ISING-TYPE MODELS WITH HOPF ALGEBRA SYMMETRIES*

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A generalisation of Ising-type models with symmetries of the interactions, which come from Hopf algebras is constructed. General features of such models and some examples are presented.

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1. Introduction

Hopf algebras appear as a natural generalisation of groups in the algebraic framework [1, 2] of noncommutative geometry. Like groups they can be seen as generalised symmetries of noncommutative spaces and physical models.

This offers an intriguing problem whether such generalised symmetries can occur in nature. Let us stress that there is no fundamental principle, which could tell us that only *groups*, a particular class of Hopf algebras, might be the symmetries of the physical models. On the contrary, quantum mechanics and the speculations on quantum gravity [3] strongly suggest that the fine structure of space-time could possibly be much different from the large-scale picture of a differentiable manifold, and, probably, described better in the language of noncommutative geometry. Thus, it would be natural to expect that symmetries at that level might be related to nontrivial Hopf algebras.

Another point of interest are finite Hopf algebras coming from quantum groups at roots of unity. They can be understood as fibres in nontrivial, "noncommutative" *coverings* of classical groups. One of the simplest examples, a covering of SL(2) at the cubic root of unity seems, surprisingly, to have some

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connections with the structure of the Standard Model of elementary particles [4–6]. In the mathematical framework of noncommutative geometry finite Hopf algebras could be used as symmetries of spectral triples [8].

In this paper we attempt to provide a class of possible model of statistical physics, which are based on symmetries represented as finite Hopf algebras. They appear to generalise the well-known Ising and Potts models. Although we cannot claim that such generalisation is relevant for a description of a known physical phenomenon, we shall suggest possible models which may exhibit such properties and links with some real phenomena. The paper is meant to be self-contained, providing all necessary (basic) definitions of mathematical structures.

2. Finite Hopf algebras and finite groups

Before we proceed with the case of finite structures, let us present the definition and the intuition of Hopf algebras. Roughly speaking, Hopf algebras are a generalisation of algebras of functions on a group or a group algebra (in the continuous case, the universal envelope of the Lie algebra).

A Hopf algebra $\mathcal{H}$ is an algebra equipped with the following data:

- an algebra homomorphism $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, a *coproduct*, to shorten the notation we shall use Sweedler's notations: $\Delta a = \sum a_{(1)} \otimes a_{(2)}$,
- an homomorphism $\epsilon : \mathcal{H} \to \mathbb{C}$, a *counit*,
- an antihomomorphism $S : \mathcal{H} \to \mathcal{H}$, *i.e.*, $S(ab) = S(b)S(a)$, an *antipode*,

which satisfy a set of compatibility relations:

- *coassociativity*: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$,
- *counit property*: $a = \epsilon(a_{(1)})a_{(2)} = a_{(1)}\epsilon(a_{(2)})$,
- *antipode property*: $\epsilon(a) = S(a_{(1)})a_{(2)} = a_{(1)}S(a_{(2)})$.

We shall see that their names are appropriate, as the coproduct corresponds to the group multiplication (hence the coassociativity means that the group multiplication is associative), the counit property says that the group has a neutral element (and the counite property just tells us what defines a neutral element of the group) and the antipode correspond to the inverse map in the group.

To see this picture clearly we shall present details of two examples, both important for the rest of the paper. Let us note first that a *finite* Hopf algebra is a Hopf algebra, which is finite dimensional as a vector space.

2.1. Example: finite groups and commutative Hopf algebras

Let G be a finite group and $\mathbb{C}(G)$ be the algebra of complex valued functions on it. It is convenient to use a basis of this algebra (its basis as a vector space) given by functions $v_g, g \in G$, such that:

$$v_g(h) = 0, \quad \text{if} \quad g \neq h, \quad v_g(g) = 1,$$

notice that the algebra multiplication rules are very simple, $v_g v_h = 0$ unless $g = h$ and then $v_g v_g = v_g$.

Then, one may easily verify that the following counit, antipode and the coproduct introduce a Hopf algebra structure on $\mathbb{C}(G)$:

$$\epsilon(v_g) = 0 \ \text{ if } \quad g \neq e, \quad \epsilon(v_e) = 1,$$

where e is the neutral element of the group G,

$$Sv_g \ = \ v_{g^{-1}},$$

and

$$\Delta v_g \ = \ \sum_{h \in G} v_h \otimes v_{h^{-1}g}.$$

This is a standard ("canonical") example of a *commutative* Hopf algebra, in fact, for C^*-algebras one may show that all commutative Hopf algebras are in fact algebras of continuous functions on some group, in a finite case, on a discrete group.

However, Hopf algebras might not necessarily be commutative, as we will show in the next (also standard) example.

2.2. Example: group algebras and cocommutative Hopf algebras

Consider a finite group G and the group algebra $\mathbb{C}G$, which can be constructed as an algebra of linear combination of group elements (with complex coefficients) with the product being the linear extension of the group multiplication. Clearly, for a nonabelian group this a noncommutative algebra. Now, with the following structure we can see that it is a Hopf algebra:

$$\Delta g = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}.$$

Note that in this case, the coproduct is *cocommutative*, *i.e.*, for any element a in the Hopf algebra $a_{(1)} \otimes a_{(2)} = a_{(2)} \otimes a_{(1)}$. Of course, this is not a coincidence, as the group algebra $\mathbb{C}G$ is dual to the above considered algebra $\mathbb{C}(G)$.

This fact (their duals are also Hopf algebras) is a general feature of Hopf algebras, with the natural correspondence between product is in the dual and the coproduct, unit and counit and the antipodes. Commutative Hopf algebras are dual to the cocommutative ones.

As a consequence we may conclude that cocommutative semisimple finite Hopf algebras are always group algebras for a certain group G.

Of course, the category of finite Hopf algebras (even semisimple ones) is much richer than this of commutative or cocommutative Hopf algebras, though the full classification is not known even for low dimensions [7].

3. The Ising and Potts models in the Hopf algebra language

Before we propose a generalisation we shall demonstrate the way in which the Ising, or more generally, the Potts model can be easily formulated in the language of Hopf algebras.

Let us remind the formulation of the Potts model. Given a lattice $\mathcal{I}$ we have a field, which can be treated as a map from $\mathcal{I}$ into a discrete group G (in most cases a cyclic group $\mathbb{Z}_n$). The interaction Hamiltonian is the sum of local terms (nearest neighbour interaction), which are of the form, for neighbours $i, j \in \mathcal{I}$:

$$\mathcal{E}_{kl} = H\left((g_k)^{-1} g_l\right), \tag{1}$$

where H is a real-valued function on G.

Note that the interaction term has a very simple form due to the fact that the field is group-valued and the interaction term has a global G symmetry.

In order to transform the above description to the realm of Hopf algebras we have to make some additional definitions and remarks.

Let J be a set of irreducible distinct (not equivalent) representation spaces of the algebra $\mathbb{C}(G)$, where G is a discrete group with n elements. We can prove the following simple proposition:

Proposition: J could be identified with G, *i.e.*, there exist n one-dimensional representation labelled by group elements g:

$$\rho_g(f)v = f(g)v, \quad f \in \mathbb{C}(G), \; v \in \mathbb{C}.$$

To prove it we take representation ρ and assume that it is irreducible. Since we have $v_g v_g = v_g$, each $\rho(v_g)$ is a projection. Clearly, since all these projections are on orthogonal subspaces, the representation is irreducible if all of them apart from one vanish identically. Therefore there exist a unique g such that only $\rho(v_g)$ does not vanish and this gives our desired identification.

Let Tr_j, $j \in J$ be a trace on the algebra associated with the representation ρ_j. Since J could be identified with G we have:

$$\mathrm{Tr}_g v_h = 0 \;\text{ if }\; g \neq h, \quad \mathrm{Tr}_g v_g = 1.$$

Next, let H be an arbitrary element of the algebra $\mathbb{C}(G)$ and let Φ be a field on the lattice $\mathcal{I}$, which takes values in J. Consider the following interaction term:

$$\mathcal{E}_{kl} = \left(\mathrm{Tr}_{\Phi(k)} \otimes \mathrm{Tr}_{\Phi(l)}\right) (S \otimes \mathrm{id})\, \Delta H. \tag{2}$$

Take $H = \sum H_g v_g$. Then, first we would have:

$$(S \otimes \mathrm{id})\Delta H = \sum_{g,h} H_g (S v_h) \otimes v_{h^{-1}g} = \sum_{g,h'} H_g v_{h'} \otimes v_{h'g},$$

where, in the last step we changed the summation index $h' = h^{-1}$.

Now, by applying the trace we obtain:

$$\mathcal{E}_{kl} = \left(\mathrm{Tr}_{\Phi(k)} \otimes \mathrm{Tr}_{\Phi(l)}\right) (S \otimes \mathrm{id})\, \Delta H = H_{\Phi(k)^{-1}\Phi(l)} = H\left(\Phi(k)^{-1}\Phi(l)\right),$$

which is exactly the formula for the interaction (1) we had presented earlier.

4. The generalised Ising model

By now, we have almost all ingredients necessary to make the generalisation. Let us assume that we have a finite dimensional Hopf algebra $\mathcal{H}$ and a set of its irreducible representations J, with associated normalised traces[1]. The field on the lattice takes values in the set J and the Hamiltonian is given by the formula (2), for some chosen element H of the Hopf algebra.

Before we present more examples let us make some simple observations.

Observation: If $\mathcal{H}$ is a star Hopf algebra (which means that $\Delta(h^*) = h^*_{(1)} \otimes h^*_{(2)}$) and the representations are star representations then for a self-adjoint H the interaction term is real.

A very interesting situation occurs (in the previously discussed case of the $\mathbb{C}(G)$ algebra) if the algebra element chosen to describe interactions is an *integral* in $\mathcal{H}$. Let us remind that at $\varepsilon \in \mathcal{H}$ is an integral (for noncommutative algebras one may distinguish between left and right ones, in a general situation, although in many cases they coincide) if for every $a \in \mathcal{H}$ we have $a\varepsilon = \epsilon(a)\varepsilon$. For the algebra $\mathbb{C}(G)$ the integral is v_e and, for its dual $\mathbb{C}G$ it is just $\frac{1}{n}\sum_g g$.

Observation: If, for an algebra $\mathbb{C}(G)$, the interaction is described by the integral $\varepsilon = v_e$ then the nearest neighbour contribution to the Hamiltonian is 1 if the values of the field Φ coincide and 0 otherwise.

The above observation has an interesting conclusion:

Corollary: If the interaction term for the generalised Ising model described above with a group G of n-elements is given by the integral in $\mathbb{C}(G)$,

[1] We use normalised traces so that always $\mathrm{Tr} 1 = 1$

then the model is equivalent to the model with $\mathbb{Z}_n$ group. Indeed, as we have shown above, the contribution to the interaction depends only on whether the neighbour values coincide or not and therefore the physical system is the same as for $\mathbb{Z}_n$.

4.1. Example: the group algebra S_3

The first nontrivial example, which we consider takes as an object the cocommutative group algebra of permutation of three elements.

The group S_3 has two generators x, y (and the neutral element e), with the following multiplication rules:

$$x^2 = e, \quad y^2 = e, \quad xyx = yxy.$$

It is convenient to rename the element $xyx = z$, $z^2 = e$ and use the rules for multiplication between x, y, z:

$$xy = yz, \quad xz = yx, \quad yz = zx$$

As an algebra $\mathbb{C}S_3$ is isomorphic to the algebra $M_2(C) \oplus \mathbb{C} \oplus \mathbb{C}$. We shall present the form of this isomorphism i for the generators x, y:

$$i(x) = \begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad i(y) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & & \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & & \\ & & 1 & \\ & & & -1 \end{pmatrix},$$

This tells us immediately that there are three irreducible representations, one 2-dimensional and 2 1-dimensional.

Now, we shall demonstrate what are the possibilities for the choice of the interaction term. Clearly, choosing the neutral element (unit of the algebra) we would contribute only to the constant, field independent term.

Proposition: The interaction term depends only on the conjugacy class of the chosen element. This follows immediately from the invariance of the trace under adjoint transformation in the group: $g \to h^{-1}gh$.

For the group S_3 we have three conjugacy classes, of the neutral element $[e]$, then $[x] = [y] = [z]$ and $[xy] = [yx]$.

We know that $[e]$ contributes to a constant term, so let us discuss the two remaining situations.

If H is x then one of the states (associated with the two-dimensional representation space) decouples and does not interact at all, whereas the remaining states interact like in the standard Ising model.

If H is xy then the states associated with one-dimensional representations do not interact with each other, however, they both interact with (giving the same contribution) the state associated the other state.

Finally, let us observe what happens if the chosen element is the integral in the group.

Observation: If the interaction is set by the integral in the group (using the dependence only on the conjugacy class and the irrelevance of the constant term we could equally well use $\frac{1}{2}x + \frac{1}{3}xy$) the energy contribution is given (up to a constant term) by the following table:

	2	+	−
2	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
+	$-\frac{1}{3}$	$\frac{5}{6}$	$-\frac{5}{6}$
−	$-\frac{1}{3}$	$-\frac{5}{6}$	$\frac{5}{6}$

where we have denote the one-dimensional representations as + and − and the two-dimensional by 2.

Such model, which seems to be a mixture of Ising and Potts models, has one absolute minimum (maximum) energy state with $\mathbb{Z}_2$ symmetry and some metastable states, which are local extrema. One example of the latter is a state with $\mathbb{Z}_3$ translational symmetry of the type (for a one-dimensional lattice) $\cdots 2 + - 2 + - 2 + - 2 \cdots$.

4.2. The Hopf algebra A_8

Among the low-dimensional finite Hopf algebras a special place is taken by the Kac algebra, an 8-dimensional Hopf algebra, which is neither commutative nor cocommutative, thus it does not correspond to a finite group.

It is generated by the elements x, y, z, which satisfy the relations:

$$x^2 = 1, \quad y^2 = 1, \quad z^2 = \tfrac{1}{2}(1 + x + y - xy), \tag{3}$$

$$xy = yx, \quad zx = yz, \quad zy = xz, \tag{4}$$

$$\Delta x = x \otimes x, \quad \Delta y = y \otimes y, \tag{5}$$

$$\Delta z = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z). \tag{6}$$

As an algebra, A_8 could be represented on $\mathbb{C}^6$ in the following form:

$$x = \operatorname{diag}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 1, 1, -1, -1\right], \tag{7}$$

$$y = \operatorname{diag}\left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, 1, 1, -1, -1\right], \tag{8}$$

$$z = \operatorname{diag}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, -1, i, -i\right]. \tag{9}$$

Therefore there are four one-dimensional representations and one 2-dimensional. Proceeding in a similar way as in the $\mathbb{C}S_3$ case, by taking the energy functional from the integral in the algebra (for the discussed algebra it is just $1+x+y+z+xy+xz+yz+xyz$) after some calculations we obtain the following table of the interaction energy between the five possible states: (which we denote here by $i=0,1,2,3,4$.

	0	1	2	3	4
0	1	-1	-1	-1	-1
1	-1	8	0	0	0
2	-1	0	8	0	0
3	-1	0	0	8	0
4	-1	0	0	0	8

Again, we observe that the model describes a Potts model slightly deformed by the presence of an additional field. Apart from the global $\mathbb{Z}_4$ symmetric states (as one might have supposed), it has some metastable local extrema of energy functional of $\mathbb{Z}_2$ symmetry.

4.3. Nonsemisimple Hopf algebras

Another interesting class of examples of finite Hopf algebras are *nonsemisiple* algebras, which contain nilpotent elements. Clearly, they cannot be realised as groups algebras or function algebras on groups. Interesting examples arise of such objects arise from finite covering of continuous groups by their quantum deformations at roots of unity, then, they could be interpreted as finite quantum groups.

The natural question is, whether such objects may provide interesting examples of the models, which we discuss here. For simplicity we shall restrict ourselves to one of the simplest possible nonsemisimple Hopf algebra introduced by Sweedler, A_2. It has two generators x, g, with:

$$g^2 = 1, \quad x^2 = 0, \quad gx = -xg, \tag{10}$$

$$\Delta g = g \otimes g, \quad \Delta x = x \otimes g + 1 \otimes x. \tag{11}$$

As an algebra A_2 has two one-dimensional representations and one two-dimensional. Let us take an arbitrary element of the algebra for the energy functional. Then, after little calculation we may observe:

Observation: For every element $h \in A_2$, the interaction between the neighbours is the same as for the Ising model and the additional field (related with the nilpotent element) decouples.

Therefore in this case the algebra A_2, which is a nonsemisimple extension of $\mathbb{Z}_2$, describes precisely the $\mathbb{Z}_2$-symmetric Ising model and a free noninteracting field.

5. Conclusions

The presented examples of models are constructed using Hopf algebras. Our understanding of them as symmetries could be formulated in the following way. Suppose that instead of using Tr_i, *i.e.*; a trace associated with the representation i we use $(\xi \otimes \mathrm{Tr}_i)\Delta$, where ξ is a character of the algebra. Let us have a look at the energy functional:

$$\begin{aligned}
&((\xi \otimes \mathrm{Tr}_i)(\xi \otimes \mathrm{Tr}_j))\,(\Delta \otimes \Delta)(S \otimes \mathrm{id})h \\
&= \mathrm{Tr}_k(Sh_{(2)})\mathrm{Tr}_i(Sh_{(1)}\mathrm{Tr}_k(h_{(3)})\mathrm{Tr}_j(h_{(4)}) \\
&= \xi(Sh_{(2)}h_{(3)})\mathrm{Tr}_i(Sh_{(1)}\mathrm{Tr}_j(h_{(4)}) \\
&= \xi(\epsilon(h_{(2)})\mathrm{Tr}_i(Sh_{(1)}\mathrm{Tr}_j(h_{(3)}) = \mathrm{Tr}_i(Sh_{(1)}\mathrm{Tr}_j(h_{(2)})),
\end{aligned}$$

which is the starting formula for the interaction, therefore such invariance might be interpreted as the symmetry.

All of the discussed examples could be constructed and analysed in the same way as the classical Potts model, in particular it would be interesting to find out their critical behaviour and critical parameters for some lattices. Then, it would be interesting to check whether they correspond in the continuum limit to any field theory models. This would throw a light on the issue what might be the physical setup for nontrivial Hopf algebra symmetries.

Particularly appealing seems the idea [9] that finite Hopf algebras appear as symmetries of quantum states obtained via the symmetry breaking, where some bigger continuous symmetry is broken to a finite-dimensional one. Another possibility is the restriction of Heisenberg spin chains with quantum symmetries to some of its nontrivial finite Hopf subalgebras.

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