SUPER STAR PRODUCTS AND QUANTUM SUPERALGEBRAS

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We prove that the super star product on a Poisson Lie supergroup leads to the structure of quantum superalgebra (triangular Hopf superalgebra) on the super quantized enveloping algebra of the corresponding Lie superalgebra and that equivalent super star products generate isomorphic quantum superalgebras.

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1. Introduction

The development of the Quantum Inverse Scattering Method (QISM) [1] intended for investigations of integrable models of the quantum field theory and statistical physics gives rise to some interesting algebraic constructions. These investigations allow to select a special class of Hopf algebras now known as quantum groups and quantum algebras [2, 3]. The nice R-matrix formulation of the quantum group theory [4], based on the fundamental relation of QISM (the FRT relation) has given an additional impulse to these investigations. The extension of the activity on quantum groups to the field of supersymmetry was started with the paper of Manin [5], where the standard multiparametric quantum deformation of the supergroup $\mathrm{GL}(m/n)$ was introduced. The study of the superalgebra in duality with the standard multiparametric deformation of $\mathrm{GL}(m/n)$ was given in [6]. Quantum superalgebras appeared naturally when the quantum inverse scattering method was generalized to the super-systems [7]. Related R-matrix were considered in [8, 9] and simple examples were presented in [10]. The works [11, 12, 15] are devoted to the q-bosonization of the q-superalgebras.The properties of the quantum superalgebras $U_Q(A(m,n)), U_Q(B(m,n)), U_Q(C(n+1))$ and $U_Q(D(m,n))$ when their deformation parameter Q goes to a root of unity, are investigated in [13, 14]. Quantum supergroups, were investigated in [15–17].

As is well known, quantum groups can be seen as noncommutative generalizations of topological spaces which have a group structure. Such a structure induces an Abelian Hopf algebra structure [18] on the algebra of smooth functions on the group. Quantum groups are defined then as a non Abelian Hopf algebras [19]. A way to generate them consists of deforming the Abelian Hopf algebra of functions into a non Abelian one (*-product), using the so called deformation quantization or star-quantization [20–24].

A star-quantization method is used also to give an h-deformed algebra (quantum Lie algebra) in [25], to realize both q-deformed Virasoro and $su_q(2)$ algebras in [26], to deform the corresponding Yangian of a simple Lie algebra in [27] and to generate quantum algebras in [28]. The notion of a super star-product on a symplectic flat supermanifold is investigated in [29] and a deformation-quantization of Fedesov type of super Poisson bracket is given in [30].

The purpose of the present paper is to show that the quantum supergroups can be generated by deforming the graded (Abelian) Hopf algebra of super functions structure into a non graded-Abelian one (super star-product). This quantization technique gives a deformed product once a Poisson superbracket on the superalgebra of super smooth functions is given. In order to ensure that the deformed superalgebra is a Hopf superalgebra, namely a quantum supergroup, the starting supergroup G has to be endowed with a super Lie-Poisson structure. Finally, using the duality procedure, this quantization leads to the structure of the quantum superalgebra on the super quantized enveloping algebra of the Lie superalgebra corresponding to the above Lie supergroup G .

This paper is organized as follows; the second section is devoted to a review of basic definitions of Lie bisuperalgebras and Lie-Poisson supergroups. In the third section we show the main result that states that a super star product on a Lie–Poisson supergroup leads to the structure of a quantum superalgebra on the quantized enveloping algebra of the Lie superalgebra corresponding to the above supergroup, we give a vector representation of the super quantum Yang–Baxter equation and we show that equivalent super star products generate isomorphic quantum superalgebras.

2. Basic definitions

Let us first recall some properties of the vector superspaces and Lie superalgebras on the complex number field.

If g is a vector super space then $g = g_0 \oplus g_1$, where we refer to g_0 and g_1 as the even and odd subspaces of g, respectively. We define the operator index

$$||: \quad g \longrightarrow \{0,1\}$$

for the homogeneous elements of g by

$$| x |= 0 \,, \quad \text{if} \quad x \quad \varepsilon \quad g_0 \,,$$

$$| x |= 1 \,, \quad \text{if} \quad x \quad \varepsilon \quad g_1$$

and call $(-1)^{|x|}$ the parity of x . The dual g^* inherits a natural super gradation $g^* = g_0^* \oplus g_1^*$, with g_a^* is isomorphic to the dual of g_a $(a = 0, 1)$. On the tensor product $g \otimes g$, there exists a natural super gradation induced from that of g, where the parity of $x \otimes y$ is related to those of the homogenous elements $x, y \in g$ through

$$(-1)^{|x \otimes y|} = (-1)^{|x|+|y|}.$$

It is also useful to define the twisting map

$$T : g \otimes g \longrightarrow g \otimes g$$

by

$$T(x \otimes y) = (-1)^{|x||y|} (y \otimes x) \tag{1}$$

for all homogenous $x, y \in g$; this definition is extended by linearity to all $g \otimes g$.

A Lie superalgebra structure on g is provided by a linear mapping

$$[,] : g \otimes g \longrightarrow g$$

satisfying the requirement of super Jacobi identity and super antisymmetry. In order to express them it is useful to introduce a basis $\{X_i\}$ in g and structure constants defined by

$$[X_i, X_j] = C_{ij}^k X_k \,.$$

Then the structure constants have to satisfy

$$C_{ij}^k = 0 \quad \text{whenever} \quad | X_i | + | X_j | \neq | X_k | \ (\text{mod } 2)$$

$$C_{ij}^k = (-1)^{|i||j|} C_{ji}^k \quad \text{(super antisymmetry)}$$

$$(-1)^{|i||l|} C_{ij}^k C_{kl}^m + (-1)^{|i||j|} C_{jl}^k C_{ki}^m + (-1)^{|j||l|} C_{li}^k C_{kj}^m = 0 \ \text{(super Jacobi identity)}.$$

A Lie bisuperalgebra structure on g is given by a linear mapping

$$\phi : g \longrightarrow g \otimes g$$

$$\phi(X_i) = f_i^{kl} X_k \otimes X_l \,,$$

where ϕ has to satisfy several requirements. First of all it makes the dual linear space g^* a Lie superalgebra *i.e.*:

$$f_k^{ij} = 0\,,$$

whenever $\mid X_i \mid + \mid X_j \mid \neq \mid X_k \mid$ (mod 2) and

$$(-1)^{|k||m|} f_i^{kj} f_j^{lm} + (-1)^{|l||k|} f_i^{lj} f_j^{mk} + (-1)^{|m||l|} f_i^{mj} f_j^{kl} = 0\,.$$

ϕ must be a superalgebra 1-cocycle

$$\phi[X,Y] = ad_X \phi(Y) - (-1)^{|X||Y|} ad_Y \phi(X)\,.$$

A cobondary Lie superalgebra is a pair (g,r), where g is a Lie superalgebra and $r \in ((g_0 \otimes g_0) \oplus (g_1 \otimes g_1))$ such that for every $X_i \in g$ we have

$$\phi(X_i) = [r, 1 \otimes X_i + X_i \otimes 1]\,,$$

where the even element r satisfies the generalized classical Yang–Baxter equation

$$[[\![r,r]\!], 1 \otimes X_i \otimes 1 + X_i \otimes 1 \otimes 1 + 1 \otimes 1 \otimes X_i] = 0 \tag{2}$$

and the super Schouten bracket is defined as follows

$$[\![r,r]\!] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]\,.$$

The coboundary superbialgebra with the r-matrix satisfying the modified classical Yang–Baxter equation describes infinitesimally Poisson–Lie supergoups which will be defined later. We now make the following definitions:

Definition 1 *A super quantized universal enveloping algebra is a topological Hopf superalgebra B with a bijective antipode over the ring of formal series $C[[h]]$, complete with respect to the h-adic topology and such that $\frac{B}{hB}$ is the universal enveloping algebra $U(g)$ of some Lie superalgebra g.*

Let $W = W_0 + W_1$ be a differentielle supermanifold and $\mathrm{Fun}(W) = \mathrm{Fun}_0(W) \oplus \mathrm{Fun}_1(W)$ be the algebra of supersmooth functions on W; $f \in \mathrm{Fun}_0(W)[\mathrm{Fun}_1(W)]$ is said to be homogenous of even [odd] parity.

Definition 2 *A super Poisson bracket $\{,\}$ on $\mathrm{Fun}(W))$ is a bilinear operation assigning to every pair of functions $f,g \in \mathrm{Fun}(W)$ a new function $\{f,g\} \in \mathrm{Fun}(W)$, such that for homogenous functions satisfies the following conditions:*

(i) — Graded preserving

$$\deg(\{f,g\} = \deg(f) + \deg(g)\,, \tag{3}$$

(ii) — Super skew-symmetry

$$\{f, g\} = -(-1)^{\deg(f)\deg(g)}\{g, f\}\,, \tag{4}$$

(iii) — Graded Leibniz rule

$$\{f, gh\} = \{f, g\}h + (-1)^{\deg(f)\deg(g)}g\{f, h\}\,, \tag{5}$$

(iv) — Super Jocobi identity

$$(-1)^{\deg(f)\deg(h)}\{f, \{g, h\}\} + (-1)^{\deg(g)\deg(f)}\{g, \{h, f\}\} + (-1)^{\deg(h)\deg(g)}\{h, \{f, g\}\} = 0\,. \tag{6}$$

Since the conditions [3–6] are just the axioms of superalgebras, the space $\mathrm{Fun}(W)$ endowed with the super Poisson bracket becomes a Poisson superalgebra and W a Poisson supermanifold.

Definition 3 *A Poisson Lie supergroup is a Lie supergroup G provided with a Poisson superbracket $\{,\}$ such that the comultiplication*

$$\Delta : \mathrm{Fun}(G) \longrightarrow \mathrm{Fun}(G) \otimes \mathrm{Fun}(G)$$

is a morphism of Poisson superbrackets:

$$\Delta\{\phi, \psi\} = \{\Delta(\phi), \Delta(\psi)\}\,,$$

where the Poisson superbracket on $\mathrm{Fun}(G) \otimes \mathrm{Fun}(G)$ is defined as

$$\{\phi_1 \otimes \psi_1, \phi_2 \otimes \psi_2\} = (-1)^{|\phi_2||\psi_1|}(\{\phi_1, \phi_2\} \otimes \psi_1\psi_2 + \phi_1\phi_2 \otimes \{\psi_1, \psi_2\})\,. \tag{7}$$

and the following rule for the multiplication of graded tensor products should be used:

$$(\psi_1 \otimes \psi_2)(\phi_1 \otimes \phi_2) = (-1)^{\deg(\psi_2}\deg(\phi_1)(\psi_1 phi_1 \otimes \psi_2 phi_2)\,.$$

3. Super star products and quantum superalgebras

3.1. Triangular Hopf superalgebra structure

Let $G = G_0 \oplus G_1$ be a Lie supergroup, g its Lie superalgebra. The enveloping algebra of the Lie superalgebra g is defined [31] to be the tensor algebra $T(g) = \oplus_{k=0}^{\infty} g^{\otimes k}$, modulo the ideal I in $T(g)$ generated by all elements in $T(g)$ of the form

$$x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y] \tag{8}$$

for $x, y \in g$. As in the classical case the Poincare–Birkoff–Witt theorem is valid for $U(g)$, indeed for $g = g_0 \oplus g_1$ and if $\{(e_i),\ i = 1, 2, \dots n\}$ is a basis of g_0 and $\{(v_i),\ i = 1, 2, \dots m\}$ is a basis of g_1 then a basis of $U(g)$ is given by

$$e_1^{k_1} \dots e_n^{k_n} . v_{i_1} \dots v_{i_j} \,, \tag{9}$$

where $k_1, \dots k_n \in N$ and $1 \le i_1 \le \dots, i_j \le m$.

Let 1 be the identity of the enveloping superalgebra. Then the morphism of degree zero g into $U(g) \otimes U(g)$ given by

$$x \longrightarrow x \otimes 1 + 1 \otimes x \tag{10}$$

extends to a morphism of degree zero

$$\Delta_0 : U(g) \longrightarrow U(g) \otimes U(g) \,. \tag{11}$$

We note that for a bisuperalgebra $A = A_0 \oplus A_1$, the coproduct preserves the parity; namely, one has

$$\begin{aligned} \Delta : A &\longrightarrow A \otimes A \\ \Delta : A_0 &\longrightarrow A_0 \otimes A_0 + A_1 \otimes A_1 \\ \Delta : A_1 &\longrightarrow A_0 \otimes A_1 + A_1 \otimes A_0 \,. \end{aligned}$$

The antipode of the enveloping superalgebra is defined as an homogenous bijective map of degree zero

$$S_0 : U(g) \longrightarrow U(g) \tag{12}$$

such that for any $x \in g$ we have

$$S_0(x) = -x \tag{13}$$

and for $u, v \in U(g)$ we have

$$S_0(uv) = (-1)^{|u||v|} S_0(v) S_0(u) \,. \tag{14}$$

Now let $r \in ((g_0 \otimes g_0) \oplus (g_1 \otimes g_1))$ be a solution of the super classical Yang–Baxter equation.

$$[\![r, r]\!] = 0\,. \tag{15}$$

Then the Lie bisuperalgebra structure on g is given by the superalgebra 1-cocycle

$$\begin{aligned} \delta : g &\longrightarrow g \otimes g\,, \\ x &\longmapsto (ad_x \otimes 1 + 1 \otimes ad_x) r\,, \end{aligned} \tag{16}$$

where ad_x stands for the adjoint representation and the super Poisson–Lie structure on Lie supergroup G is given by [32]

$$\{\phi, \psi\} = \sum_{i,j} (-1)^{|\phi||j|} r^{ij} (X_i^{\mathrm{r}}(\phi) X_j^{\mathrm{r}}(\psi) - X_i^{\mathrm{l}}(\phi) X_j^{\mathrm{l}}(\psi))\,, \tag{17}$$

where $X_i^{\mathrm{r}} = (R_g)_* X_i$ and $X_i^{\mathrm{l}} = (L_g)_* X_i$ are the right and left vectors fields on the supergroup G, (X_i) is a basis of g with $(R_g)_*$ and $(L_g)_*$ the derivative maping corresponding to the right and left translation respectively .

If we denote by $R(G)(L(G))$ the set of all right(left)-invariant vector fields on G, then using elementary properties of derivative mapings [33] one may show that each of $L(G)$ and $R(G)$ is a vector superspace with a bracket operation that satisfies the super Jacobi identity. Since every element of $L(G)$ or $R(G)$ is completely determined by its value at the identity element of G it follows that $L(G)$ and $R(G)$ are isomorphic to the Lie superalgebra (the tangent space to G at the identity (e)).

Such morphisms can be extended to graded algebra morphisms

$$U(g) \longrightarrow D^{\mathrm{l}}(G)\,, \tag{18}$$
$$A \longmapsto A^{\mathrm{l}}\,, \tag{19}$$
$$U(g) \longrightarrow D^{\mathrm{r}}(G)\,, \tag{20}$$
$$A \longmapsto A^{\mathrm{r}}\,, \tag{21}$$

where $D^{\mathrm{l}}(G)$ and $D^{\mathrm{r}}(G)$ are respectively the superalgebra of left-invariant differential operators and the superalgebra of right-invariant differential operators, such that the action of $U(g)$ on $\boldsymbol{F}(G)$ will be given by

$$\langle X, Y^{\mathrm{l}}(\phi)\rangle = \langle XY, \phi\rangle\,, \tag{22}$$
$$\langle X, Y^{\mathrm{r}}(\phi)\rangle = (-1)^{|X||Y|}\langle S_0(Y)X, \phi\rangle\,. \tag{23}$$

We now make the following definitions

Definition 4 *A super star product on the Poisson Lie supergroup is a bilinear map*

$$F(G) \times F(G) \longrightarrow F(G)[[h]]\,,$$

$$(\phi, \psi) \longmapsto \phi * \psi = \sum_j h^j C_j(\phi, \psi) \tag{24}$$

such that

(i) when the above map is extended to $F(G)[[h]]$*, it is formally associative*

$$(\phi * \psi) * \chi = \phi * (\psi * \chi) \tag{25}$$

(ii) $C_0(\phi, \psi) = \phi.\psi = (-1)^{|\phi||\psi|}\psi.\phi$

(iii) $C_1(\phi, \psi) = \{\phi, \psi\}$

(iv) the two-cochains $C_k(\phi, \psi)$ *are bidifferential operators , homogeneous of degree zero on* $F(G)$ *.*

The problem is to get a super star-product on G such that the compatibility relation

$$\Delta(\phi * \psi) = (\Delta(\phi) * \Delta(\psi)) \tag{26}$$

is satisfied. The super star-product on the right side is canonically defined on $F(G) \otimes F(G)$ by

$$(\phi \otimes \psi) * (\phi^{'} \otimes \psi^{'}) = (-1)^{|\psi||\phi^{'}|}(\phi * \phi^{'}) \otimes (\psi * \psi^{'})\,. \tag{27}$$

Remark: If all C_k are a left (right)-invariant even bidifferential operators then the corresponding super star product is called left (right)-invariant.

Definition 5 *Two super star-products* $*_1$ *and* $*_2$ *defined on the supergroup* G *are said to be formally equivalent if there exists a series*

$$T = id + \sum_{i=1}^{\infty} h^i T_i\,, \tag{28}$$

where the T_i *are even differential operators, such that*

$$T(\phi *_1 \psi) = T(\phi) *_2 T(\psi)\,. \tag{29}$$

Thanks to the morphisms(16),(18), we see that if C_i is a left-invariant even two cochain then there is an homogeneous element of degree zero $F_i \in U(g) \otimes U(g)$ such that:

$$C_i^{\mathrm{l}}(\phi, \psi) = F_i^{\mathrm{l}}(\phi \otimes \psi)\,. \tag{30}$$

Similarly for the right invariant even two cochain there exists an homogeneous element of degree zero $H_i \in U(g) \otimes U(g)$ such that:

$$C_j^{\mathrm{r}}(\phi, \psi) = H_j^{\mathrm{r}}(\phi \otimes \psi)\,. \tag{31}$$

If we introduce the two homogeneous elements of degree zero of $U(g) \otimes U(g)[[h]]$

$$F = 1 + \sum_{i \geq 1} F_i h^i\,,$$

$$H = 1 + \sum_{j \geq 1} H_j h^j$$

then we obtain the following result

Proposition 1 *The associativity of the left-invariant super star-product implies*

$$(\Delta_0 \otimes id)F.(F \otimes 1) = (1 \otimes \Delta_0)F.(1 \otimes F) \tag{32}$$

and the associativity of the right-invariant super star-product leads to the following equality

$$(S_0^{\otimes 2}(H) \otimes 1).(\Delta_0 \otimes id)S_0^{\otimes 2}(H) = (1 \otimes S_0^{\otimes 2}(H)).(1 \otimes \Delta_0)S_0^{\otimes 2}(H)\,. \tag{33}$$

Proof: writing the right-invariant super star product in the following form

$$(\phi *^{\mathrm{r}} \psi) = m(H^{\mathrm{r}}(\phi \otimes \psi))\,,$$

where $H = 1 + \frac{\mathrm{h}}{2} r + \sum_{i \geq 2} H_i h^i$
we have for any homogeneous element X in the enveloping superalgebra,

$$\begin{aligned} &\langle X, \phi *^{\mathrm{r}} (\psi *^{\mathrm{r}} \chi)\rangle \\ = \ &\langle X, m(id \otimes m)((id \otimes \Delta_0)H^{\mathrm{r}}.H_{23}^{\mathrm{r}}(\phi \otimes \psi \otimes \chi))\rangle \\ = \ &\langle (id \otimes \Delta_0)\Delta_0(X), (id \otimes \Delta_0)H^{\mathrm{r}}.H_{23}^{\mathrm{r}}(\phi \otimes \psi \otimes \chi)\rangle \\ = \ &\langle (1 \otimes (S_0^{\otimes 2}))H(id \otimes \Delta_0)((S_0^{\otimes 2})H)(id \otimes \Delta_0)\Delta_0(X), (\phi \otimes \psi \otimes \chi)\rangle\,. \end{aligned} \tag{34}$$

Similarly, we have

$$\begin{aligned} & \langle X, (\phi *^{\mathrm{r}} \psi) *^{\mathrm{r}} \chi\rangle \\ = & \langle ((S_0^{\otimes 2}) \otimes 1) H(\Delta_0 \otimes id)((S_0^{\otimes 2}) H)(\Delta_0 \otimes id)\Delta_0(X), (\phi \otimes \psi \otimes \chi)\rangle \end{aligned} \tag{35}$$

so, from (34) (35) we deduce easily the result (33).

An analogous proof establishes the left-invariant case.

Proposition 2 *Assume that F is a left-invariant super star product on the supergroup G, then $S_0^{\otimes 2}(F)$ is a right-invariant super star product on the supergroup G.*

Proof: by applying the operator $(S_0 \otimes S_0 \otimes S_0)$ to the equation (29) and using the fact that $(S_0 \otimes S_0) \circ \Delta_0^{\mathrm{op}} = \Delta_0 \circ S_0$, we find obviously the equation(30).

The super star product on the Poisson–Lie supergroup will be given by the following expression

$$\phi * \psi = \mu((S_0^{\otimes 2})^{-1}(F^{-1})^{\mathrm{r}}.F^{\mathrm{l}}(\phi \otimes \psi))\,, \tag{36}$$

where μ is the usual mutiplication on the superalgebra of smooth functions on the supergroup. In fact, the product defined in this way is associative

$$\begin{aligned} & (\phi * \psi) * \chi = \mu\left((S_0^{\otimes 2})^{-1}(F^{-1})^{\mathrm{r}}.F^{\mathrm{l}}(\mu\left((S_0^{\otimes 2})^{-1}(F^{-1})^{\mathrm{r}}.F^{\mathrm{l}}(\phi \otimes \psi)\right) \otimes \chi)\right) \\ = & \mu(\mu \otimes id)\big((\Delta_0 \otimes 1)\left((S_0^{\otimes 2})^{-1}(F^{-1})^{\mathrm{r}}\right).(\Delta_0 \otimes 1)F^{\mathrm{l}}.\left((S_0^{\otimes 2})^{-1}(F^{-1})^{\mathrm{r}} \otimes 1\right). \\ & (F^{\mathrm{l}} \otimes 1)(\phi \otimes \psi \otimes \chi)\big) \\ = & \mu(\mu \otimes id)\big((\Delta_0 \otimes id)\left((S_0^{\otimes 2})^{-1}(F^{-1})^{\mathrm{r}}\right).\left((S_0^{\otimes 2})^{-1}(F^{-1})^{\mathrm{r}} \otimes 1\right).(\Delta_0 \otimes id)F^{\mathrm{l}}. \\ & (F^{\mathrm{l}} \otimes 1)(\phi \otimes \psi \otimes \chi)\big) \\ = & \mu(\mu \otimes id)\big((id \otimes \Delta_0)\left((S_0^{\otimes 2})^{-1}(F^{-1})^{\mathrm{r}}\right).(1 \otimes \left(S_0^{\otimes 2}\right)^{-1})(F^{-1})^{\mathrm{r}}.(id \otimes \Delta)F^{\mathrm{l}}. \\ & (1 \otimes F^{\mathrm{l}})(\phi \otimes \psi \otimes \chi)\big) \\ = & \mu(id \otimes \mu)\big((id \otimes \Delta_0)\left((S_0^{\otimes 2})^{-1}(F^{-1})^{\mathrm{r}}\right).(1 \otimes \left(S_0^{\otimes 2}\right)^{-1})(F^{-1})^{\mathrm{r}}.(id \otimes \Delta_0)F^{\mathrm{l}}. \\ & (1 \otimes F^{\mathrm{l}})(\phi \otimes \psi \otimes \chi)\big) \\ = & \mu(id \otimes \mu)\big((id \otimes \Delta_0)\left((S_0^{\otimes 2})^{-1}(F^{-1})^{\mathrm{r}}\right).(id \otimes \Delta_0)F^{\mathrm{l}}.(1 \otimes \left(S_0^{\otimes 2}\right)^{-1}(F^{-1})^{\mathrm{r}}). \\ & (1 \otimes F^{\mathrm{l}})(\phi \otimes \psi \otimes \chi)\big) \\ = & \mu((S_0^{\otimes 2})^{-1}(F^{-1})^{\mathrm{r}}.F^{\mathrm{l}}.(\phi \otimes \mu\left((S_0^{\otimes 2})^{-1}(F^{-1})^{\mathrm{r}}.F^{\mathrm{l}}(\psi \otimes \chi)\right)) \\ = & \phi * (\psi * \chi)\,. \end{aligned}$$

For the compatibility relation, the proof is a graded version of the proof given in [23].

Actually a super star-product does not only define a deformation of the superalgebra of the super smooth functions on the supergroup $\boldsymbol{F}(G)$ but also of a quotient superalgebra $\boldsymbol{F}_e(G)$ defined as the set of element of $\boldsymbol{F}(G)$ in a neighbourhood containing the identity of G modulo the equivalence relation

$$\phi \sim \psi \quad \text{if} \quad \langle X, \phi - \psi \rangle = 0 \text{ for any} X \in U(g)\,,$$

where $\langle , \rangle$ is the pairing between $\boldsymbol{F}_e(G)$ and $U(g)$.

Let us recall now that two bialgebras U, A are said to be in duality if there exists a doubly nondegenerate bilinear form

$$\langle , \rangle : U \times A \longrightarrow C, \langle , \rangle : (u, a) \longrightarrow \langle u, a \rangle, u \in U, a \in A$$

such that for any $u, v \in U$ and $a, b \in A$ we have:

$$\begin{aligned} \langle u, ab \rangle &= \langle \Delta_A(u), a \otimes b \rangle\,, \\ \langle uv, a \rangle &= \langle u \otimes v, \Delta_U(a) \rangle\,, \\ \langle 1_U, a \rangle &= \varepsilon_A(a), \langle u\,, 1_U \rangle = \varepsilon_U(u)\,. \end{aligned}$$

All this extends to bisuperalgebras [5]. The only subtlety is that the tensor product is also graded, and, if (using Sweedlers notation) $\Delta_U(u) = \sum u_1 \otimes u_2, \Delta_A(a) = \sum a_1 \otimes a_2$, then

$$\begin{aligned} \langle u, ab \rangle &= (-1)^{|u_2||a|} \sum \langle u_1, a \rangle \langle u_2, b \rangle\,, \\ \langle uv, a \rangle &= (-1)^{|a_1||v|} \sum \langle u, a_1 \rangle \langle v, a_2 \rangle\,. \end{aligned}$$

The duality between bisuperalgebras may be used to obtain the unknown superalgebra from a known one if the two are in duality. So, the deformation we talk about is a deformation of the $\boldsymbol{F}_e(G)$ as a bialgebra ; this allows us to provide by the duality the deformed superalgebra $\boldsymbol{F}^*_e(G)[[h]]$, where $\boldsymbol{F}^*_e(G)$ is the set of distributions on G with support at the unit element (e). Indeed, as in the case of ordinary Lie groups, the set of distributions on G with support at the identity element is the enveloping superalgebra of the Lie superalgebra of the Lie supergroup, and we deduce that a super star product provide a deformation of the enveloping superalgebra.

The super quantized enveloping algebra $U(g)[[h]]$ is endowed with the structure of a Hopf superalgebra, where the multiplication superalgebra is the ordinary convolution on $\boldsymbol{F}^*_e(G)$ and the coproduct Δ_F is given by [34]

$$\langle \Delta_F(X), \phi \otimes \psi \rangle = \langle X, \phi * \psi \rangle \tag{37}$$

for all $\phi, \psi \;\in\; \boldsymbol{F}_e(G)$, and $X \in U(g)$.

In fact using the equations (19),(20) we obtain:

$$\begin{aligned} &\langle \Delta_F(X), \phi \otimes \psi \rangle = \langle X, \mu((S_0^{\otimes 2})^{-1}(F^{-1})^{\mathrm{r}}.F^{\mathrm{l}}(\phi \otimes \psi)) \rangle \\ &= \langle \Delta_0(X), (S_0^{\otimes 2})^{-1}(F^{-1})^{\mathrm{r}}.F^{\mathrm{l}}(\phi \otimes \psi) \rangle = \langle F^{-1}.\Delta_0(X).F, (\phi \otimes \psi) \rangle\,, \end{aligned}$$

and this implies

$$\Delta_F(X) = F^{-1}.\Delta_0(X)\,.F\,. \tag{38}$$

For the antipode of the super quantized enveloping algebra, we recall first that the antipode S_0 of $U(g)$ satisfies the following equation

$$m(S_0 \otimes id)\Delta_0(X) = m(id \otimes S_0)\Delta_0(X) = \varepsilon(X)1\,, \tag{39}$$

where m is the usual multiplication on the super enveloping algebra $U(g)$.

F and F^{-1} can be respectively split as

$$F = \sum_k a_k \otimes b_k\,, \quad F^{-1} = \sum_k c_k \otimes d_k$$

and if setting $u = m(id \otimes S_0)(F^{-1})$ as an invertible homogeneous element of $U(g)[[h]]$ of degree zero, then we can easily show that the antipode of the super quantized enveloping algebra $U(g)[[h]]$ is given by:

$$S_F(X) = u.S_0(X).u^{-1}\,, \tag{40}$$

where $u^{-1} = m(S_0 \otimes id)F$.

We will give the proof for the simple case when

$$\mid a_k \mid=\mid b_k \mid=\mid c_k \mid=\mid d_k \mid= 0$$

since for other cases the generalization is obvious. In fact

$$\begin{aligned} m(S_F \otimes id)\Delta_F(X) &= m(uS_0u^{-1} \otimes id)(F^{-1}\Delta_0(X)F) \\ &= \sum_{i,j,k} uS_0(a_i)S_0(X'_k)S_0(c_j)u^{-1}d_jX''_kb_i \end{aligned}$$

with $\Delta_0(X) = \sum_k X'_k \otimes X''_k$. Owing to the fact that S_0 satisfies the equation (39) and that

$$\sum_j S_0(c_j)u^{-1}d_j = m(S_0 \otimes id)(F.F^{-1}) = 1$$

we obtain that

$$m(S_F \otimes id)\Delta_F(X) = \sum_i uS_0(a_i)b_i\varepsilon(X)1 = \varepsilon(X)1\,.$$

Similarly, we can prove that:

$$m(id \otimes S_F)\Delta_F(X) = \varepsilon(X)1\,.$$

Now if we define the following even element, as Drinfeld does in [35] for the non graded case

$$R_F = F_{21}^{-1}.F\,, \tag{41}$$

where $F_{21} = T.F_{12}.T$, then we can easily show that R_F defines a quasitriangular structure on the super quantized enveloping algebra $U(g)[[h]]$.
In fact, applying the operator $T^{23}T^{13}$ to the equation (32) and using the fact that $\Delta_0^{\mathrm{op}} = T \circ \Delta_0$ we obtain the following relation

$$F_{12}^{-1}(\Delta_0 \otimes id)R_F F^{12} = (R_F)_{13}.(R_F)_{23}$$

which implies that

$$(\Delta_F \otimes id)R_F = (R_F)_{13}.(R_F)_{23}\,. \tag{42}$$

Similarly, applying $T^{12}T^{23}$ to the same equation(29), we obtain

$$(id \otimes \Delta_F)R_F = (R_F)_{13}.(R_F)_{12}\,. \tag{43}$$

From the fact that $\phi * 1 = 1 * \phi = \phi$ for all $\phi \ \in \boldsymbol{F}_e(G)$, we deduce that

$$(id \otimes \varepsilon)F = (\varepsilon \otimes id)F = 1\,; \tag{44}$$

consequently

$$(\varepsilon \otimes id)(R_F) = (id \otimes \varepsilon)(R_F) = 1 \tag{45}$$

and from the definition(41) we deduce that

$$(R_F)_{21}.R_F = 1\,. \tag{46}$$

Using again the expression (38) we obtain that

$$\begin{aligned}(\Delta_F)^{\mathrm{op}} &= T(\Delta_F) = T(F^{-1}).\Delta_0.T(F)\\ &= T(F^{-1}).F.\Delta_0.F^{-1}.T(F)\end{aligned}$$

then

$$(\Delta_F)^{\mathrm{op}} = R_F.\Delta_F.(R_F)^{-1}. \tag{47}$$

From (42) and (47), we see that R_F satisfies the super quantum Yang–Baxter equation

$$(R_F)_{12}.(R_F)_{13}.(R_F)_{23} = (R_F)_{23}.(R_F)_{13}.(R_F)_{12}. \tag{48}$$

3.2. Representation of the super quantum Yang–Baxter equation

Consider a graded space $W^{(n/m)}$ consisting of n bosons and m fermions. Let ρ be a representation of the Lie superalgebra g on $W^{(n/m)}$, then

$$R = (\rho \otimes \rho)(R_F) \quad \in (W^{(n/m)} \otimes W^{(n/m)})$$

satisfies the super quantum Yang–Baxter equation

$$R_{12}.R_{13}.R_{23} = R_{23}.R_{13}.R_{12}. \tag{49}$$

If we choose $\{w_i\}$ as a basis of $W^{(n/m)}$, where

$$\begin{aligned} \mid w_i \mid &= 0\text{for} \quad i = 1, 2, \ldots, n\,, \\ \mid w_i \mid &= 1\text{for} \quad i = n+1, n+2, \ldots, n+m \end{aligned}$$

then the equation (46)can be rewritten as:

$$\begin{aligned} &(-1)^{|m|(|c|+|n|)}(-1)^{|d|(|m|+|n|+|e|+|f|)} R^{ab}_{im}.R^{ic}_{dn}.R^{mn}_{ef} \\ =\ &(-1)^{|m|(|k|+|f|)}(-1)^{|a|(|b|+|c|+|m|+|k|)} R^{bc}_{mk}.R^{ak}_{lf}.R^{lm}_{de}\,, \end{aligned} \tag{50}$$

where $\mid i \mid = \mid w_i \mid$, and if we introduce the matrix $S = PR$, where P is the super permutation operator on the tensor vector space $W^{(n/m)} \otimes W^{(n/m)}$ with

$$P^{ij}_{kl} = (-1)^{|i||j|} \delta^i_l \delta^j_k$$

then S satisfies

$$\begin{aligned} &(-1)^{|a|(|b|+|c||j|+|n|)}(-1)^{|d|(|m|+|m|+|n|+|e|+|f|)} S^{bc}_{jm}.S^{aj}_{dm}.S^{mn}_{ef} \\ =\ &(-1)^{|l|(|j|+|c||m|+|f|)} S^{ab}_{lj}.S^{jc}_{mf}.S^{lm}_{de} \end{aligned} \tag{51}$$

which can be rewritten in a compact form as

$$(S \otimes id).(id \otimes S).(S \otimes id) = (id \otimes S).(S \otimes id).(id \otimes S)\,. \tag{52}$$

This gives rise to a representation of the symmetric group S_n.

3.3. Equivalents super star products on a supergroup

Let F and $\bar{F}$ be two super star products *i.e.*, two homogeneous elements of degree zero of the Hopf superalgebra $(U(g)[[h]]$ and let $A = U((g)[[h]], \Delta_F, R_F, S_F)$ and $\bar{A} = (U((g)[[h]], \Delta_{\bar{F}}, R_{\bar{F}}, S_{\bar{F}})$ be the resulting quantum supergroups, where

$$\begin{aligned} \Delta_F &= F.\Delta_0.F^{-1}\,, \quad R_F = F_{21}^{-1}.F \\ \Delta_{\bar{F}} &= \bar{F}.\Delta_0.\bar{F}^{-1}\,, \quad R_{\bar{F}} = \bar{F}_{21}^{-1}.\bar{F} \end{aligned}$$

then it is easily seen that $\bar{A}$ can be obtained from A by applying the twist $\hat{F} = F^{-1}.\bar{F}$. In fact

$$\Delta_{\bar{F}} = \hat{F}.\Delta_F.\hat{F}^{-1} \tag{53}$$

and

$$R_{\bar{F}} = \hat{F}_{21}.R_F.\hat{F}\,. \tag{54}$$

If the two star product are equivalent *i.e.* the corresponding elements F and $\bar{F}$ are related by the following expression

$$\bar{F} = \Delta_0(E^{-1}).F.(E \otimes E) \tag{55}$$

for some invertible homogeneous element E of degree zero of $U(g)[[h]]$, then the coproduct $\Delta_{\bar{F}}$ can be rewritten as

$$\Delta_{\bar{F}}(X) = (E^{-1} \otimes E^{-1})\Delta_F(E.X.E^{-1}).(E \otimes E)\,. \tag{56}$$

Similarly, the quasitriangular structures are related by

$$R_{\bar{F}} = (E^{-1} \otimes E^{-1}).R_F.(E \otimes E)\,. \tag{57}$$

And the two twisted antipodes are related by the following expression

$$S_{\bar{F}} = E^{-1}S_0(E^{-1}).S_F.S_0(E).E\,. \tag{58}$$

Then the inner automorphism of degree zero of the superalgebra structure $E(.)E^{-1}$ defines now a Hopf superalgebra isomorphism of degree zero. Finally, from (53) we see that the induced isomorphism of degree zero maps the quasitriangular structures into each other as well.

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