta(x) + \eta(1-x)]\,.$$

The counterterm function $\gamma_\infty(k_1,k_2,k_3)$ must remove the regularization dependence from the above expression. The regularization effects are independent of $\kappa_{12}^\perp$. Dropping all parts that are independent of regularization, we conclude that

$$\gamma_\infty(k_1,k_2,k_3) + \left(\frac{g}{2}\right)^3 \frac{1}{(4\pi)^3}$$
$$\times \left[3\log\frac{\Delta}{\mu} - 4\left[\int_{x_1}^1 dx \frac{1-x}{x_2}\log c_\eta + (x_1 \leftrightarrow x_2)\right] + 3\int_0^1 dx\, x(1-x)\log d_\eta\right]. \tag{4.12}$$

must be independent of regularization. We see that the diverging regularization dependence of the interaction vertex, *i.e.* the term proportional to $\log\Delta$, is independent of the particle momenta and one can remove the divergence by introducing a $\gamma_\infty(k_1,k_2,k_3)$ that is equivalent to changing the

initial coupling constant g in Eq. (4.2). Thus, no diverging x-dependent counterterms are required — a different situation than in [17]. However, it is also visible that the vertex contains a finite regularization dependent part that is a function of x_1. The function depends on our choice for η. For example, if $\eta = 1$ one has $c_\eta = 4 + 2(x_1/x)^2$ and $d_\eta = 4$. The resulting integral is a function of x_1, and needs to be subtracted. But this would not assure us that the whole ultraviolet regularization dependence is removed, because we work with a specific functional form of the regulating function (4.3).

Since the whole regularization effect is independent of λ and $\kappa_{12}^{\perp}$, it can be completely removed from the effective interaction by subtracting its value for $\kappa_{12}^{\perp} = 0$ at an arbitrarily chosen finite λ_0. However, one has to add back the finite regularization independent part of the effective vertex, which is a function of x_1, denoted below by $\gamma_0(x_1)$. The function $\gamma_0(x_1)$ is necessary to recover Poincaré symmetry of observables, because our regularization spoiled the symmetry. The symmetry may be restored once counterterms remove the regularization effects, but one is not allowed to change terms independent of the regularization, which were given by the initial covariant Lagrangian density unambiguously. Therefore, the function $\gamma_0(x_1)$ must be reinserted. This function is not altered when λ changes and could be considered marginal in analogy with usual renormalization group analysis. The ultimate adjustment of the function $\gamma_0(x_1)$ requires 4th order calculations. For there exists in ϕ^3 theory no 3rd order scattering amplitude one could use to find out what function $\gamma_0(x_1)$ renders Poincaré symmetry of scattering observables with our choice of r_Δ in Eq. (4.2). However, it should be pointed out that the function does not influence the way the 3rd order running coupling constant in effective Hamiltonians depends on λ.

So, in Eq. (4.6), the counterterm function $\gamma_\infty(k_1, k_2, k_3) \equiv \gamma_\infty(x_1, \kappa_{12}^{\perp})$, which removes the regularization dependence from the effective vertex reads

$$\gamma_\infty(x_1, \kappa_{12}^{\perp}) = -\,\gamma_{\lambda_0}(x_1, 0^{\perp}) \;+\; \gamma_0(x_1)\,. \tag{4.13}$$

This result is used to define the new regularization dependent coupling constant g_Δ in the initial Hamiltonian in Eq. (4.2). We select a convenient value of $x_1 = x_0$ and obtain

$$\frac{g_\Delta}{2} = \frac{g}{2} \;+\; \gamma_\infty(x_0, 0^{\perp}) = \frac{g}{2} \;-\; \gamma_{\lambda_0}(x_0, 0^{\perp}) + \gamma_0\,, \tag{4.14}$$

where $\gamma_0 \equiv \gamma_0(x_0)$. We see that the initial coupling g is replaced by a new Δ-dependent quantity

$$g_\Delta = g\left[1 - g^2 \frac{3}{4(4\pi)^3} \log\frac{\Delta}{m_0}\right] + o(g^5)\,, \tag{4.15}$$

with certain free constant m_0. Thus, the theory exhibits asymptotic freedom in 3rd order terms. Our result agrees with literature, say Eq. (7.1.26) from [16], in the sense that when we change Δ, the change required in the coupling constant in the initial Hamiltonian for obtaining Δ-independent effective Hamiltonians matches the change implied by Feynman diagrams and dimensional regularization.

Having established the structure of counterterms we can proceed to evaluation of the finite similarity flow of effective Hamiltonians towards small widths λ. The effective kinetic energy term in narrow Hamiltonians is

$$H_{\lambda 11} = \int [k] \, \frac{k^{\perp\,2} + \mu_\lambda^2}{k^+} a^\dagger_{\lambda k} a_{\lambda k} \,, \tag{4.16}$$

where

$$\begin{aligned} \mu_\lambda^2 &= \mu^2 + \delta\mu_\lambda^2 \\ &= \mu^2 + \delta\mu_0^2 + \left(\frac{g}{2}\right)^2 \frac{1}{2(2\pi)^5} \int\limits_0^1 \frac{dx}{x(1-x)} \int d^4\kappa^\perp \\ &\quad \times \frac{2}{\mathcal{M}^2 - \mu^2} \left[f_\lambda^2(\mathcal{M}^2, \mu^2) - f_{\lambda_0}^2(\mathcal{M}^2, \mu^2) \right] . \end{aligned} \tag{4.17}$$

The above result is particularly simple for $\mu = 0$ and in that case it reads ($\delta\mu_0^2$ is proportional to g^2)

$$\mu_\lambda^2 = \delta\mu_0^2 + g^2 \frac{1}{(4\pi)^3} \, \frac{1}{24} \sqrt{\frac{\pi}{2}} \, (\lambda^2 - \lambda_0^2) \,. \tag{4.18}$$

Logarithmic dependence on λ arises for $\mu > 0$. The value of $\delta\mu_0^2$ could be found, for example, by solving a single physical boson eigenvalue problem, expressing the physical boson mass in terms of $\delta\mu_0^2$ and adjusting the latter to obtain the gedanken experimental mass value for bosons. Note a change in the mass function of cutoff parameter, from the case of the mass counterterm, dependent on Δ, to the case of running mass term, dependent on the width λ (independent of Δ). The change corresponds to a transition from the initial side of a fixed point (bare canonical Hamiltonian with regularization) to the other side of the fixed point (renormalization group trajectory of effective Hamiltonians in the similarity procedure, *cf.* [2]).

The effective vertex reads

$$\begin{aligned} H_{\lambda 21} &= \int [k_1 k_2 k_3] \, 2(2\pi)^5 \delta^5(k_1 + k_2 - k_3) \\ &\quad \times f_\lambda[(k_1 + k_2)^2, k_3^2] \; V_\lambda(x_1, \kappa_{12}^\perp) \; a^\dagger_{\lambda k_1} a^\dagger_{\lambda k_2} a_{\lambda k_3} \,, \end{aligned} \tag{4.19}$$

where $V_\lambda(x_1, \kappa_{12}^\perp)$ is the effective vertex function and f_λ is the similarity vertex form factor. The vertex function is given by an integral over loop variables x and $\kappa^\perp$ of a known function [13].

We define the running coupling constant as the value of $2V_\lambda(x_1, \kappa_{12}^\perp)$ at a chosen configuration of momentum variables, denoted by $(x_{10}, \kappa_{120}^\perp)$. In other words, $g_\lambda = 2V_\lambda(x_{10}, \kappa_{120}^\perp)$. A possible choice for massless bosons is $x_{10} = 0$ and $\kappa_{120}^\perp = 0$. This is a natural definition, analogous to the standard Thomson limit in the case of electron charge in QED. This choice greatly simplifies the integrand, giving $V_\lambda(0, 0^\perp)$, so that the result can be fully produced here (g_0 is the value of g_{λ_0} required by phenomenology done using H_{λ_0})

$$\begin{aligned} g_\lambda &= g_0 + g_0^3 \frac{1}{24} \frac{1}{(4\pi)^3} \int_0^\infty \frac{dz}{z} \\ &\times \left[2(f_\lambda - f_\lambda^3) - 2(f_0 - f_0^3) + 20(f_\lambda^3 - f_\lambda^2) - 20(f_0^3 - f_0^2) + 9(f_0^2 - f_\lambda^2) \right], \end{aligned} \tag{4.20}$$

where $f_\lambda = \exp -z^2/\lambda^4$ and $f_0 = \exp -z^2/\lambda_0^4$. A straightforward integration gives

$$g_\lambda = g_0 \; - \; g_0^3 \; \frac{3}{4(4\pi)^3} \log \frac{\lambda}{\lambda_0}, \tag{4.21}$$

which exhibits asymptotic freedom. Differentiating with respect to λ and keeping terms up to order g_λ^3 one obtains

$$\frac{d}{d\lambda} g_\lambda = -g_\lambda^3 \; \frac{3}{256\pi^3} \; \frac{1}{\lambda}. \tag{4.22}$$

This equation demonstrates the same β function for coupling constants in effective Hamiltonians as obtained in Lagrangian approaches using Feynman diagrams and dimensional regularization, when one identifies the renormalization scale with the Hamiltonian width λ. This is encouraging but one needs to remember that for comparison of perturbative scattering amplitudes in Hamiltonian and Lagrangian approaches it is necessary to make additional calculations and at least of fourth order in g. Beyond model matrix studies such as in [8], 4th order similarity calculations have so far been carried out only in a simplified Yukawa model by Masłowski and Więckowski [18] (the latter model calculations should be helpful in setting up a light-front theory of nucleons and pions).

Integrating Eq. (4.10), one obtains ($\alpha = g^2/4\pi$)

$$\alpha_\lambda = \frac{\alpha_0}{1 + \alpha_0 (3/32\pi^2) \log \lambda/\lambda_0}, \tag{4.23}$$

which shows our result for a boost invariant running coupling constant in effective Hamiltonians. Our procedure explains how the running coupling constant can be included in quantum mechanics of effective particles, which is given by the Schrödinger equation with the corresponding Hamiltonian H_λ. One can evaluate matrix elements of the small width Hamiltonian in a limited subspace of Fock states built of the effective particles. The form factor in the interaction vertex (4.19) secures a small range of the interactions on the energy scale and one can expect a rapid convergence of wave functions in the effective particle basis.

Note three characteristic features of the Hamiltonian calculation.

(1) No field renormalization constant appeared, since the similarity transformation did not eliminate (or integrated out) any degrees of freedom.

(2) No vacuum effect played any role, since we used the light-front form of dynamics. Extensive literature concerning the vacuum issue can be traced through reference [19]. ϕ^3 theory is unstable due to a possibility that the field ϕ takes an infinitely large, negative value. It would be interesting to check if the perturbatively evaluated effective Hamiltonians of small widths have any tendency to develop eigenstates that deviate in that direction.

(3) The Hamiltonian structure is invariant with respect to boosts, including boosts from the rest frame of any bound state to the infinite momentum frame. This suggests the approach outlined above should be tried in QCD, and in effective theories of strong interactions in nuclear physics, to connect low energy observables, such as binding energies, radii or magnetic moments of bound states, with high energy ones, such as parton distributions, form factors or jets.

5. QCD gluon vertex counterterm

We come back to Eq. (1.1) in QCD and repeat the same analysis as we did for the scalar theory. Most of the procedure remains the same, but an important complication arises. The vertex function in the canonical Hamiltonian has now the form (c refers to color and ε to polarization of gluons)

$$Y_{123} = if^{c_1c_2c_3}\left[\varepsilon_1^{*\perp}\varepsilon_2^{*\perp}\cdot\varepsilon_3^{\perp}\kappa_{12}^{\perp} - \frac{\varepsilon_1^{*\perp}\varepsilon_3^{\perp}\cdot\varepsilon_2^{*\perp}\kappa_{12}^{\perp}}{x_2} - \frac{\varepsilon_2^{*\perp}\varepsilon_3^{\perp}\cdot\varepsilon_1^{*\perp}\kappa_{12}^{\perp}}{x_1}\right], \tag{5.1}$$

in which the characteristic factors of $\kappa^\perp/x$ tend to infinity when $x \to 0$. Ultraviolet coupling constant divergences in ϕ^3 theory in 6 dimensions resulted from transverse momentum integration $\int d^4\kappa^\perp/\kappa^4$, where $1/\kappa^4$ came

from the two denominators of third order perturbation theory. In QCD in 4 dimensions, we have instead $\int d^2\kappa^\perp (\kappa^\perp/x)^2/\kappa^4$. Therefore, in QCD (more generally, in gauge theories), we have to introduce a separate regularization of small x behavior of interaction vertices in the Hamiltonians. For example, in the QCD counterpart of Eq. (4.2), we have to insert a factor, denoted by r_δ, in addition to r_Δ, that will effectively cut-off the region where one of the gluons 1 or 2 carries a smaller fraction of k_3^+ than the size of a small parameter δ.

The small x regularization function r_δ may appear to be only a technical detail. But it was pointed out by Perry that singularities at small k^+ may be related to effective confining potentials in H_λ, calculable already in second order perturbation theory. [20] In short, the canonical light-front QCD Hamiltonian contains terms that are singular at small k^+ and the singularity contributes to the effective Hamiltonians H_λ, providing potentials that grow with distance between color charges. This is quite different a situation from other formulations of the theory, where second order calculations are not expected to tell us anything about confinement. Therefore, the small x features of QCD in the light-front Hamiltonian approach deserve extensive studies. Here, we merely report some initial results for third order gluon vertex counterterm, indicating x-dependent features.

The whole analysis of the previous Section can be repeated step by step and one can derive the interaction term for effective gluons, in the narrow Hamiltonian H_λ for QCD. The condition that the effective vertex is independent of regularization parameter Δ gives us the diverging triple-gluon vertex counterterm in the initial QCD Hamiltonian. The new features appear in the coefficient of $\log\Delta$. Namely, the diverging part of the vertex, to be compensated by the ultraviolet counterterm, has the form

$$\frac{g^3}{4\pi^2}\left[\frac{11}{12}N_c - \frac{1}{6}n_f + N_c f(x_1,x_2)\right]\log\Delta\cdot Y_{123}\,, \tag{5.2}$$

where the function $f(x_1,x_2)$ is symmetric in its arguments. This function originates from three successive actions of the triple gluon interaction, the gluon mass correction providing only a constant contribution, and it depends on the regularization factor r_δ in the initial Hamiltonian. In fact,

$$f(x_1,x_2) = \frac{1}{2}\left[-\log x_1 + \int_0^1 dx\left(\frac{2}{x}+\frac{1}{1-x}\right)(r_{\delta 3}-r_{\delta 2}) + (1\leftrightarrow 2)\right]\,,$$

where $r_{\delta 3} = r_{\delta 3}(x,x_1)$ is a product of three factors r_δ for the three successive triple-gluon interactions, and $r_{\delta 2} = r_\delta^2(x)$, is a product of two factors r_δ from the gluon mass counterterm. For $r_\delta(x) = \theta(x-\delta)\theta(1-x-\delta)$ in the bare

vertex, for gluons carrying x and $1-x$ of the single gluon momentum k_3^+ in Eq. (1.1), $f(x_1,x_2)=\log[\min(x_1,x_2)]$, which is negative and hence reduces the rate at which the initial coupling depends on Δ. The latter feature comes about as follows. Since the counterterm must contain the term opposite in sign to (5.2), the coupling constant in the regularized $H_\infty(q_\infty)$ in QCD is changed to [*cf.* Eq. (4.15)]

$$g_\Delta = g - \frac{g^3}{4\pi^2}\left[\frac{11}{12}N_c - \frac{1}{6}n_f + N_c f(x_0,1-x_0)\right]\log\Delta\,. \tag{5.3}$$

The coefficient of $\log\Delta$ is independent of the parameter δ, but it depends on the small x regularization in a finite way. Having derived the function $f(x_1,x_2)$, for certain choices of $x_1=x_0$ in the definition of the Hamiltonian coupling constant, one could obtain triviality instead of asymptotic freedom. This is a peculiar result for the regularization given in Eq. (4.3) and the sharp cutoff on x at δ. In the limit when both $r_{\delta 3}$ and $r_{\delta 2}$ are replaced by 1, one would obtain $f(x_1,x_2)=-\log\sqrt{x_1x_2}$, which is always positive, and would accelerate the asymptotic freedom rate of change of g_Δ with Δ. One can seek choices of regulating functions r_Δ and r_δ that eliminate the unusual logarithm (the other terms are standard) but it is not known if a finite function of x_1 is not necessary in place of $f(x_1,x_2)$ in light-front Hamiltonians anyway, to restore symmetries for physical quantities. In addition, as in the scalar theory, the ultraviolet finite part of the counterterm involves an unknown function of x_1. One could say that the structure of the model from Ref. [17] is closer to QCD than to scalar theory in 6 dimensions. Evaluation of the effective coupling constant g_λ in QCD, in analogy to Eq. (4.20) in scalar theory, may shed some light on how to disentangle genuine ultraviolet from small x singularities in Hamiltonians.

It is clear from the above example that effective light-front Hamiltonians of QCD require careful studies employing various types of regulators before we will know the optimal ways of calculating window Hamiltonians. The interplay of transverse and longitudinal momentum variables may lead to surprising results. However, the calculations are certainly doable and the resulting matrices will tell us about details of QCD dynamics in the Fock space of effective quarks and gluons. The similarity renormalization group procedure for Hamiltonians is able to reveal new features of effective particle dynamics which standard Lagrangian approaches do not reveal.

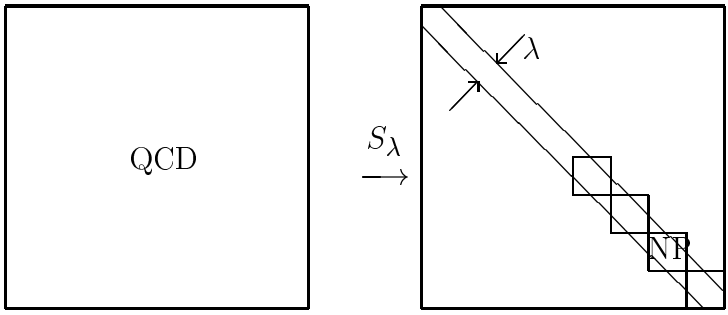
6. Transition to new degrees of freedom

We may hope to make a transition from the effective QCD degrees of freedom to nuclear physics hadronic interactions, such as pion–nucleon cou-

pling, after we achieve understanding of the narrow H_λ in QCD. To see the basis for this hope, let us come back again to the matrix picture.

Once we have the narrow Hamiltonian matrix, we can divide it into the boxes as it is illustrated in the figure below, neglecting the small triangles in the band outside of them, which are initially left out. We can find eigenstates of the boxes (they correspond to different invariant mass states) and calculate the band diagonal matrix matrix elements in the basis built from those eigenstates. The states corresponding to the middle energy scale of each box will not interact very strongly with neighboring (in energy, or mass) states, but matrix elements sensitive to the left-out triangles will lead to strong interactions. We can imagine that the lowest box corresponds to nucleons interacting through potential forces, the next box corresponds to nucleons plus one meson, the second box to nucleons and up to two mesons, *etc.* This is how one could make a connection between the QCD dynamics and nuclear physics through similarity renormalization group for Hamiltonians (*cf.* [21]).

Changing degrees of freedom



However, this is not the only possibility one can try to explore for the change of basis. One can consider new basis states built of quarks and gluons, possibly open gluon string bits with quarks at the ends or closed rings of gluons, and evaluate matrix elements of the effective Hamiltonians between such objects. Since one has a perturbative expression for the operators $\mathcal{U}_\lambda$, see Eqs. (3.1) and (3.2), one can attempt evaluation of matrix elements between states that are constructed in a variety of ways, using quarks and gluons corresponding to different scales λ. One could even ask if there is a way to calculate a connection between the quark and gluon matrices of intermediate widths and reggeized gluon interactions, once one restricts the space of states to those that dominate in multi-Regge kinematics [22].

7. Conclusion

Asymptotically free theories can be analyzed using Hamiltonian approach. The analysis can be based on the similarity renormalization group procedure for effective particles. The evaluation of running couplings in the effective Hamiltonians can be carried out without introduction of wave function renormalization constants and without invoking any properties of the vacuum state (in the light-front form of Hamiltonian dynamics). In third order calculations, one obtains familiar asymptotic results in scalar ϕ^3 theory, plus an x-dependent finite counterterm. In QCD, the standard asymptotic freedom form of triple-gluon vertex counterterm is supplemented by an ultraviolet diverging and x-dependent counterterm, and by an ultraviolet finite x-dependent counterterm. Effects predicted for QCD by the power counting in $k^\perp$ and k^+ [12] are confirmed but the analysis is changed by transition to boost invariant variables $\kappa^\perp$ and x, and detailed calculations may produce results that are not expected to emerge from Feynman diagrams. Mixing between the small x and large $\kappa^\perp$ cutoffs indicates a need for a new precise definition of the ultraviolet domain in the Hamiltonian approach. Nevertheless, one obtains well-defined expressions for effective Hamiltonian interactions without necessity to calculate scattering matrix elements for quarks and gluons as if they were observable particles.

The effective particle calculus preserves cluster properties and allows for evaluation of effective Hamiltonians without limitation to any particular set of matrix elements. In other words, we can derive integral expressions for matrix elements of effective Hamiltonians in the whole Fock space spanned by basis states of effective particles. The renormalization group equations are integrated analytically using Gaussian similarity form factors and one fully controls off-shell behavior of effective vertices that correspond to the initial theory. The effective dynamics is invariant with respect to boosts and allows simultaneous analysis of the rest frame and infinite momentum frame structure of bound states.

The effective particle Fock space expansion can converge thanks to the similarity form factors in the interaction vertices. The form factors dampen interactions changing invariant masses by more than λ and thus can tame the spread of eigenstate wave functions for low lying eigenvalues into regions of high relative momenta of constituents. This feature may lead to exponential convergence of the eigenstate expansion in the effective particle basis. Such convergence is not expected in the case of bare particles. The fine structure of effective particles would then unfold in the transformation $\mathcal{U}_{\lambda_1}\mathcal{U}^\dagger_{\lambda_2}$ relating effective degrees of freedom at two different scales, one corresponding to the binding scale and the other to the high momentum transfer probe in question.

The near-diagonal Hamiltonian matrices allow for transition to new degrees of freedom by turning to basis states that are eigenstates of small block matrices on the diagonal. In principle, these new degrees of freedom could correspond to mesons and baryons built from constituent quarks and gluons, in case of QCD. One can also consider other changes of basis states and seek most efficient degrees of freedom, such as strings of gluons with quarks at the ends, for solving the nonperturbative eigenvalue problems for narrow effective Hamiltonians.

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