THE TWO-DIMENSIONAL QUANTUM GALILEI GROUPS*

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The Poisson structures on two-dimensional Galilei Group, classified in author's previous paper are quantized. The dual quantum Galilei Lie algebras are found.

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1. Introduction

In the present paper we continue the study of deformed nonrelativistic symmetries. In the previous paper [1] all Lie–Poisson structures on two-dimensional Galilei Group were classified up to automorphism. Below we quantize these structures showing that the consistent Hopf algebras are obtained. We find also the corresponding quantum Lie algebras by straightforward application of duality rules.

As a result we obtain two families of quantum groups and quantum Lie algebras, one depends on two and the other depends on three parameters. Various limiting cases appear after sending appropriate subsets of parameters to infinity.

2. Poisson structures on two-dimensional Galilei Group

Recently all Lie bialgebra structures on two-dimensional Galilei algebra have been found and their Lie-Poisson counterparts have been classified [1]. It appeared that, up to the automorphisms, there are nine inequivalent bialgebra structures on two-dimensional Galilean Lie algebra (see Table I in Ref. [1]). The corresponding Lie–Poisson structures on two-dimensional Galilei Group read:

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	$\{a,v\}$	$\{a,\tau\}$	$\{v,\tau\}$	Remarks
1	$-\frac{\tau_0 v^2}{2}$	$\tau_0 a+\varepsilon\tau_0^2 v$	$\tau_0 v$	$\varepsilon\in R$
2	$-v_0^2\tau-\varepsilon v_0\tau_0 v$	$-\varepsilon v_0\tau_0\tau+\tau_0^2 v$	0	$\varepsilon\geq 0$
3	$v_0^2\tau-\varepsilon v_0\tau_0 v$	$-\varepsilon v_0\tau_0\tau-\tau_0^2 v$	0	$\varepsilon\geq 0$
4	$v_0^2\tau-\varepsilon v_0\tau_0 v$	$-\varepsilon v_0\tau_0\tau+\tau_0^2 v$	0	$\varepsilon\geq 0$
5	0	$\tau_0^2 v$	0	
6	$-v_0\tau_0 v$	$-v_0\tau_0\tau+\tau_0^2 v$	0	
7	0	$-\tau_0^2 v$	0	
8	$-v_0\tau_0 v$	$-v_0\tau_0\tau-\tau_0^2 v$	0	
9	$-v_0\tau_0 v$	$-\tau_0 v_0\tau$	0	

Let us note that the first structure can be rewritten in the form

$$\begin{aligned}\{a,v\}&=\frac{-1}{2\kappa}v^2\,,\\ \{a,\tau\}&=\frac{1}{\kappa}a+\frac{1}{\rho}v\,,\\ \{v,\tau\}&=\frac{1}{\kappa}v\,,\end{aligned}\tag{A}$$

and the remaining structures in the following form

$$\begin{aligned}\{a,v\}&=\frac{1}{\alpha}\tau-\frac{1}{\sigma}v\,,\\ \{a,\tau\}&=\frac{1}{\sigma}\tau+\frac{1}{\lambda}v\,,\\ \{v,\tau\}&=0\,,\end{aligned}\tag{B}$$

where $\alpha,\kappa,\rho,\sigma,\lambda$, are parameters chosen appropriately. This forms will be use latter and now let us go to the Table. As it is seen from this Table, in order to impose Lie-Poisson structures on Galilei Group two dimensionful constants v_0,τ_0 are needed. They can attain arbitrary nonzero values, different choice being related by automorphisms. The only relevant free parameter is the dimensionless parameter ε; different values of ε correspond to nonequivalent Lie–Poisson structures.

It is worth to note that this relatively rich family of nonequivalent Lie–Poisson structures contains only one coboundary. It is in a sharp contrast with semisimple case [2] as well as the case of four-dimensional Poincare Group [3].

The Lie–Poisson structures described provide the starting point for obtaining two-dimensional quantum Galilei groups. These groups will be here constructed by applying the naive quantization procedure consisting in replacing the Poisson brackets by commutators (and supplying the resulting commutation rules with imaginary unit and appropriate dimensionful constants). It is obvious from the Table that no ordering problems can appear.

As it was mentioned, the above classification of Lie–Poisson structures is complete up to the automorphisms. However, this can not be a priori taken for granted in the quantum case due to the noncommutativity of generators. This phenomenon is well known in quantum mechanics: not every canonical transformation can be lifted to the unitary one.

We do not attempt here to classify all nonequivalent quantum structures; rather we find the quantum counterparts of "canonical" Poisson structures described in the Table.

3. Two-dimensional quantum Galilei Groups

We apply the standard quantization procedure to the Poisson structures given in Ref. [1]. The result can be summarised as follows. There are two families of quantum groups, one depending on two and the other depending on three dimensionful parameters. The relevant commutation rules read, respectively:

$$\begin{aligned}
[a,v] &= \frac{-i}{2\kappa}v^2\,, \\
[a,\tau] &= \frac{i}{\kappa}a+\frac{i}{\rho}v\,, \\
[v,\tau] &= \frac{i}{\kappa}v\,,
\end{aligned} \tag{A}$$

and

$$\begin{aligned}
[a,v] &= \frac{i}{\alpha}\tau-\frac{i}{\sigma}v\,, \\
[a,\tau] &= \frac{i}{\sigma}\tau+\frac{i}{\lambda}v\,, \\
[v,\tau] &= 0\,.
\end{aligned} \tag{B}$$

The algebra (A) corresponds to the case 1 of our Table while the algebra (B) to all remaining structures. (Some of them can be obtained by taking an appropriate parameters to infinity.)

The dimensions of constans $\alpha, \kappa, \rho, \sigma, \lambda$, are as follows:

$$[\alpha] = \frac{s^2}{m^2}\,, \qquad [\kappa] = \frac{1}{s}\,, \qquad [\lambda] = \frac{1}{s^2}\,, \qquad [\rho] = \frac{1}{s^2}\,, \qquad [\sigma] = \frac{1}{m}\,. \quad (3.1)$$

The commutation rules (A) and (B) are supplied with the standard coproduct, antipode, counit and *- structures,

$$\begin{aligned} \Delta(a) &= a \otimes I + I \otimes a + v \otimes \tau\,, \\ S(a) &= -a + v\tau \\ \varepsilon(a) &= 0 \\ a^* &= a\,, \end{aligned} \qquad (3.2)$$

$$\begin{aligned} \Delta(v) &= v \otimes I + I \otimes v\,, \\ S(v) &= -v\,, \\ \varepsilon(v) &= 0\,, \\ v^* &= v\,, \end{aligned} \qquad (3.3)$$

$$\begin{aligned} \Delta(\tau) &= \tau \otimes I + I \otimes \tau\,, \\ S(\tau) &= -\tau\,, \\ \varepsilon(\tau) &= 0\,, \\ \tau^* &= \tau\,. \end{aligned} \qquad (3.4)$$

We have checked that all relations providing our commutation rules with the structure of *-Hopf algebra are fulfilled. Therefore, we obtain two Hopf algebra structures ((A) and (B)).

4. Duality and quantum Lie algebras

Having found the quantum Galilei groups one can ask what is the structure of their dual Hopf algebras, *i.e.* the quantum Lie algebras.

In the present section we find them by straightforward application of duality rules. It is well known that the dual Hopf algebra can be defined by the following duality rules

$$\langle XY, \Phi \rangle = \langle X \otimes Y, \Delta\Phi \rangle\,, \qquad (4.1)$$

$$\langle X, \Phi\Psi \rangle = \langle \Delta X, \Phi \otimes \Psi \rangle\,; \qquad (4.2)$$

also the *-structure can be defined by the formulae [4]

$$\langle X^*, \Phi \rangle = \langle X, S^{-1}(\Phi^*) \rangle \qquad (4.3)$$

provided the following identity holds

$$S^{-1}(\Phi) = [S(\Phi^*)]^*. \tag{4.4}$$

It is easy to check that the equation (4.4) is in our case fulfilled.

In order to find explicit form of quantum Lie algebras we used the following scheme [4]. First, we define the Lie algebra generators by adopting the classical duality relations

$$\langle X, \Phi \rangle \ = \ -i \frac{d}{dt} \Phi(\mathrm{e}^{itX}) \ |_{t=0} \,, \tag{4.5}$$

i.e.

$$\begin{aligned} \langle H, \tau^k a^l v^m \rangle \ &= \ i\delta_{1k}\delta_{0m}\delta_{0l} \,, &(4.6)\\ \langle P, \tau^k a^l v^m \rangle \ &= \ i\delta_{0k}\delta_{0m}\delta_{1l} \,, &(4.7)\\ \langle K, \tau^k a^l v^m \rangle \ &= \ i\delta_{0k}\delta_{1m}\delta_{0l} \,. &(4.8)\end{aligned}$$

These rules can be compactly summarised by introducing the functions

$$\text{(A)} \qquad f(\mu, \eta, \lambda) = \mathrm{e}^{\mu a}\mathrm{e}^{\eta v}\mathrm{e}^{\lambda \tau} \,, \tag{4.9}$$

$$\text{(B)} \qquad f(\mu, \lambda, \eta) = \mathrm{e}^{\mu a}\mathrm{e}^{\lambda \tau}\mathrm{e}^{\eta v} \,. \tag{4.10}$$

The choice of $f(\mu, \eta, \lambda)$ in both cases was dictated by simplicity of calculations. It is now obvious that any element X of quantum Lie algebra is uniquely determined by the numerical function $f_x(\mu, \eta, \lambda)$ defined as

$$f_x(\mu, \eta, \lambda) \equiv \langle X, f(\mu, \eta, \lambda) \rangle. \tag{4.11}$$

By applying the duality rules (4.1)–(4.4) and by multiple use of Hausdorff formula and some other tricks (*cf.* Appendix) we arrive at the following quantum Lie algebra structures.

— Case (A)

$$\begin{aligned} \Delta(H) \ &= \ H \otimes I + I \otimes H \,, \\ S(H) \ &= \ -H \,, \\ \varepsilon(H) \ &= \ 0 \,, \end{aligned} \tag{4.12}$$

$$\begin{aligned} \Delta(K) \ &= \ K \otimes I + \mathrm{e}^{(-1/\kappa)H} \otimes K - \frac{1}{\rho} H \mathrm{e}^{(-1/\kappa)H} \otimes P \,, \\ S(K) \ &= \ -K\mathrm{e}^{(-1/\kappa)H} - \frac{1}{\rho} HP\mathrm{e}^{(-1/\kappa)H} \,, \\ \varepsilon(K) \ &= \ 0 \,, \end{aligned} \tag{4.13}$$

$$\begin{aligned}\Delta(P) &= P\otimes I + \mathrm{e}^{(-1/\kappa)H}\otimes P\,,\\ S(P) &= -P\mathrm{e}^{(-1/\kappa)H}\,,\\ \varepsilon(P) &= 0\,,\end{aligned}\tag{4.14}$$

$$\begin{aligned}[H,P] &= 0\,,\\ [K,P] &= \frac{-i}{2\kappa}P^2\,,\\ [K,H] &= iP\,.\end{aligned}\tag{4.15}$$

— Case (B)

$$\begin{aligned}\Delta(H) &= I\otimes H + H\otimes \mathrm{e}^{-P/\sigma}\cosh\left(\frac{P}{\sqrt{\alpha\lambda}}\right)\\ &\quad -\frac{\alpha}{\sqrt{\alpha\lambda}}K\otimes\mathrm{e}^{-P/\sigma}\sinh\left(\frac{P}{\sqrt{\alpha\lambda}}\right)\,,\\ S(H) &= -H\mathrm{e}^{P/\sigma}\cosh\left(\frac{P}{\sqrt{\alpha\lambda}}\right)-\frac{\lambda}{\sqrt{\alpha\lambda}}K\mathrm{e}^{P/\sigma}\sinh\left(\frac{P}{\sqrt{\alpha\lambda}}\right)\,,\\ \varepsilon(H) &= 0\,,\end{aligned}\tag{4.16}$$

$$\begin{aligned}\Delta(K) &= I\otimes K + K\otimes \mathrm{e}^{-P/\sigma}\cosh\left(\frac{P}{\sqrt{\alpha\lambda}}\right)\\ &\quad -\frac{\lambda}{\sqrt{\alpha\lambda}}H\otimes\mathrm{e}^{-P/\sigma}\sinh\left(\frac{P}{\sqrt{\alpha\lambda}}\right),\\ S(K) &= -K\mathrm{e}^{P/\sigma}\cosh\left(\frac{P}{\sqrt{\alpha\lambda}}\right)-\frac{\alpha}{\sqrt{\alpha\lambda}}H\mathrm{e}^{P/\sigma}\sinh\left(\frac{P}{\sqrt{\alpha\lambda}}\right)\,,\\ \varepsilon(K) &= 0\,,\end{aligned}\tag{4.17}$$

$$\begin{aligned}\Delta(P) &= I\otimes P + P\otimes I\,,\\ S(P) &= -P\,,\\ \varepsilon(P) &= 0\,,\end{aligned}\tag{4.18}$$

$$\begin{aligned}[H,P] &= 0\,,\\ [K,P] &= 0\,,\\ [K,H] &= \frac{i\sigma}{-2}\left(\mathrm{e}^{-2P/\sigma}-1\right)\end{aligned}\tag{4.19}$$

and, in both cases, H, P, and K are Hermitian. Some examples of actual calculations are given in Appendix.

5. The Lyakhovsky–Mudrov formalism

In order to find the quantum Lie algebras dual to our groups we can also use the formalism developed by Lyakhovsky and Mudrov [5,6]. It is based on following theorem (Lyakhovsky–Mudrov):

Let $\{I, H_1, \ldots, H_n, X_1, \ldots, X_m\}$ be a basis of an associative algebra E over C verifying the conditions

$$[H_i, H_j] = 0, \qquad i,j = 1,\ldots,n. \tag{5.1}$$

Let $\mu_i, \nu_j (i,j = 1,\ldots,n)$ be a set of $m \times n$ complex matrices such that

$$[\mu_i, \nu_j] = [\mu_i, \mu_j] = [\nu_i, \nu_j] = 0, \qquad i,j = 1,\ldots,n. \tag{5.2}$$

Let $\vec{X}$ be a vector with components $X_l (l = 1,\ldots,m)$.
The co-product

$$\Delta(I) = I \otimes I\,, \qquad \Delta(H_i) = I \otimes H_i + H_i \otimes I\,,$$

$$\Delta(\vec{X}) = \exp\left(\sum_{i=1}^{n} \mu_i H_i\right) \dot{\otimes} \vec{X} + \sigma\left(\exp(\sum_{i=1}^{n} \nu_i H_i) \dot{\otimes} \vec{X}\right) \tag{5.3}$$

and the counit

$$\varepsilon(I) = I\,, \qquad \varepsilon(H_i) = 0\,, \qquad i = 1,\ldots,n\,;$$

$$\varepsilon(X_l) = 0\,, \qquad l = 1,\ldots,m\,; \tag{5.4}$$

endow (E, Δ, ε) with a coalgebra structure.

With the help of this theorem we can find coalgebra structure. To this end we recall that the co-commutator δ corresponds to the leading part of the co-antisymmetric part of the co-products

$$\delta(\vec{X}) = \Delta_{(1)}(\vec{X}) - \sigma \circ \Delta_{(1)}(\vec{X}), \tag{5.5}$$

where

$$\Delta_{(1)}(\vec{X}) = \left(\sum_{i=1}^{n} \mu_i H_i\right) \dot{\otimes} \vec{X} + \sigma\left(\left(\sum_{i=1}^{n} \nu_i H_i\right) \dot{\otimes} \vec{X}\right) \tag{5.6}$$

is the first order term in all the parameters of (5.6).

Therefore matrices μ_i and ν_i can be determined from the known form of $\delta(X)$.

Now, if one is able to find a commutation rules compatible with the co-product one obtains a quantum algebra. By applying this formalism to our case (which, actually, has been done in Ref. [6]) we arrive at the same form of co-product as given by duality rules, Eqs. (4.12)–(4.14); (4.16)–(4.18).

Therefore, we have an alternative way to construct our quantum Galilei algebras.

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Appendix

We present some sample calculations concerning the dual structures. Let us consider the (B) case,

$$\begin{aligned}
[a,v] &= \frac{i}{\alpha}\tau - \frac{i}{\sigma}v\,,\\
[a,\tau] &= \frac{i}{\sigma}\tau + \frac{i}{\lambda}v\,,\\
[v,\tau] &= 0\,,
\end{aligned} \tag{A.1}$$

$\varepsilon^2\sigma^2 = \alpha\lambda$.

Let us calculate

$$\begin{aligned}
ff' &= \mathrm{e}^{\mu a}\mathrm{e}^{\lambda\tau}\mathrm{e}^{\eta v}\mathrm{e}^{\mu' a}\mathrm{e}^{\lambda'\tau}\mathrm{e}^{\eta' v}\\
&= \mathrm{e}^{\mu a+\mu' a}\mathrm{e}^{\lambda(\mathrm{e}^{-\mu' a}\tau\mathrm{e}^{\mu' a})}\mathrm{e}^{\eta(\mathrm{e}^{-\mu' a}v\mathrm{e}^{\mu' a})}\mathrm{e}^{\lambda'\tau}\mathrm{e}^{\eta' v}\,;
\end{aligned} \tag{A.2}$$

here prime means the second factor of tensor product and the tensor product symbol $\otimes$ has been omitted. Denoting

$$x(\mu') = \mathrm{e}^{-\mu' a}\tau\mathrm{e}^{\mu' a}\,, \qquad x(0) = \tau\,, \tag{A.3}$$

$$y(\mu') = \mathrm{e}^{-\mu' a}v\mathrm{e}^{\mu' a}\,, \qquad y(0) = v\,, \tag{A.4}$$

we obtain the following differential equations

$$\dot{x}(\mu') = \mathrm{e}^{-\mu' a}[\tau,a]\mathrm{e}^{\mu' a} = \frac{i}{\sigma}x(\mu') - \frac{i}{\lambda}y(\mu')\,, \tag{A.5}$$

$$\dot{y}(\mu') = \mathrm{e}^{-\mu' a}[v,a]\mathrm{e}^{\mu' a} = \frac{i}{\sigma}y(\mu') - \frac{i}{\alpha}x(\mu') \tag{A.6}$$

or, in matrix form

$$\dot{\begin{pmatrix} x(\mu') \\ y(\mu') \end{pmatrix}} = \begin{pmatrix} \frac{i}{\sigma} & \frac{-i}{\lambda} \\ \frac{-i}{\alpha} & \frac{i}{\sigma} \end{pmatrix}\begin{pmatrix} x(\mu') \\ y(\mu') \end{pmatrix}. \tag{A.7}$$

Thus the solution to Eq. (A.7) reads

$$\begin{pmatrix} x(\mu') \\ y(\mu') \end{pmatrix} = \mathrm{e}^{i\mu' A} \begin{pmatrix} \tau \\ v \end{pmatrix}, \tag{A.8}$$

where

$$A = \begin{pmatrix} \frac{1}{\sigma} & \frac{-1}{\lambda} \\ \frac{-1}{\alpha} & \frac{1}{\sigma} \end{pmatrix}.$$

It is easy to check that

$$\mathrm{e}^{i\mu' A} = \mathrm{e}^{i\mu'/\sigma} \begin{pmatrix} \cosh\left(\frac{\mu'}{\sqrt{\alpha\lambda}}\right) & \frac{-\alpha}{\sqrt{\alpha\lambda}}\sinh\left(\frac{\mu'}{\sqrt{\alpha\lambda}}\right) \\ \frac{-\lambda}{\sqrt{\alpha\lambda}}\sinh\left(\frac{\mu'}{\sqrt{\alpha\lambda}}\right) & \cosh\left(\frac{\mu'}{\sqrt{\alpha\lambda}}\right) \end{pmatrix} \tag{A.9}$$

and, consequently

$$\begin{aligned} x &= \mathrm{e}^{i\mu'/\sigma}\left(\tau\cosh\left(\frac{\mu'}{\sqrt{\alpha\lambda}}\right) - \frac{\alpha}{\sqrt{\alpha\lambda}} v\sinh\left(\frac{\mu'}{\sqrt{\alpha\lambda}}\right)\right), \\ y &= \mathrm{e}^{i\mu'/\sigma}\left(\frac{-\lambda}{\sqrt{\alpha\lambda}}\tau\sinh\left(\frac{\mu'}{\sqrt{\alpha\lambda}}\right) + v\cosh\left(\frac{\mu'}{\sqrt{\alpha\lambda}}\right)\right). \end{aligned} \tag{A.10}$$

Therefore

$$\begin{aligned} ff' &= \exp(\mu+\mu')a \\ &\quad\times \exp\left[\mathrm{e}^{i\mu'/\sigma}\left(\frac{-\lambda}{\sqrt{\alpha\lambda}}\eta\sinh\left(\frac{\mu'}{\sqrt{\alpha\lambda}}\right) + \lambda\cosh\left(\frac{\mu'}{\sqrt{\alpha\lambda}}\right)\right) + \lambda'\right]\tau \\ &\quad\times \exp\left[\mathrm{e}^{i\mu'/\sigma}\left(\eta\cosh\left(\frac{\mu'}{\sqrt{\alpha\lambda}}\right) - \frac{\alpha}{\sqrt{\alpha\lambda}}\lambda\sinh\left(\frac{\mu}{\sqrt{\alpha\lambda}}\right)\right) + \eta'\right]v \end{aligned} \tag{A.11}$$

and from this formula we obtain

$$\begin{aligned} \Delta(H) &= I\otimes H + H\otimes \mathrm{e}^{\frac{-P}{\sigma}}\cosh\left(\frac{P}{\sqrt{\alpha\lambda}}\right) - \frac{\lambda}{\sqrt{\alpha\lambda}}K\otimes \mathrm{e}^{\frac{-P}{\sigma}}\sinh\left(\frac{P}{\sqrt{\alpha\lambda}}\right), \\ \Delta(K) &= I\otimes K + K\otimes \mathrm{e}^{\frac{-P}{\sigma}}\cosh\left(\frac{P}{\sqrt{\alpha\lambda}}\right) - \frac{\alpha}{\sqrt{\alpha\lambda}}H\otimes \mathrm{e}^{\frac{-P}{\sigma}}\sinh\left(\frac{P}{\sqrt{\alpha\lambda}}\right), \\ \Delta(P) &= I\otimes P + P\otimes I. \end{aligned} \tag{A.12}$$

On the other hand

$$\Delta(f) = \mathrm{e}^{\mu(a+a'+v\tau')}\mathrm{e}^{\lambda(\tau+\tau')}\mathrm{e}^{\eta(v+v')}. \tag{A.13}$$

In order to calculate $\Delta(f)$ we use the following trick: we write

$$\Delta(f) = \mathrm{e}^{\mu(a+a')}\chi \mathrm{e}^{\lambda(\tau+\tau')}\mathrm{e}^{\eta(v+v')}\,, \tag{A.14}$$

where

$$\chi = \mathrm{e}^{-\mu(a+a')}\mathrm{e}^{\mu(a+a'+v\tau')}\,, \qquad \chi(0) = 1\,. \tag{A.15}$$

Again differentiating both sides with respect to μ we got

$$\dot{\chi} = (\mathrm{e}^{-\mu a} v \mathrm{e}^{\mu a})(\mathrm{e}^{-\mu a'}\tau' \mathrm{e}^{\mu a'})\chi \tag{A.16}$$

and

$$\begin{aligned} \frac{\dot{\chi}}{\chi} &= \mathrm{e}^{\frac{2i\mu}{\sigma}}\left(v\cosh\left(\frac{\mu}{\sqrt{\alpha\lambda}}\right) - \frac{\lambda}{\sqrt{\alpha\lambda}}\tau\sinh\left(\frac{\mu}{\sqrt{\alpha\lambda}}\right)\right) \\ &\quad\times\left(\tau'\cosh(\frac{\mu}{\sqrt{\alpha\lambda}}) - \frac{\alpha}{\sqrt{\alpha\lambda}}v'\sinh\left(\frac{\mu}{\sqrt{\alpha\lambda}}\right)\right). \end{aligned} \tag{A.17}$$

The terms vv' and $\tau\tau'$ do not contribute to the product of different generators, so they can be neglected in what follows. Therfore, up to irrelevant terms

$$\begin{aligned} \Delta(f) =& \mathrm{e}^{\mu a}\mathrm{e}^{\mu a'}\mathrm{e}^{\lambda\tau}\mathrm{e}^{\lambda\tau'}\mathrm{e}^{\eta v}\mathrm{e}^{\eta v'} \\ &\times \mathrm{e}^{\tau v'\left[-i\sqrt{\alpha\lambda}\left(\frac{\mathrm{e}^{(\frac{2i\mu}{\sigma})}}{2(1-\frac{\alpha\lambda}{\sigma^2})}\left(\sinh(\frac{i\mu}{\sqrt{\alpha\lambda}})\left(\cosh(\frac{i\mu}{\sqrt{\alpha\lambda}})+\frac{\sqrt{\alpha\lambda}}{\sigma}\sinh(\frac{i\mu}{\sqrt{\alpha\lambda}})\right)-\frac{\sigma}{2\sqrt{\alpha\lambda}}\right)\right)+\frac{\sigma}{4\sqrt{\alpha\lambda}(1-\frac{\alpha\lambda}{\sigma^2})}\right]} \\ &\times \mathrm{e}^{\tau' v\left[-i\sqrt{\alpha\lambda}\left(\frac{\mathrm{e}^{(\frac{2i\mu}{\sigma})}}{2(1-\frac{\alpha\lambda}{\sigma^2})}\left(\cosh(\frac{i\mu}{\sqrt{\alpha\lambda}})\left(\sinh(\frac{i\mu}{\sqrt{\alpha\lambda}})+\frac{\sqrt{\alpha\lambda}}{\sigma}\cosh(\frac{i\mu}{\sqrt{\alpha\lambda}})\right)+\frac{\sigma}{2\sqrt{\alpha\lambda}}\right)\right)-\frac{\sigma(1-2\frac{\sqrt{\alpha\lambda}}{\sigma^2})}{4\sqrt{\alpha\lambda}(1-\frac{\alpha\lambda}{\sigma^2})}\right]}. \end{aligned} \tag{A.18}$$

Hence, the duality rule (4.1) gives

$$\begin{aligned} [H,P] &= 0\,, \\ [K,P] &= 0\,, \\ [K,H] &= \frac{i\sigma}{2}\left(1-\mathrm{e}^{-2P/\sigma}\right)\,. \end{aligned} \tag{A.19}$$

The antipode and counit can be obtained either from the relevant duality relations or directly from the Hopf algebra rules:

$$(id\otimes\varepsilon)\Delta = id\,, \tag{A.20}$$

$$m\circ(id\otimes S)\Delta = \varepsilon\,. \tag{A.21}$$

In this way we have

$$\begin{aligned}
\varepsilon(X) &= 0\,, \qquad\qquad (A.22)\\
S(H) &= -H\mathrm{e}^{(P/\sigma)}\cosh\left(\frac{P}{\sqrt{\alpha\lambda}}\right) - \frac{\lambda}{\sqrt{\alpha\lambda}}K\mathrm{e}^{(P/\sigma)}\sinh\left(\frac{P}{\sqrt{\alpha\lambda}}\right)\,,\\
S(K) &= -K\mathrm{e}^{(P/\sigma)}\cosh\left(\frac{P}{\sqrt{\alpha\lambda}}\right) - \frac{\alpha}{\sqrt{\alpha\lambda}}H\mathrm{e}^{(P/\sigma)}\sinh\left(\frac{P}{\sqrt{\alpha\lambda}}\right)\,,\\
S(P) &= -P\,; \qquad\qquad (A.23)
\end{aligned}$$

moreover co-product is homomorphism of this algebra *i.e.*

$$\Delta([,]) = [\Delta, \Delta]\,.$$

Let us now check Eq. (4.4):

$$\begin{aligned}
S^{-1}\left(\left(\mathrm{e}^{\lambda\tau}\mathrm{e}^{\mu a}\mathrm{e}^{\eta v}\right)^*\right) &= S^{-1}\left(\mathrm{e}^{\eta^* v}\mathrm{e}^{\mu^* a}\mathrm{e}^{\lambda^*\tau}\right)\\
&= \mathrm{e}^{-\lambda^*\tau}\mathrm{e}^{\mu^*(-a+\tau v)}\mathrm{e}^{-\eta^* v}. \qquad (A.24)
\end{aligned}$$

$$\begin{aligned}
\left[S\left(\left(\mathrm{e}^{\lambda\tau}\mathrm{e}^{\mu a}\mathrm{e}^{\eta v}\right)^*\right)\right]^* &= \left[S\left(\mathrm{e}^{\eta^* v}\mathrm{e}^{\mu^* a}\mathrm{e}^{\lambda^*\tau}\right)\right]^*\\
&= \left(\mathrm{e}^{-\lambda^*\tau}\mathrm{e}^{\mu^*(-a+\tau v)}\mathrm{e}^{-\eta^* v}\right)^*\\
&= \mathrm{e}^{-\eta v}\mathrm{e}^{\mu(-a+v\tau)}\mathrm{e}^{-\lambda\tau}\\
&= S^{-1}\left(\mathrm{e}^{\lambda\tau}\mathrm{e}^{\mu a}\mathrm{e}^{\eta v}\right). \qquad (A.25)
\end{aligned}$$

By applying the formula (4.3) we get

$$\begin{aligned}
\langle H^*, \mathrm{e}^{\lambda\tau}\mathrm{e}^{\mu a}\mathrm{e}^{\eta v}\rangle &= \langle H, \mathrm{e}^{-\lambda^*\tau}\mathrm{e}^{\mu^*(-a+\tau v)}\mathrm{e}^{-\eta^* v}\rangle^*\\
&= (-i\lambda^*)^* = i\lambda\\
&= \langle H, \mathrm{e}^{\lambda\tau}\mathrm{e}^{\mu a}\mathrm{e}^{\eta v}\rangle\,, \qquad (A.26)
\end{aligned}$$

hence

$$H^* = H\,.$$

In this same way one obtains $K^* = K$, $P^* = P$.

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