

THE LINEAR POTENTIAL AND HARMONIC OSCILLATOR IN RELATIVISTIC QUANTUM MECHANICS

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It is a nontrivial problem to formulate a Poincaré invariant quantum theory, that describes the binding of two particles in a confining potential. Four attempts at such theories are discussed and subsequently used to calculate the spectrum of two particles, which are bound in an harmonic oscillator potential or in a linear potential. These theories are described by the following equations

1. The so called “Relativistic Schrödinger equation”.
2. The Klein–Gordon equation.
3. The Dirac equation.
4. RQM (Relativistic Quantum Mechanics), the author’s private theory, which is of the “quasiparticle” type.

For each of these theories the Regge trajectories are calculated, both for the linear and for the harmonic potential. Since in RQM the interaction potential is the carrier, not only of energy, but also of momentum and hence of angular momentum, the Regge slopes differ from their usual values. Along the way it is shown how confining potentials can be handled in a theory which is formulated in the momentum representation, in spite of the fact that their Fourier transforms do not exist. For other quasiparticle theories the spectrum of the relativistic harmonic oscillator has not been calculated.

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1. Introduction

In nonrelativistic quantum mechanics it is possible to separate the motion of the centre of mass from the relative motion. For two particles one is then left with the Schrödinger equation for a single particle, in which the mass is replaced by the reduced mass.

For relativistic theories this separation presents a problem. The Relativistic Schrödinger equation, the Klein–Gordon equation and the Dirac equation are single particle equations from the start. Although they have a certain relativity flavour, they cannot be derived from a Poincaré invariant two-body equation. Some quasipotential theories do better in this respect. A particular one, using Dirac’s point form [1], was constructed by the author [2]. In the following it will be referred to as RQM for Relativistic Quantum Mechanics.

The main purpose of the present paper is to demonstrate that this RQM is well suited for calculating the spectrum of two bound particles. In particular the linear potential, but also the harmonic oscillator, will be treated in detail.

In [3] it was pointed out that the slopes of the linear Regge trajectories, as calculated by any of the three (static) theories, deviate from the values expected from QCD. The authors ascribe this discrepancy to the nonlocal character of the effective interaction, which they then try to explain by assigning an extra rotational degree of freedom to the gluon string between the constituent quarks.

In RQM, to be presented in Section 5, the potential, being defined in the momentum representation, is also nonlocal. As a result we will find Regge slopes which in some cases differ from the ones obtained with the static theories. Moreover, it will be shown that the deviations occur because of a modification of the centrifugal term in the potential. This suggests a relation with the effect found in [3].

In order to make this paper self contained, we first discuss the static theories in Sections 2, 3 and 4. The results are not new, but the methods to derive them may be of some interest. They are essential for understanding the calculations of Section 5.

In the remaining part of this first section we show how to derive nonsingular equations in the momentum representation. We also briefly recall the Bohr–Sommerfeld quantisation rule, which will be used extensively.

1.1. Harmonic potential

Since RQM is formulated in the momentum representation, it will be necessary first to show how we handle the singularities connected with the nonexistence of the Fourier transform of the potential. The method to be used will be demonstrated by applying it to the nonrelativistic Schrödinger equation

$$\left[-\frac{\hbar^2}{2m} \Delta + \widetilde{W}(r) \right] \psi(\vec{r}) = E \psi(\vec{r}). \quad (1)$$

To begin with we will consider the harmonic oscillator potential

$$\widetilde{W}(r) = \frac{1}{2}m\omega^2 r^2. \tag{2}$$

In terms of spherical waves

$$\psi(\vec{r}) = \frac{\chi_l(r)}{r} Y_{lm}(\theta, \varphi) \tag{3}$$

Eq. (1) becomes

$$-\frac{\hbar^2}{2m} \left[\chi_l''(r) - \frac{l(l+1)}{r^2} \chi_l(r) \right] + \widetilde{W}(r) \chi_l(r) = E \chi_l(r). \tag{4}$$

Introducing dimensionless variables

$$y = r \sqrt{\frac{m\omega}{\hbar}} \quad \text{and} \quad \varepsilon = \frac{2E}{\hbar\omega}, \tag{5}$$

this can be written as

$$\frac{d^2 \chi_l}{dy^2} - \frac{l(l+1)}{y^2} \chi_l - y^2 \chi_l = -\varepsilon \chi_l. \tag{6}$$

The well known spectrum is

$$\begin{aligned} \varepsilon_{nl} &= 4n + 2l - 1 & \text{or} & & E_{nl} &= (2n + l - \frac{1}{2}) \hbar\omega \\ \text{with } l &= 0, 1, \dots & \text{and} & & n &= 1, 2, \dots \end{aligned} \tag{7}$$

So far this is the standard treatment of the harmonic oscillator in the coordinate representation.

In momentum space it can most easily be described by replacing $-r^2/\hbar^2$ in Eq. (2) by the Laplace operator in momentum space Δ_p and considering $-\hbar^2\Delta$ in (1) as multiplication by p^2 . In this way one finds for the wave function

$$\phi(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\vec{k}\cdot\vec{r}} \psi(\vec{r}) d\vec{r} \tag{8}$$

the following equation (from now on units are such that $\hbar = 1$)

$$\frac{k^2}{2m} \phi(\vec{k}) - \frac{1}{2}m\omega^2 \Delta \phi(\vec{k}) = \frac{\kappa^2}{2m} \phi(\vec{k}), \quad \text{with} \quad E = \frac{\kappa^2}{2m}. \tag{9}$$

In terms of spherical waves

$$\phi(\vec{k}) = \frac{H_l(k)}{k} Y_{lm}(\theta, \varphi) \tag{10}$$

this reads

$$\frac{d^2 H_l}{dy^2} - \frac{l(l+1)}{y^2} H_l - y^2 H_l = -\varepsilon H_l, \tag{11}$$

with the new dimensionless variables

$$y = \frac{k}{\sqrt{m\omega}} \quad \text{and} \quad \varepsilon = \frac{\kappa^2}{m\omega}. \tag{12}$$

Eq. (11) is the same as Eq. (6) and the spectrum is again given by Eq. (7).

1.2. General confining potential

For a more general potential the Schrödinger equation in momentum space takes the form of an integral equation

$$(k^2 - \kappa^2) \phi(\vec{k}) + 2m \int W(\vec{k}' - \vec{k}) \phi(\vec{k}') d\vec{k}' = 0, \tag{13}$$

with

$$W(\vec{q}) = \frac{1}{(2\pi)^3} \int e^{-i\vec{q}\cdot\vec{r}} \widetilde{W}(r) d\vec{r}. \tag{14}$$

Since the Fourier transform of the harmonic potential (2) does not exist, we first replace it by

$$\widetilde{W}_R(r) = m\omega^2 R^2 \left[1 - e^{-\frac{r^2}{2R^2}} \right], \tag{15}$$

and only at the end of the calculation the limit $R \rightarrow \infty$ will be taken. The Fourier transform (14) of this potential is equal to

$$W_R(\vec{q}) = W_R^{(1)}(\vec{q}) + W_R^{(2)}(\vec{q}), \tag{16}$$

with

$$W_R^{(1)}(\vec{q}) = m\omega^2 R^2 \delta(\vec{q}) \quad \text{and} \quad W_R^{(2)}(\vec{q}) = -\frac{m\omega^2 R^5}{(2\pi)^{3/2}} e^{-\frac{1}{2}q^2 R^2}. \tag{17}$$

Substitution into Eq. (13) gives

$$\left[\frac{k^2}{2m} + m\omega^2 R^2 \right] \phi(\vec{k}) - \frac{m\omega^2 R^5}{(2\pi)^{3/2}} \int e^{-\frac{1}{2}|\vec{k}'-\vec{k}|^2 R^2} \phi(\vec{k}') d\vec{k}' = \frac{\kappa^2}{2m} \phi(\vec{k}). \tag{18}$$

For large values of R the main contribution to the integral comes from values of \vec{k}' close to \vec{k} . It can therefore be evaluated by expanding the wave function as

$$\phi(\vec{k}') = \phi(\vec{k}) + \sum_i (k'_i - k_i) \frac{\partial \phi}{\partial k_i} + \frac{1}{2} \sum_{i,j} (k'_i - k_i)(k'_j - k_j) \frac{\partial^2 \phi}{\partial k_i \partial k_j} + \dots \tag{19}$$

The remaining integrations can easily be performed, leading to

$$\left[\frac{k^2}{2m} + m\omega^2 R^2 \right] \phi(\vec{k}) - m\omega^2 R^5 \left[\frac{1}{R^3} \phi(\vec{k}) + \frac{1}{2R^5} \Delta \phi(\vec{k}) + \mathcal{O} \left(\frac{1}{R^7} \right) \right] = \frac{\kappa^2}{2m} \phi(\vec{k}). \tag{20}$$

The terms of $\mathcal{O}(R^2)$ cancel, so that the final equation is again the same as Eq. (9).

For the harmonic oscillator we therefore have succeeded to write the eigenvalue problem (13) in the form of a nonsingular differential equation.

1.3. Linear potential

For other confining potentials Eq. (13) will in general not take the form of a differential equation, so that it must still be shown how to remove the singularity.

For that purpose we take a linear potential $\widetilde{W}(r) = \sigma r$ as an example. This potential is considered as the limit of

$$\widetilde{W}_\mu(r) = \frac{\sigma}{\mu} (1 - e^{-\mu r}) \quad \text{for } \mu \rightarrow 0. \tag{21}$$

Its Fourier transform can again be calculated explicitly and is equal to

$$W_\mu(\vec{q}) = \frac{\sigma}{\mu} \delta(\vec{q}) - \frac{\sigma}{\pi^2} \frac{1}{(q^2 + \mu^2)^2}. \tag{22}$$

The integral equation (13) then takes the form

$$(k^2 - \kappa^2 + \frac{2m\sigma}{\mu}) \phi(\vec{k}) = \frac{2m\sigma}{\pi^2} J_\mu(\vec{k}), \tag{23}$$

in which $J_\mu(\vec{k}) = J_\mu^{\text{reg}}(\vec{k}) + J_\mu^{\text{sin}}(\vec{k})$ is the sum of a regular and a singular integral, defined by

$$J_\mu^{\text{reg}}(\vec{k}) = \int \frac{\phi(\vec{k}') - \phi(\vec{k})}{\left[|\vec{k}' - \vec{k}|^2 + \mu^2 \right]^2} d\vec{k}' \tag{24}$$

and

$$J_\mu^{\text{sin}}(\vec{k}) = \phi(\vec{k}) \int \frac{d\vec{k}'}{\left[|\vec{k}' - \vec{k}|^2 + \mu^2 \right]^2} = \frac{\pi^2}{\mu} \phi(\vec{k}). \tag{25}$$

This latter integral, which approaches infinity when $\mu \rightarrow 0$, is canceled against the term $\frac{2m\sigma}{\mu} \phi(\vec{k})$ in the left-hand side of Eq. (23). In order to show the existence of the limit of $J_{\mu}^{\text{reg}}(\vec{k})$ when $\mu \rightarrow 0$, the contribution of all \vec{k}' for which the length of the vector $\vec{q} = \vec{k}' - \vec{k}$ is less than μ is considered separately. In this region the function $\phi(\vec{k}') - \phi(\vec{k})$ can be approximated by a power series in the components of \vec{q} . After integrating over all directions of \vec{q} one is left with

$$2\pi \Delta\phi(\vec{k}) \int_0^{\mu} \frac{q^4}{(q^2 + \mu^2)^2} dq = 2\pi \Delta\phi(\vec{k}) \mu \int_0^1 \frac{x^4}{(1 + x^2)^2} dx, \tag{26}$$

which tends to zero when $\mu \rightarrow 0$.

Finally Eq. (23) takes the form

$$(k^2 - \kappa^2) \phi(\vec{k}) = \frac{2m\sigma}{\pi^2} \int^* \frac{\phi(\vec{k}') - \phi(\vec{k})}{|\vec{k}' - \vec{k}|^4} d\vec{k}'. \tag{27}$$

Because of the spherical \vec{k}' integration around \vec{k} , indicated by \int^* , this integral equation is nonsingular, which we set out to prove.

For a linear and harmonic potential we have now written the Schrödinger equation as a nonsingular integral equation in momentum space. For arbitrary confining potentials, however, this reduction must still be handled on an individual basis.

For the numerical calculation of the spectrum of the Schrödinger equation for a linear potential, an expansion in partial waves is more convenient. With the same subtraction method as used above, I find for the eigenvalue equation of a spherical wave amplitude

$$\left(k^2 - \kappa^2 + \frac{2m\sigma}{\pi k}\right) \phi_l(k) + \frac{2m\sigma}{\pi k^2} \int_0^{\infty} \left[Q_l'(z) \phi_l(k') + \frac{k^2 \phi_l(k)}{(k' - k)^2} \right]_P dk' = 0, \tag{28}$$

in which $Q_l(z)$ is a Legendre function of the second kind with argument $z = \frac{1}{2}(k'/k + k/k')$. The integral is not singular in the point $k' = k$. By using a symmetric integration, indicated by P , the numerical integration is speeded up, however. The same equation (28) has been found by a number of authors, among them Maung *et al.* [4] and Hersbach [5].

Eq. (28) is not very useful for calculating the spectrum for large angular momenta, but will be important for the connection with the relativistic theory.

In order to establish the asymptotic behaviour of the spectrum for the present nonrelativistic case it is better to start from the coordinate representation and use the Bohr–Sommerfeld quantisation rule. This method will also be useful when considering other theories.

1.4. The Bohr–Sommerfeld method

By taking $\widetilde{W}(r) = \sigma r$ in Eq. (4), the eigenvalue equation for the spherical amplitudes becomes

$$-\frac{d^2 \chi_l}{d\eta^2} + \left(\eta + \frac{l(l+1)}{\eta^2} \right) \chi_l = \eta_l \chi_l, \tag{29}$$

in which we have introduced the following abbreviations

$$\eta = (2m\sigma)^{1/3} r \quad \text{and} \quad E = m \left(\frac{\sigma^2}{2m^4} \right)^{1/3} \eta_l. \tag{30}$$

For $l = 0$ the exact solution of Eq. (29) is given by the Airy function $\text{Ai}(\eta - \eta_0)$ and the possible values of η_0 are the zeros of $\text{Ai}(-\eta_0)$. The ground state has

$$\eta_0 = 2.338 \dots \tag{31}$$

However, for arbitrary values of l it is not possible to give a closed form for the eigenvalues η_{nl} .

In the Bohr–Sommerfeld approximation the values of η_l should be found by solving the equation

$$\int_{\eta_-}^{\eta_+} \sqrt{\eta_l - \eta - \frac{l(l+1)}{\eta^2}} d\eta = \left(n + \frac{1}{2} \right) \pi. \tag{32}$$

The classical turning points are denoted by η_- and η_+ . With

$$\eta = \eta_l x \quad \text{and} \quad \lambda = \frac{l(l+1)}{\eta_l^3} \quad \text{and} \quad \eta_{\pm} = \eta_l x_{\pm} \tag{33}$$

this becomes

$$\int_{x_-}^{x_+} \sqrt{1 - x - \frac{\lambda}{x^2}} dx = \frac{(n + \frac{1}{2})}{\eta_l^{3/2}} \pi. \tag{34}$$

The left-hand side is an elliptic integral, which for each value of λ can be calculated numerically. However, in order to get an analytic expression in λ , it is more convenient to replace it by an approximation, which in the limit $l \rightarrow \infty$, becomes exact.

Define x_c and λ_c as the values of x and λ for which the function $x^2 - x^3$ is tangent to the line $\lambda = cst$. These values are easily found to be

$$x_c = \frac{2}{3} \quad \text{and} \quad \lambda_c = \frac{4}{27}. \quad (35)$$

We now expand around this point

$$x = x_c + y \quad \text{and} \quad \lambda = \lambda_c - \mu, \quad (36)$$

and assume that in the whole integration region $|y| \ll 1$. For large l this assumption will turn out to be correct. The integral in Eq. (34) now becomes elementary. The remaining algebra is trivial and eventually leads to the following expression for η_l

$$\eta_l = \frac{3}{2^{2/3}} l^{2/3} \left[1 + \frac{2}{\sqrt{3}} \frac{n}{l} + \dots \right] \quad \text{for large } l \text{ and fixed } n. \quad (37)$$

From Eq. (30) we see that the ensuing Regge trajectories are not straight lines.

$$E_{nl}^2 = \frac{9}{4} \left(\frac{\sigma^2}{2m} \right)^{2/3} l^{4/3} \left[1 + \frac{4}{\sqrt{3}} \frac{n}{l} + \dots \right]. \quad (38)$$

Moreover, for each l the distance between any two neighbouring levels has the same value, which increases with l like $l^{1/3}$.

2. The Relativistic Schrödinger equation

Formally the Relativistic Schrödinger equation is defined by the two-particle Hamiltonian

$$H = \sqrt{\vec{p}_1^2 + m_1^2} - m_1 + \sqrt{\vec{p}_2^2 + m_2^2} - m_2 + \widetilde{W}(|\vec{r}_1 - \vec{r}_2|). \quad (39)$$

Since for this equation the separation of the centre of mass motion is impossible, one only considers the case of vanishing total momentum, *i.e.*, $\vec{p}_1 = -\vec{p}_2 = \vec{p}$. It is not known, however, how to construct the boost operator for this two-particle system. Thus one cannot transform to arbitrary momentum and the Relativistic Schrödinger equation cannot be considered to be a proper Lorentz invariant theory.

Ignoring this fact, one can only hope that in some sense it will be a good approximation of a correct theory. In particular one can study the case in

which the mass of one of the particles approaches infinity. Then Eq. (39) simplifies and the Relativistic Schrödinger equation takes the form

$$\left[\sqrt{\vec{p}^2 + m^2} - m + \widetilde{W}(r) \right] \psi(\vec{r}) = E \psi(\vec{r}), \tag{40}$$

where m is the mass of the other particle and \vec{r} is the relative distance.

2.1. Harmonic potential

For the harmonic oscillator $\widetilde{W}(r) = \frac{1}{2}m\omega^2 r^2$ it can again be written in the momentum representation by writing r^2 as $-\Delta$. In this way Eq. (40) becomes

$$\left[\sqrt{k^2 + m^2} - m - E \right] \chi(\vec{k}) = \frac{1}{2}m\omega^2 \Delta \chi(\vec{k}). \tag{41}$$

For spherical waves

$$\chi(\vec{k}) = \frac{H_l(k)}{k} Y_{lm}(\theta, \varphi) \tag{42}$$

the eigenvalue equation is

$$\left[\sqrt{k^2 + m^2} - m - E \right] H_l(k) = \frac{1}{2}m\omega^2 \left[\frac{d^2 H_l(k)}{dk^2} - \frac{l(l+1)}{k^2} H_l(k) \right]. \tag{43}$$

A simple numerical calculation using the Numerov method gives the mass spectrum $M_{nl} = m + E_{nl}$. For $n = 1, 2, 3$ and for $g \equiv \hbar\omega/mc^2 = 1$, I have plotted the square of this mass versus l . These three lowest Regge trajectories (figure 1) show a slight upward curvature.

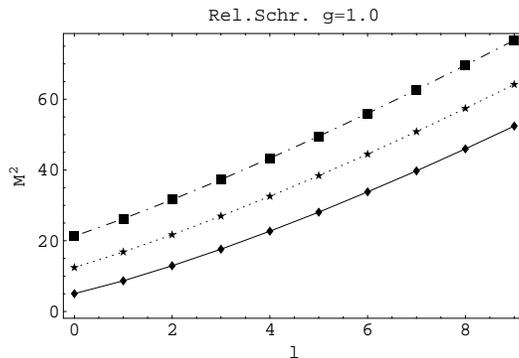


Fig. 1. Regge trajectories for Relativistic Schrödinger equation with harmonic potential.

For fixed n and $l \rightarrow \infty$, the asymptotic behaviour of this mass spectrum can be obtained by using the Bohr–Sommerfeld quantisation rule. The result is

$$M_{nl}^2 \simeq \frac{9}{4} (\hbar\omega)^{4/3} (mc^2)^{2/3} l^{4/3} \left[1 + \frac{4}{\sqrt{3}} \frac{n}{l} + \mathcal{O}(l^{-2}) \right] \quad \text{for large } l \text{ and fixed } n. \quad (44)$$

This confirms the upward bending and the slow separation of the Regge trajectories as seen in figure 1.

In the high energy limit $k \gg m$ Eq. (43) has the same form as the nonrelativistic Schrödinger equation in the coordinate representation for a linear potential $\widetilde{W}(r) = \sigma r$. This explains why the Regge trajectories for the two cases have the same l and n dependence, as is seen by comparing the equations (38) and (44).

2.2. Linear potential

The Relativistic Schrödinger equation with a linear potential can again be put in the form of Eq. (27), but with $k^2/2m$ in the first term replaced with $\varepsilon(k) = \sqrt{k^2 + m^2}$ and $\kappa^2/2m$ with ε_n . So

$$[\varepsilon(k) - \varepsilon_n] \phi(\vec{k}) = \frac{\sigma}{\pi^2} \int^* \frac{\phi(\vec{k}') - \phi(\vec{k})}{|\vec{k}' - \vec{k}|^4} d\vec{k}'. \quad (45)$$

However, in order to find the asymptotic behaviour of M_{nl}^2 , the simplest method is again to apply the Bohr–Sommerfeld method to Eq. (40), but now with $\widetilde{W}(r) = \sigma r$. This amounts to calculating the integral for the action variable associated with the radial variable

$$J_r = \frac{1}{\pi} \int_{r_-}^{r_+} \sqrt{(\varepsilon_n - \sigma r)^2 - m^2 - \frac{l^2}{r^2}} dr \quad (46)$$

between the classical turning points r_- and r_+ . The energy spectrum ε_{nl} is given by requiring that

$$J_r = n + \frac{\alpha}{4}. \quad (47)$$

In this case we have added the customary term $\frac{\alpha}{4}$, in order to allow a comparison with numerical calculations for small values of l and n . This Maslov index α should be chosen so as to guarantee the correct behaviour of the phase of the wave function in the turning points. For the linear potential $\alpha = -1$. See Ref. [6].

For high energies the integral (46) can be written as

$$J_r = \frac{\varepsilon_n^2}{\pi\sigma} \int_{x_-}^{x_+} \sqrt{(1-x)^2 - \frac{\lambda^2}{x^2}} dx, \tag{48}$$

with

$$r = \frac{\varepsilon_n}{\sigma} x \quad \text{and} \quad \lambda = \frac{\sigma l}{\varepsilon_n^2} \quad \text{and} \quad x_{\pm} = \frac{1}{2}(1 \pm \sqrt{1-4\lambda}). \tag{49}$$

A numerical calculation shows that the integral in Eq. (48), plotted in Fig. 2 can, to a very good approximation, be represented by a linear function of λ .

Therefore

$$J_r \simeq \frac{2\varepsilon_n^2}{\pi\sigma} \left(\frac{1}{4} - \lambda \right). \tag{50}$$

With this expression one eventually derives from Eq. (47) that

$$\varepsilon_{nl}^2 \simeq 4\sigma l + 2\pi\sigma n - \frac{\pi}{2}\sigma. \tag{51}$$

This formula is in very good agreement with Eq. (2.8) of [7], which was derived by interpolation of the exact values for $l = 0 - 3$ and $n = 1 - 5$. It shows that, for a linear potential, linear equidistant Regge trajectories can be obtained from the Relativistic Schrödinger equation, which was not possible with the nonrelativistic theory.

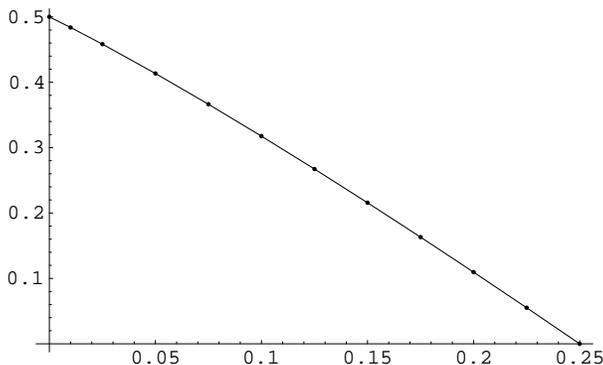


Fig. 2. The integral in Eq. (48) as a function of λ

At this point I want to make a remark about the coefficient $2\pi\sigma$ in front of n in equation (51). For high energies, *i.e.*, for large values of l , this coefficient is incorrect.

This can be seen by writing the integral in Eq. (48) in the form

$$I(\mu) = \int_{-\mu}^{\mu} \frac{\sqrt{(\frac{1}{2} - \mu^2 - y^2)(\mu^2 - y^2)}}{\frac{1}{2} + y^2} dy, \quad (52)$$

where

$$y = x - \frac{1}{2} \quad \text{and} \quad \mu^2 = \frac{1}{4} - \lambda. \quad (53)$$

Using Eq. (50), it follows from the fact that J_r is finite (Eq. (47)), that λ is very close to $\frac{1}{4}$, and that therefore $\mu \ll 1$. In this limit the integral becomes

$$I(\mu) \simeq \sqrt{2} \int_{-\mu}^{\mu} \sqrt{\mu^2 - y^2} \sqrt{2} \left(\frac{1}{4} - \lambda \right). \quad (54)$$

This means that instead of Eq. (51), one obtains for the correct asymptotic behaviour of the Regge trajectories

$$\varepsilon_{nl}^2 \simeq 4\sigma l + 4\sqrt{2}\sigma \left(n - \frac{1}{4} \right). \quad (55)$$

An exact calculation of low lying states may therefore be misleading, when used as an indication of the n -dependence of the energies of states with high angular momentum.

3. The Klein-Gordon equation

A relativistic spinless particle, coupled to another particle with infinite mass, is usually described by the Klein-Gordon equation

$$[-\Delta + m^2 + U^2(r)] \psi(\vec{r}) = (E - \widetilde{W}(r))^2 \psi(\vec{r}), \quad (56)$$

in which $U(r)$ and $\widetilde{W}(r)$ are the potentials for respectively the scalar- and vector-coupling. The physical relevance of this equation is in doubt, however, because even when both $U(r)$ and $\widetilde{W}(r)$ are confining potentials, Eq. (56) may not have stable solutions.

The first difficulties with the Klein-Gordon equation are encountered when, as in the nonrelativistic Schrödinger equation, one tries to prove the orthogonality of the eigenfunctions and the reality of the eigenvalues. Starting from Eq. (56) one easily proves that for two eigenfunctions $\psi_n(\vec{r})$ and $\psi_m(\vec{r})$ and their corresponding eigenvalues the following relation is valid

$$(E_m^2 - E_n^2) \int \psi_n^*(\vec{r}) \psi_m(\vec{r}) d\vec{r} = 2(E_m - E_n) \int \psi_n^*(\vec{r}) \widetilde{W}(r) \psi_m(\vec{r}) d\vec{r}. \quad (57)$$

Taking $m = n$ this shows that the eigenvalues are not necessarily real. Moreover, eigenfunctions belonging to different eigenvalues are not always orthogonal.

The problems with the possible instability of bound states are illustrated by taking a linear potential both for the scalar and for the vector couplings

$$U(r) = \sigma_1 r \quad \text{and} \quad \widetilde{W}(r) = \sigma_2 r. \tag{58}$$

For spherical waves the Klein–Gordon equation then becomes

$$\frac{d^2 \chi_l(y)}{dy^2} = \left[\frac{l(l+1)}{y^2} + 1 + g_1^2 y^2 - (\varepsilon - g_2 y)^2 \right] \chi_l(y), \tag{59}$$

with

$$y = mr, \quad g_1 = \frac{\sigma_1}{m^2}, \quad g_2 = \frac{\sigma_2}{m^2} \quad \text{and} \quad E = m\varepsilon. \tag{60}$$

In the limit $y \rightarrow \infty$ Eq. (59) becomes

$$\frac{d^2 \chi_l(y)}{dy^2} = (g_1^2 - g_2^2) y^2 \chi_l(y) \quad \text{for large } y. \tag{61}$$

From this equation we see immediately that bound states can exist only when $g_2^2 < g_1^2$. When the vector coupling is stronger than the scalar coupling the solution becomes oscillatory and there will be no stable bound states.

For a number of other theories this effect is discussed in [8]. For the pure vector potential, *i.e.*, when $U(r) \equiv 0$, the connection with the Klein paradox was explained by Fulling [9], who found complex eigenvalues for a particle in a square well potential.

We therefore restrict ourselves to vanishing vector potentials, in which case problems of this kind do not occur.

3.1. Linear potential

For a linear potential $U(r) = \sigma r$ Eq. (56) reduces to the nonrelativistic Schrödinger equation with an harmonic interaction

$$\left[\frac{p^2}{2m} + \frac{\sigma^2}{2m} r^2 \right] \psi(\vec{r}) = \frac{E^2 - m^2}{2m} \psi(\vec{r}). \tag{62}$$

From the known eigenvalues one then obtains

$$E_{nl}^2 = m^2 + 2\sigma \left(l + 2n - \frac{1}{2} \right), \tag{63}$$

which again describes linear equidistant Regge trajectories.

3.2. Harmonic potential

For the harmonic interaction $U(r) = \frac{1}{2}m\omega^2 r^2$ the Klein–Gordon equation takes the form of the Schrödinger equation

$$\left[\frac{p^2}{2m} + V(r) \right] \psi(\vec{r}) = \mathcal{E} \psi(\vec{r}), \quad (64)$$

for an anharmonic oscillator with

$$V(r) = \frac{1}{8}m\omega^4 r^4 \quad \text{and} \quad \mathcal{E} = \frac{E^2 - m^2}{2m}. \quad (65)$$

Applying again the Bohr–Sommerfeld quantisation rule for states with high angular momentum, the action integral

$$J_r = \frac{1}{\pi} \int_{r_-}^{r_+} \sqrt{2m(\mathcal{E} - V(r)) - m^2 - \frac{l^2}{r^2}} dr \quad (66)$$

can be calculated and is found to be equal to

$$J_r = \frac{3^{3/4}}{2^{11/4}} \frac{m}{\omega \varepsilon_0^{1/4}} \left[\frac{8}{3\varepsilon_0} (\varepsilon - \varepsilon_0)^2 + \frac{32}{3} (\varepsilon - \varepsilon_0) \right], \quad (67)$$

with

$$\varepsilon_0 = \frac{3}{2^{7/3}} \left(\frac{\omega l}{m} \right)^{4/3} \quad \text{and} \quad \varepsilon = \frac{\mathcal{E}}{m}. \quad (68)$$

The spectrum is then determined by putting $J_r = n$. This finally gives

$$M_{nl}^2 \simeq \frac{3}{2^{4/3}} (\hbar\omega)^{4/3} (mc^2)^{2/3} l^{4/3} \left[1 + \sqrt{\frac{2}{3}} \frac{n}{l} + \mathcal{O}(l^{-2}) \right] \quad (69)$$

for the squared masses of an harmonic oscillator as described by the Klein–Gordon equation. Apart from the numerical constants, this is the same as Eq. (44), obtained when using the Relativistic Schrödinger equation. For the latter the Regge trajectories are steeper and more widely separated.

4. The Dirac equation

For the Dirac equation the orthogonality of eigenfunctions and the reality of the eigenvalues can be proved in the standard way, so that the objections against the Klein–Gordon equation about this point do not exist.

However, since the work of Plesset [10], it is known that bound states of the Dirac equation with a confining potential and pure vector coupling,

cannot exist. We therefore consider only scalar couplings and take the linear and harmonic potentials as examples

$$U(r) = \sigma r \quad \text{or} \quad U(r) = \frac{1}{2}m\omega^2 r^2. \tag{70}$$

4.1. Reduction of the Dirac equation

In a standard way the two-component wave equation for spherical waves can be derived

$$\begin{aligned} \frac{dF(y)}{dy} &= -\frac{k}{y}F(y) + (W(y) + \varepsilon)G(y), \\ \frac{dG(y)}{dy} &= \frac{k}{y}G(y) + (W(y) - \varepsilon)F(y). \end{aligned} \tag{71}$$

Here the following abbreviations have been introduced

$$\begin{aligned} W(y) &= 1 + gy \quad \text{or} \quad W = 1 + \frac{1}{2}g^2y^2, & y &= mr \\ g &= \frac{\sigma}{m^2} \quad \text{or} \quad g = \frac{\omega}{m}, & \varepsilon &= \frac{E}{m} \end{aligned} \tag{72}$$

and

$$k = j + \frac{1}{2} \quad \text{if} \quad l = j + \frac{1}{2} \quad \text{and} \quad k = -(j + \frac{1}{2}) \quad \text{if} \quad l = j - \frac{1}{2}. \tag{73}$$

Defining $P(y)$ and $Q(y)$ by

$$F(y) = y^{-k}P(y) \quad \text{and} \quad G(y) = y^kQ(y), \tag{74}$$

Eq. (71) becomes

$$\begin{aligned} \frac{dP(y)}{dy} &= (W + \varepsilon)y^{2k}Q(y), \\ \frac{dQ(y)}{dy} &= (W - \varepsilon)y^{-2k}P(y). \end{aligned} \tag{75}$$

The function $Q(y)$ can be eliminated and this then leads to

$$\frac{d^2P(y)}{dy^2} = (W^2 - \varepsilon^2)P(y) + 2T(y)\frac{dP(y)}{dy}, \tag{76}$$

in which

$$T(y) = \frac{1}{2(W + \varepsilon)}\frac{dW}{dy} + \frac{k}{y}. \tag{77}$$

The first derivative in Eq. (76) can be eliminated by writing $P(y) = S(y)R(y)$, in which $S(y)$ is a solution of

$$\frac{dS}{dy} = T(y) S(y). \quad (78)$$

This then leads to a nonrelativistic Schrödinger equation

$$-\frac{d^2 R(y)}{dy^2} + V(y) R(y) = \varepsilon^2 R(y) \quad (y \geq 0), \quad (79)$$

with the potential

$$V(y) = W^2(y) + T^2(y) - \frac{dT(y)}{dy}. \quad (80)$$

In the Bohr–Sommerfeld approximation the spectrum can be calculated from

$$\int_{y_-}^{y_+} \sqrt{\varepsilon^2 - V(y)} dy = \left(n + \frac{1}{2}\right) \pi, \quad (81)$$

in which y_- and y_+ are the classical turning points, satisfying $\varepsilon^2 = V(y_{\pm})$.

4.2. Linear potential

For high energies $\varepsilon \gg m$ the linear potential in Eq. (72) may be replaced with $W(y) = gy$. The function $T(y)$ then becomes

$$T(y) = \frac{g}{2(\varepsilon + gy)} + \frac{k}{y} \quad (82)$$

and from Eq. (80) we get for the potential

$$V(y) = g^2 y^2 + \frac{1}{(\varepsilon + gy)^2} \left[g^2 \left(k + \frac{3}{4} \right) + gk \frac{\varepsilon}{y} \right] + \frac{k(k+1)}{y^2}. \quad (83)$$

If μ and x are defined by

$$\varepsilon^2 = g\mu \quad \text{and} \quad y = \sqrt{\frac{\mu}{g}} x, \quad (84)$$

Eq. (81) can be written as

$$\int_{x_-}^{x_+} \sqrt{\mu^2(1-x^2) - \left[\frac{k + \frac{3}{4} + \frac{k}{x}}{(1+x)^2} + \frac{k(k+1)}{x^2} \right]} dx = \left(n + \frac{1}{2}\right) \pi. \quad (85)$$

This is the equation from which $\mu(n, k)$ must be solved.

For $|k| \gg 1$ we can neglect all terms between the square brackets, except the term which is $\mathcal{O}(k^2)$. In this case Eq. (85) becomes

$$\int_{x_-}^{x_+} \sqrt{\mu^2(1-x^2) - \frac{k^2}{x^2}} dx = \left(n + \frac{1}{2}\right) \pi. \tag{86}$$

The integral can be evaluated in closed form (see Gradshteyn and Ryzhik 2.267) and gives

$$\frac{\mu}{2} \left(\frac{1}{2} - \frac{|k|}{\mu}\right) \pi = \left(n + \frac{1}{2}\right) \pi, \tag{87}$$

so that

$$\mu(n, k) \simeq 4n + 2 + 2|k| \quad \text{for} \quad |k| \gg 1. \tag{88}$$

This finally results in

$$M^2 \simeq m^2 \varepsilon^2 = m^2 g \mu = \sigma \mu = 2\sigma (2n + 1 + |k|) = 2\sigma \left(2n + j + \frac{3}{2}\right), \tag{89}$$

in agreement with [3] and with the numerical study made by Critchfield [11].

Eq. (89) should be compared with Eq. (63), which gives linear Regge trajectories for a mass- and spinless particle, described by the Klein–Gordon equation with a linear scalar coupling.

4.3. Harmonic potential

In spite of the fact that in the case of pure scalar coupling the eigenvalues can be shown to be real, Ram and Halasa [12] found low eigenvalues with a complex component for the Dirac equation with an harmonic potential and scalar coupling. A new calculation of the spectrum showed that they missed the first three states and that all energies are indeed real.

For high energies $\varepsilon \gg m$ the harmonic potential in Eq. (72) may be replaced with $W(y) = \frac{1}{2}g^2y^2$.

The function $T(y)$ then becomes

$$T(y) = \frac{g^2y}{2\varepsilon + g^2y^2} + \frac{k}{y} \tag{90}$$

and from Eq. (80) we get for the potential

$$V(y) = \frac{1}{4}g^4y^4 + \frac{3g^4y^2}{(2\varepsilon + g^2y^2)^2} + \frac{g^2(2k-1)}{2\varepsilon + g^2y^2} + \frac{k(k+1)}{y^2}. \tag{91}$$

Defining μ and the new variable x by

$$\varepsilon = \left(\frac{g^2}{2}\right)^{\frac{1}{3}} \mu \quad \text{and} \quad y = \left(\frac{4\mu^3}{g^4}\right)^{\frac{1}{6}} x, \quad (92)$$

the Bohr–Sommerfeld equation (81) takes the form

$$\int_{x_-}^{x_+} \sqrt{\mu^3(1-x^4) - \left[\frac{3x^2}{(1+x^2)^2} + \frac{2k-1}{1+x^2} + \frac{k(k+1)}{x^2} \right]} dx = \left(n + \frac{1}{2}\right) \pi. \quad (93)$$

For high energies, *i.e.*, for $|k| \gg 1$, this equation becomes

$$\int_{x_-}^{x_+} \sqrt{\mu^3(1-x^4) - \frac{k^2}{x^2}} dx = \left(n + \frac{1}{2}\right) \pi. \quad (94)$$

Using the fact that $|k|$ is large, the integral can be calculated and $\mu(n, k)$ can be solved. Eventually the following expression is obtained for the square of the masses:

$$M^2 = \frac{3}{2^{4/3}} m^{2/3} \omega^{4/3} |k|^{4/3} \left[1 + 2\sqrt{\frac{2}{3}} \frac{(n + \frac{1}{2})}{|k|} + \dots \right]. \quad (95)$$

This equation shows that, as for the Klein–Gordon equation and for the Relativistic Schrödinger equation, also for the Dirac equation the Regge trajectories are slowly separating, and not straight, but bending upwards.

5. RQM

5.1. Recapitulation

Since all details of RQM were given in [2], a brief resumé should suffice here.

The problem is to construct a nontrivial representation of the generators P_μ and $J_{\mu\nu}$ of the Poincaré group, which means that they should include the interparticle interaction and satisfy the commutation relations

$$\begin{aligned} [P_\mu, P_\nu] &= 0, & [P_\mu, J_{\nu\lambda}] &= i(g_{\mu\nu}P_\lambda - g_{\mu\lambda}P_\nu), \\ [J_{\mu\nu}, J_{\lambda\sigma}] &= i(g_{\mu\lambda}J_{\sigma\nu} - g_{\mu\sigma}J_{\lambda\nu} + g_{\nu\lambda}J_{\mu\sigma} - g_{\nu\sigma}J_{\mu\lambda}). \end{aligned} \quad (96)$$

Moreover, in the case of two particles, these P_μ and $J_{\mu\nu}$ should be operators acting in the Hilbert space spanned by a complete set of free 2-particle states.

For a review of the attempts made to solve this problem, I want to refer to the paper by Keister and Polyzou [13].

In my opinion, however, all of these attempts, that insisted on the construction of the dynamical generators with a local interaction, lead to insuperable problems. To quote Wigner [14]:

Finally, we had to recognise, every attempt to provide a precise definition of a position coordinate stands in direct contradiction with special relativity.

Another problem previous authors had to face, arose from the fact that they chose either the instant form or the light front form, as defined by Dirac [1]. In both cases the interaction between particles changed at least one of the $J_{\mu\nu}$ operators. This leads to very complicated equations, which can only be handled in an approximate way.

In RQM both problems are circumvented. The Hilbert space is spanned by the products of states of a single particle with a given momentum. The interaction, which is then given in momentum space, will turn out not to be strictly local. It was shown, however, that this nonlocality is not measurable [2, 15].

The second problem was avoided by using Dirac's point form [1]. In this case the Lorentz generators $J_{\mu\nu}$ are unaffected by the interaction, and only P_0 and \vec{P} acquire extra terms. This means that the mediating potential between the particles carries not only energy, as in the nonrelativistic theory, but also momentum.

In order to construct the operators P_μ , we first consider the total four-velocity

$$u = \frac{1}{\sqrt{1 - |\vec{v}|^2}}(1, \vec{v}), \quad (97)$$

where for two particles the total 3-velocity is given by

$$\vec{v} = \frac{\vec{p}_1 + \vec{p}_2}{p_1^0 + p_2^0} \quad \text{with} \quad p_1^0 = \sqrt{|\vec{p}_1|^2 + m_1^2} \quad \text{and} \quad p_2^0 = \sqrt{|\vec{p}_2|^2 + m_2^2}. \quad (98)$$

The main assumptions of RQM now are

1. The operators P_μ have nonvanishing matrix elements only between states with the same 4-velocity.
2. The kinetic part of P_μ and the interaction part of P_μ each transform like a 4-vector under Lorentz transformations.

From this follows that P_μ must be of the form

$$\langle \alpha | P^\mu | \beta \rangle = [K_{\alpha\beta} + W_{\alpha\beta} L^{(3)}(\vec{v}_\alpha, \vec{v}_\beta)] u^\mu \quad \text{with} \quad \vec{v}_\alpha = \vec{v}_\beta = \vec{v}, \quad (99)$$

in which $K_{\alpha\beta}$ is given by the momentum of the free particles and

$$L^{(3)}(\vec{v}_\alpha, \vec{v}_\beta) = (1 - |\vec{v}|^2)^2 \delta_3(\vec{v}_\alpha - \vec{v}_\beta) \quad (100)$$

is the Lorentz invariant propagator, which ensures the conservation of the total 4-velocity. In [2] a detailed description is given of how to construct the potential $W_{\alpha\beta}$, starting from the desired nonrelativistic limit.

The generators P_μ and $J_{\mu\nu}$ constructed in this way can then easily be shown to satisfy the commutation relations (96).

5.2. The equation and the potential

On several occasions [2, 15] it was shown that the RQM-equation from which the bound states and their masses should be calculated, is

$$\left(\sqrt{P_\alpha^2} - M_n \right) \psi_\alpha^n + \int_\beta^* \widetilde{W}_{\alpha\beta} L^{(3)}(\vec{v}_\alpha, \vec{v}_\beta) \psi_\beta^n = 0 \quad \text{for} \quad \vec{v}_\alpha = \vec{v}. \quad (101)$$

Since for two particles the integration element is given by

$$\int_\beta^* \dots = \frac{4}{m_1^2 m_2^2} \int dp_1 dp_2 \prod_{j=1}^2 \delta(p_j^2 - m_j^2) \theta(p_j^0) \dots, \quad (102)$$

it is clear that in the intermediate states β the particle masses do not go off mass shell, but keep their free particle values, as in the nonrelativistic theory. This implies that the energy and also the momentum of the states β are allowed to differ from their values in the state α .

As an example we consider the potential as derived from a one boson exchange, as shown in figure 3.

This potential is given by

$$V_{\alpha\beta} = \frac{2\alpha m_1 m_2}{\pi^2(\bar{t} - \mu^2)}, \quad (103)$$

in which \bar{t} is the usual Mandelstam variable

$$\bar{t} = q_1 \cdot q_2 \quad \text{with} \quad q_1 = p'_1 - p_1 \quad \text{and} \quad q_2 = p_2 - p'_2. \quad (104)$$

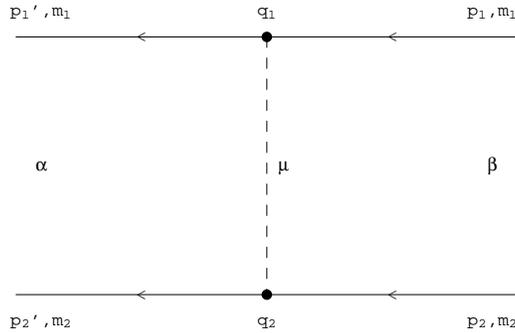


Fig. 3. One-boson exchange diagram for Yukawa interaction. The mass of the exchanged particle is μ .

Alternatively this result can be obtained from the nonrelativistic Yukawa potential $V(r) = -\alpha e^{-\mu r}/r$, by first calculating its Fourier transform

$$V(|\vec{k}' - \vec{k}|) = -\frac{\alpha}{2\pi^2(|\vec{k}' - \vec{k}|^2 + \mu^2)}, \tag{105}$$

and then replacing the square of the momentum transfer $|\vec{k}' - \vec{k}|^2$ with its relativistic form $-\bar{t}$.

This will also be the rule which we will adopt for the general extension of a nonrelativistic potential to the relativistic case. The extra factor $4m_1m_2$ derives from differences in normalisation.

In field theory $q_1 = q_2$, because $p'_1 + p'_2 = p_1 + p_2$ in each elementary interaction, *i.e.*, both energy and momentum are conserved. In RQM, however, this is replaced by the conservation of total velocity

$$\frac{\vec{p}'_1 + \vec{p}'_2}{p^{0'}_1 + p^{0'}_2} = \frac{\vec{p}_1 + \vec{p}_2}{p^0_1 + p^0_2}, \tag{106}$$

which can also be written in terms of the four-vectors

$$\frac{p'_1 + p'_2}{\sqrt{s'}} = \frac{p_1 + p_2}{\sqrt{s}}, \tag{107}$$

with $s' = (p'_1 + p'_2)^2$ and $s = (p_1 + p_2)^2$. The energy-momentum transfer in the upper and lower vertex are in general not equal, $q_1 \neq q_2$, and also $s' \neq s$. Therefore the potential must also be defined for the case where the energy-momentum of the initial and final states are different (although they have the same velocity).

This is done by adopting the same expression (103), still with \bar{t} given by Eq. (104), but where it is now understood that q_1 and q_2 may be different.

This off-shell-ness replaces the assumptions made in other quasi-potential theories, in which one or both particles are allowed to go off mass-shell. This can lead to difficulties, because the wave function for such a particle will no longer satisfy the free particle wave equation.

In the following sections this new prescription for calculating the potential will be applied to the linear and harmonic potentials.

5.3. Harmonic potential

The simplest form of the eigenvalue problem (101) is obtained when writing it in the centre of momentum system. Taking into account the proper factors, which are given in Eq. (1.45) of [2], and a new normalisation of the wave function (Eq. (1.58) of [2]), the equation turns into

$$\left[\sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2} - M_n \right] \chi_n(\vec{k}) + \frac{1}{4} \int \frac{V(\vec{k}, \vec{k}')}{\sqrt{(k'^2 + m_1^2)(k'^2 + m_2^2)}} \chi_n(\vec{k}') d\vec{k}' = 0. \tag{108}$$

The functions $\chi_n(\vec{k})$ are orthonormal in the sense that

$$\frac{1}{4} \int \frac{\chi_{n'}^*(\vec{k}) \chi_n(\vec{k})}{\sqrt{(k^2 + m_1^2)(k^2 + m_2^2)}} d\vec{k} = \delta_{n'n}. \tag{109}$$

This equation will now be used for the harmonic oscillator potential $\widetilde{W}(r) = \frac{1}{2}m\omega^2 r^2$, where we take for m the reduced mass of m_1 and m_2 . By first introducing a cut-off, as in Eq. (15), we can calculate the Fourier transform Eqs. (16) and (17). When we then apply the rule described in the previous section, we obtain for the potential in Eq. (108) the following expression

$$V(\vec{k}, \vec{k}') = 4m_1 m_2 \left[m\omega^2 R^2 \delta_s(\vec{k}' - \vec{k}) - \frac{m\omega^2 R^5}{(2\pi)^{3/2}} e^{\frac{1}{2}\vec{t}R^2} \right]. \tag{110}$$

In this formula $\delta_s(\vec{k}' - \vec{k})$ is obtained from

$$\delta(\vec{k}' - \vec{k}) = \delta(x_1) \delta(x_2) \delta(x_3) \quad \text{with} \quad \vec{x} = \vec{k}' - \vec{k}, \tag{111}$$

by replacing $|\vec{k}' - \vec{k}|^2 = x^2 = x_1^2 + x_2^2 + x_3^2$ with $-\vec{t}$, which in the centre of momentum system is equal to

$$-\vec{t} = \left| \vec{k}' - \vec{k} \right|^2 + \left\{ \sqrt{k'^2 + m_1^2} - \sqrt{k^2 + m_1^2} \right\} \left\{ \sqrt{k'^2 + m_2^2} - \sqrt{k^2 + m_2^2} \right\}. \tag{112}$$

For \vec{k}' close to \vec{k} we expand this expression in powers of \vec{x} , which gives

$$-\bar{t} = x^2 + a(\vec{k} \cdot \vec{x})^2 + \mathcal{O}(x^3) \quad \text{with} \quad a(k) = \frac{1}{\sqrt{(k^2 + m_1^2)(k^2 + m_2^2)}}. \quad (113)$$

Choosing the coordinate system in such a way that the positive x_3 -axis points in the direction of \vec{k} , Eq. (113) can also be written as

$$-\bar{t} = x_1^2 + x_2^2 + \beta^2 x_3^2 + \mathcal{O}(x^3), \quad \text{with} \quad \beta(k) = \sqrt{1 + ak^2}. \quad (114)$$

The function $\beta(k)$ varies between $\beta = 1$ for $k = 0$ and $\beta = \sqrt{2}$ for $k \rightarrow \infty$.

The replacement $|\vec{k}' - \vec{k}|^2 \rightarrow -\bar{t}$ therefore amounts to $x_1 \rightarrow x_1$, $x_2 \rightarrow x_2$ and $x_3 \rightarrow \beta x_3$. Consequently it follows from Eq. (111) that

$$\delta_s(\vec{k}' - \vec{k}) = \frac{1}{\beta(k)} \delta(\vec{k}' - \vec{k}). \quad (115)$$

If we now substitute the potential from Eq. (110) into Eq. (108), we obtain

$$\begin{aligned} & \left[\sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2} - M_n \right] \chi_n(\vec{k}) \\ & + \frac{m_1 m_2 m \omega^2 R^2}{\beta(k) \sqrt{(k^2 + m_1^2)(k^2 + m_2^2)}} \chi_n(\vec{k}) \\ & = \frac{m_1 m_2 m \omega^2 R^5}{(2\pi)^{3/2}} \int \frac{e^{\frac{1}{2}\bar{t}R^2}}{\sqrt{(k'^2 + m_1^2)(k'^2 + m_2^2)}} \chi_n(\vec{k}') d\vec{k}'. \end{aligned} \quad (116)$$

For $R \rightarrow \infty$ the main contribution to the integral comes from \vec{k}' -values close to \vec{k} . Therefore \bar{t} can be approximated by Eq. (114) and also for $\chi_n(\vec{k}')$ it suffices to use the first three terms in the Taylor expansion

$$\chi_n(\vec{k}') = \chi_n(\vec{k}) + \sum_i x_i \frac{\partial \chi_n(\vec{k})}{\partial k_i} + \frac{1}{2} \sum_{i,j} x_i x_j \frac{\partial^2 \chi_n(\vec{k})}{\partial k_i \partial k_j} + \dots \quad (117)$$

On substitution into Eq. (116) we are left with the following integrals

$$A(k) = \int e^{-\frac{1}{2}R^2(x_1^2 + x_2^2 + \beta^2 x_3^2)} d\vec{x} \quad (118)$$

and

$$B_{ij}(k) = \int x_i x_j e^{-\frac{1}{2}R^2(x_1^2 + x_2^2 + \beta^2 x_3^2)} d\vec{x}. \quad (119)$$

These integrals are equal to

$$A(k) = \frac{(2\pi)^{3/2}}{\beta R^3}, \quad B_{11}(k) = B_{22}(k) = \frac{(2\pi)^{3/2}}{\beta R^5}, \quad B_{33}(k) = \frac{(2\pi)^{3/2}}{\beta^3 R^5} \tag{120}$$

and $B_{ij}(k) = 0$ for $i \neq j$.

The term proportional to $A(k)$ cancels against the R^2 -term in the left-hand side of Eq. (116), while the R^{-5} dependence of the B -terms is cancelled by the factor R^5 in front of the integral.

The final equation for the relativistic harmonic oscillator now reads

$$\begin{aligned} & \left[\sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2} - M_n \right] \chi_n(\vec{k}) \\ &= \frac{1}{2} m\omega^2 \frac{m_1 m_2}{\beta(k) \sqrt{(k^2 + m_1^2)(k^2 + m_2^2)}} \\ & \times \left\{ \frac{\partial^2 \chi_n(\vec{k})}{\partial k_1^2} + \frac{\partial^2 \chi_n(\vec{k})}{\partial k_2^2} + \frac{1}{\beta^2(k)} \frac{\partial^2 \chi_n(\vec{k})}{\partial k_3^2} \right\}. \end{aligned} \tag{121}$$

This equation seems not to be rotational invariant. This is not the case, however, because the first two terms between the curly brackets derive from the variation of $\chi_n(\vec{k})$ in the directions transverse to \vec{k} , while the last term refers to the longitudinal direction. These are rotational invariant concepts.

On comparison with Eq. (41) we see that for low energies our theory reduces to the Relativistic Schrödinger equation. For high energies, however, *i.e.*, for highly excited states, the two theories are completely different. Because of the extra factor on the right hand side of Eq. (121) the coupling $\frac{1}{2}m\omega^2$ is reduced by a factor which tends to zero for high energies.

For numerical calculations it is easier to work with partial waves. By writing

$$\chi_n(\vec{k}) = \frac{\phi_{nl}(k)}{k} Y_{lm}(\theta, \varphi) \tag{122}$$

one then obtains for the harmonic oscillator in RQM

$$\begin{aligned} & \left[\sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2} - M_{nl} \right] \phi_{nl}(k) \\ &= \frac{1}{2} m\omega^2 \frac{m_1 m_2}{\beta(k) \sqrt{(k^2 + m_1^2)(k^2 + m_2^2)}} \left\{ \frac{1}{\beta^2(k)} \frac{d^2 \phi_{nl}(k)}{dk^2} - \frac{l(l+1)}{k^2} \phi_{nl}(k) \right\}. \end{aligned} \tag{123}$$

This equation resembles the Relativistic Schrödinger equation (43). The reduction of Eq. (108) to the form of a differential equation, so far only

succeeds for the harmonic oscillator. For other potentials such a reduction will in general be impossible or amount to an approximation.

In the following subsections we will discuss Eq. (123) for the static case and for the case of two equal masses.

5.3.1. The static case $m_2 \rightarrow \infty$

If the total mass $M = m_1 + m_2$ becomes arbitrarily large, while the reduced mass $m = \frac{m_1 m_2}{M}$ is kept fixed, Eq. (123) takes on a much simpler form. With $\varepsilon(k)$ and ε_{nl} defined by

$$\varepsilon(k) = \sqrt{k^2 + m^2} \quad \text{and} \quad M_{nl} = M - m + \varepsilon_{nl}, \tag{124}$$

Eq. (123) becomes

$$[\varepsilon(k) - \varepsilon_{nl}] \phi_l(k) = \frac{1}{2} m \omega^2 \frac{m}{\varepsilon(k)} \left\{ \frac{d^2 \phi_l(k)}{dk^2} - \frac{l(l+1)}{k^2} \phi_l(k) \right\}. \tag{125}$$

Although it differs from the Relativistic Schrödinger equation (43) only in the factor $m/\varepsilon(k)$, this has a large effect on the slope of the Regge trajectories.

In order to show this effect we first introduce dimensionless variables y , y_l and g by

$$k = my \quad \varepsilon_{nl} = m \sqrt{1 + y_l^2}, \quad g = \frac{\omega}{m}. \tag{126}$$

We furthermore assume that l is large enough, so that $y \gg 1$ and $y_l \gg 1$. Eq. (125) then becomes

$$-\frac{d^2 \phi_l(y)}{dy^2} = [E(y) - V(y)] \phi_l(y), \tag{127}$$

where

$$E(y) = ty, \quad \left(t \equiv \frac{2y_l}{g^2} \right) \quad \text{and} \quad V(y) = \frac{2y^2}{g^2} + \frac{l^2}{y^2}. \tag{128}$$

This looks like the nonrelativistic Schrödinger equation, except that the eigenvalue is replaced by a linear function in y .

Still, for $l \gg 1$, the spectrum can be calculated by again applying the Bohr–Sommerfeld quantisation rule, as described in Section 1.4. We use the fact that the classical turning points are very close to the point $y = y_c$, where the function $E(y) = ty$ is tangent to the function $V(y)$. This happens when

$$t = t_c = \frac{8}{3g^2} y_c \quad \text{with} \quad y_c = \left(\frac{3}{2} \right)^{1/4} \sqrt{gl}. \tag{129}$$

By expanding $V(y)$ in a power series around y_c , an expression for the classical turning points is found and the integral in

$$\int_{y_-}^{y_+} \sqrt{E(y) - V(y)} dy = n\pi \tag{130}$$

can be calculated explicitly as a function of t . Solving Eq. (130) for t gives

$$t = \frac{8}{3g^2}y_c + \frac{4n}{gy_c} + \dots \quad \text{for } l \gg n. \tag{131}$$

The spectrum therefore is

$$\frac{\varepsilon_{nl}^2}{m^2} = y_l^2 = \frac{1}{4}g^4 t^2 = \frac{16}{9}\sqrt{\frac{3}{2}}gl + \frac{16}{3}gn + \dots \tag{132}$$

These are linear equidistant Regge trajectories for the static case of the harmonic oscillator. In Eq. (55) such trajectories were also found with the Relativistic Schrödinger equation, but then for a linear potential.

Also for small values of l and n this linear dependence on l and n is obtained from a numerical solution of Eq. (125). This is shown in figure 4.

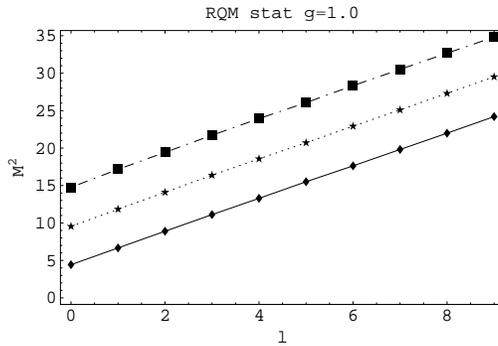


Fig. 4. Regge trajectories for RQM with harmonic potential. The static case.

5.3.2. The case of equal masses $m_1 = m_2$

For the case of two equal masses $m_1 = m_2 = 2m$, and in the limit $l \gg 1$, Eq. (123) can again be written in the form

$$-\frac{d^2\phi_l(y)}{dy^2} = [E(y) - V(y)] \phi_l(y). \tag{133}$$

The difference with Eq. (127) lies in the definitions of $E(y)$ and $V(y)$, which now are

$$E(y) = ty^2, \quad \left(t \equiv \frac{\sqrt{2}yl}{g^2}\right) \quad \text{and} \quad V(y) = \frac{2\sqrt{2}y^3}{g^2} + \frac{l^2}{y^2} \quad \text{with} \quad M_{nl} = my_l. \tag{134}$$

Again we can determine the values of $t = t_c$ and $y = y_c$ so that the function $E(y) = t_c y^2$ is tangent to $V(y)$ in the point $y = y_c$. This is the case when

$$t = t_c = \frac{5}{\sqrt{2}g^2} y_c \quad \text{and} \quad y_c = 2^{3/10} (gl)^{2/5}. \tag{135}$$

In the same way as in the static case the Bohr–Sommerfeld integral can again be calculated as a function of t . Putting the integral equal to $n\pi$ gives an equation for t , which can be solved to give

$$t = \frac{5}{\sqrt{2}g^2} y_c + \frac{2n}{y_c^2} + \dots \quad \text{for} \quad l \gg n. \tag{136}$$

The spectrum therefore is

$$\frac{M_{nl}^2}{m^2} = y_l^2 = \frac{1}{2} g^4 t^2 = \frac{5^2}{2^{7/5}} (gl)^{4/5} + 5 \cdot 2^{1/5} g^{8/5} \frac{n}{l^{2/5}} + \dots \tag{137}$$

These Regge trajectories are rising slower than linearly and the distance between them tends to zero with $l \rightarrow \infty$.

The same behaviour is obtained from a numerical solution of Eq. (123) for $m_1 = m_2$, as can be seen in figure 5.

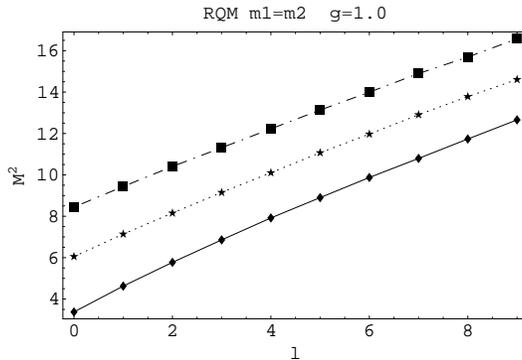


Fig. 5. Regge trajectories for RQM with harmonic potential. The case of equal masses.

5.4. Linear potential

In this section we will use the basic equation (108) of RQM for the calculation of Regge trajectories of two particles, bound by a linear potential. The convergence properties of the integral in Eq. (108) are different for the static case and the case of equal masses. For this reason these two cases require a separate treatment.

5.4.1. The static case $m_2 \rightarrow \infty$

For the construction of the relativistic linear potential $V(\vec{k}, \vec{k}')$ from its nonrelativistic form (22), we apply the general prescription of Sec. 5.2. Using Eq. (115) we find

$$V(\vec{k}, \vec{k}') = 4m_1 m_2 \sigma \left[\frac{1}{\mu \beta(k)} \delta(\vec{k}' - \vec{k}) - \frac{1}{\pi^2 (-\bar{t} + \mu^2)^2} \right], \quad (138)$$

in which \bar{t} and $\beta(k)$ are defined in Eqs. (112)–(114) and where in the end the limit $\mu \rightarrow 0$ should be taken.

For $m_2 \rightarrow \infty$ it is seen that \bar{t} approaches the nonrelativistic limit $\bar{t} \rightarrow -|\vec{k}' - \vec{k}|^2$ and $\beta(k) \rightarrow 1$. We introduce the total mass M and the reduced mass m and write

$$M_n = M - m + \varepsilon_n \quad \text{and} \quad \sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2} = M + \varepsilon(k) - m$$

with $\varepsilon(k) = \sqrt{k^2 + m^2}$. (139)

After substituting the potential (138) into Eq. (108) and taking the limit $M \rightarrow \infty$, we obtain the RQM equation for the static case. As for the nonrelativistic theory of Section 1.3 the integral equation is singular again. In order to make it regular, however, we apply the same method as used there. In this way Eq. (108) becomes

$$[\varepsilon(k) - \varepsilon_n] \phi(\vec{k}) = \frac{m \sigma}{\pi^2 \varepsilon(k)} \int^* \frac{\phi(\vec{k}') - \phi(\vec{k})}{|\vec{k}' - \vec{k}|^4} d\vec{k}', \quad (140)$$

where we put

$$\chi_n(\vec{k}) = \varepsilon(k) \phi(\vec{k}). \quad (141)$$

An exact solution of this eigenvalue problem is impossible and also the Bohr–Sommerfeld method cannot be used, because Eq. (140) has no equivalent Schrödinger form. Also the expansion method, which was used for the harmonic oscillator in Section 5.3, cannot be applied, because the coefficients of the derivatives are now expressed by divergent integrals.

We therefore apply another method, which consists of comparing Eq. (140) with Eqs. (27) and (45). These were solved previously and gave the spectrum of the linear potential in the case of the nonrelativistic and Relativistic Schrödinger equation.

For Eq. (27) we define

$$\vec{k} = \kappa \vec{u}, \quad \vec{k}' = \kappa \vec{u}', \quad \lambda_{\text{NR}} = \frac{m \sigma}{\kappa^3}. \quad (142)$$

The nonrelativistic Schrödinger equation then turns into

$$(u - 1) \phi(\vec{u}) = \frac{2\lambda_{\text{NR}}}{\pi^2(u + 1)} \int^* \frac{\phi(\vec{u}') - \phi(\vec{u})}{|\vec{u}' - \vec{u}|^4} d\vec{u}'. \quad (143)$$

In Eqs. (45) and (140) we approximate $\varepsilon(k)$ by k and define \vec{u} by $\vec{k} = \varepsilon_n \vec{u}$. Eq. (45) for the Relativistic Schrödinger equation then becomes

$$(u - 1) \phi(\vec{u}) = \frac{\lambda_{\text{RS}}}{\pi^2} \int^* \frac{\phi(\vec{u}') - \phi(\vec{u})}{|\vec{u}' - \vec{u}|^4} d\vec{u}' \quad \text{with} \quad \lambda_{\text{RS}} = \frac{\sigma}{\varepsilon_n^2}. \quad (144)$$

Eq. (140) takes the form

$$(u - 1) \phi(\vec{u}) = \frac{\lambda_{\text{RQM}}}{\pi^2 u} \int^* \frac{\phi(\vec{u}') - \phi(\vec{u})}{|\vec{u}' - \vec{u}|^4} d\vec{u}' \quad \text{with} \quad \lambda_{\text{RQM}} = \frac{m \sigma}{\varepsilon_n^3}. \quad (145)$$

Notice the close similarity between the last three equations.

In Section 1.4 it was shown, see Eq. (36), that for high energies the distance between the classical turning points, when measured in units determined by the eigenvalue, was much less than unity. The same happened in Section 2.2, Eq. (49) and below Eq. (53). Therefore, also the potential energy varies very little between these turning points. For this reason we expect that for high energies the relevant values of \vec{k} and \vec{k}' lie in a very narrow spherical shell with radius equal to the eigenvalue κ or ε_n . This means that the relevant region of the variable u is a very small interval around $u = 1$. In the three equations (143), (144) and (145) we are therefore allowed to replace u by $u = 1$ in the factors in front of the integrals. By doing so the equations become identical and the spectral values of λ_{NR} , λ_{RS} and λ_{RQM} become equal.

The values of κ were calculated in Section 1.4. From Eqs. (30) with $E = \kappa^2/2m$ and (37) we derive

$$\kappa_{nl}^2 = 3m^2 \left(\frac{\sigma}{m^2} \right)^{2/3} l^{2/3} \left[1 + \frac{2}{\sqrt{3}} \frac{n}{l} \right]. \quad (146)$$

By putting from Eqs. (142) and (144)

$$\frac{\sigma}{\varepsilon_{nl}^2} = \frac{m\sigma}{\kappa_{nl}^3}, \quad (147)$$

we get

$$\varepsilon_{nl}^2 = \sqrt{27} \sigma l + 9 \sigma n. \quad (148)$$

Comparing this result with Eq. (55), we see that the linear equidistant Regge trajectories are reproduced, but that there are small differences in the slope and the distance.

With this method we can now also calculate the spectrum of Eq. (145). By comparing with the spectrum of the nonrelativistic theory we see that we have to put

$$\frac{m\sigma}{\varepsilon_{nl}^3} = \frac{m\sigma}{\kappa_{nl}^3}. \quad (149)$$

This gives for the spectrum with RQM

$$\varepsilon_{nl}^2 = \kappa_{nl}^2 = 3m^2 \left(\frac{\sigma}{m^2} \right)^{2/3} l^{2/3} \left[1 + \frac{2}{\sqrt{3}} \frac{n}{l} \right]. \quad (150)$$

This spectrum can also be calculated by comparing with ε_{RS} from Eq. (55) by putting

$$\frac{m\sigma}{\varepsilon_{nl}^3} = \frac{\sigma}{\varepsilon_{RS}^2}. \quad (151)$$

This gives

$$\varepsilon_{nl}^2 = 2^{4/3} m^2 \left(\frac{\sigma}{m^2} \right)^{2/3} l^{2/3} \left[1 + \frac{2\sqrt{2}}{3} \frac{n}{l} \right]. \quad (152)$$

Eqs. (150) and (152) show the same dependence of ε_{nl}^2 on l , n and σ/m^2 . The coefficients differ from each other by not more than 25%.

This result differs from the linear Regge trajectories as obtained from the Relativistic Schrödinger equation, the Klein–Gordon equation and the Dirac equation. Experimental information about the correct behaviour could not be found.

For mesons consisting of two light quarks the square of the masses do lie on linear Regge trajectories. In the next section we will show that this phenomenon is reproduced in RQM.

5.4.2. The case of equal masses $m_1 = m_2$

In the previous section and also in Section 5.3, where the harmonic oscillator was considered, it was shown that the only contribution to the integral in Eq. (108) was given by \vec{k}' -values close to \vec{k} . For this reason the wave

function $\chi_n(\vec{k}')$ could be represented by the first three terms of its Taylor series expansion (117). This eventually led to the differential equation (121), which strongly resembles the Relativistic Schrödinger equation (41).

For the present case of the linear potential we will now apply the same expansion (117). In this way we replace the integral equation (108) by the following differential equation

$$\left[\sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2} - M_n + \frac{m_1 m_2 \sigma}{\mu \beta(k) \sqrt{(k^2 + m_1^2)(k^2 + m_2^2)}} \right] \chi_n(\vec{k}) = \frac{m_1 m_2 \sigma}{4\pi^2 k^2} \left[I_0(k) \chi_n(\vec{k}) + \frac{1}{2} \sum_{i=1}^3 I_i(k) \frac{\partial^2 \chi_n(\vec{k})}{\partial k_i^2} \right]. \tag{153}$$

The coefficients are defined by the positive integrals

$$I_0(k) = \int \frac{d\vec{k}'}{k'^2 \sqrt{(k'^2 + m_1^2)(k'^2 + m_2^2)} (z - \cos \theta)^2}, \tag{154}$$

$$I_1(k) = I_2(k) = \int \frac{k_x'^2 d\vec{k}'}{k'^2 \sqrt{(k'^2 + m_1^2)(k'^2 + m_2^2)} (z - \cos \theta)^2}, \tag{155}$$

$$I_3(k) = \int \frac{(k_z' - k)^2 d\vec{k}'}{k'^2 \sqrt{(k'^2 + m_1^2)(k'^2 + m_2^2)} (z - \cos \theta)^2}. \tag{156}$$

The coordinate system has been chosen in such a way that the positive z -axis points in the direction of \vec{k} . The angle between \vec{k} and \vec{k}' is denoted by θ and the function $z = z(k, k')$ is defined by writing

$$-\bar{t} + \mu^2 = 2kk'(z - \cos \theta), \tag{157}$$

so that

$$z(k, k') = \frac{1}{2} \left(\frac{k'}{k} + \frac{k}{k'} \right) + \frac{1}{2kk'} \left\{ \sqrt{k'^2 + m_1^2} - \sqrt{k^2 + m_1^2} \right\} \times \left\{ \sqrt{k'^2 + m_2^2} - \sqrt{k^2 + m_2^2} \right\} + \frac{\mu^2}{2kk'}. \tag{158}$$

In the static case of Section 5.4.1 this method could not be applied, because the integrals corresponding to $I_0(k)$ and $I_i(k)$ would not converge.

In the present case they are finite, although their calculation in closed form could only be performed for $m_1 = m_2 = 0$. After a lengthy calculation (and repeated mistakes) I found

$$I_0(k) = \frac{4\pi^2}{\mu\sqrt{2}} - \frac{2\pi}{k} \quad \text{and} \quad I_1(k) = I_2(k) = 4\pi k J_t \quad \text{and} \quad I_3(k) = 4\pi k J_l \tag{159}$$

with

$$J_t = \frac{3\pi^2}{8} - \frac{\pi}{2} - \frac{1}{2} = 1.6303 \quad \text{and} \quad J_l = \frac{\pi}{2} + 1 = 2.5708. \tag{160}$$

Since $\beta(k) = \sqrt{2}$ when $m_1 = m_2 = 0$, the term $4\pi^2/(\mu\sqrt{2})$, which blows up for $\mu \rightarrow 0$, cancels against the corresponding term on the left-hand side of Eq. (153). We are then left with

$$\begin{aligned} & \left[\sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2} - M_n + \frac{m_1 m_2 \sigma}{2\pi k^3} \right] \chi_n(\vec{k}) \\ &= \frac{m_1 m_2 \sigma}{2\pi k} \left[J_t \left(\frac{\partial^2 \chi_n(\vec{k})}{\partial k_x^2} + \frac{\partial^2 \chi_n(\vec{k})}{\partial k_y^2} \right) + J_l \frac{\partial^2 \chi_n(\vec{k})}{\partial k_z^2} \right]. \end{aligned} \tag{161}$$

This shows that J_t is the coefficient in front of the transverse derivative, while J_l multiplies the longitudinal derivative.

Eq. (161) can again be written for spherical waves of the form of Eq. (122). If, moreover, we take $m_1 = m_2 = 2m$ and neglect the masses on the left-hand side of the equation, it becomes

$$[2k - M_{nl}] \phi_{nl}(k) = \frac{2m^2 \sigma}{\pi k} \left\{ J_l \frac{d^2 \phi_{nl}(k)}{dk^2} - J_t \frac{l(l+1)}{k^2} \phi_{nl}(k) \right\}. \tag{162}$$

If we now introduce dimensionless variables $y = k/m$ and $y_l = M_{nl}/(2m)$, this differential equation can again be written in the form (127)

$$-\frac{d^2 \phi_l(y)}{dy^2} = [E(y) - V(y)] \phi_l(y), \tag{163}$$

but now with

$$E(y) = ty \quad (t \equiv \frac{\pi m^2 y_l}{\sigma J_l}) \quad \text{and} \quad V(y) = \frac{\pi m^2}{\sigma J_l} y^2 + \frac{J_t l^2}{J_l y^2}. \tag{164}$$

The coefficients differ from those in Eq. (128), but the y - and l -dependence of $E(y)$ and $V(y)$ are the same.

Apart from some numerical factors, the eigenvalue spectrum will therefore be the same as in the static case of the harmonic oscillator. Repeating the calculation of Section 5.3.1, again produces linear Regge trajectories, which are now given by the equation

$$\frac{M_{nl}^2}{m^2} = \frac{64}{9} \sqrt{\frac{3J_t}{\pi}} \sqrt{\frac{\sigma}{m^2}} l + \frac{64 J_l}{3\pi} \frac{\sigma}{m^2} n. \quad (165)$$

For smaller and more realistic values of the angular momentum Hersbach [16] used RQM to calculate a large number of meson masses. In addition to a one-gluon exchange potential, he used a linear potential, containing a scalar and a vector part. He obtained very good agreement with experimental mass values, the square of which lie on linear Regge trajectories.

6. Conclusions

The conclusions of the RQM calculations for two-particle bound states, can be summarised as follows.

1. The slopes of Regge trajectories and their mutual distances strongly depend on the interaction-linear or harmonic-and on possible recoil effects, *i.e.*, equal masses or the static case with $m_2 \rightarrow \infty$. The results for M_{nl}^2/m^2 are collected in the table. We have also included the results of the four other theories. The numerical values of the constants a and b appearing in each entry can be taken from the corresponding equations. The dimensionless coupling constants are given by $g = \sigma/m^2$ for the linear potential and by $g = \omega/m$ for the harmonic oscillator. Notice that in RQM with a linear potential and $m_1 = m_2$ the Regge trajectories are straight equidistant lines, but that their slope is proportional to $\sqrt{\sigma}$ and not to the usual σ . Straight Regge trajectories also occur in the static case of the harmonic potential, but not in the static linear potential, where the other theories give linear trajectories. It must be added, however, that for this case we have not performed any numerical calculations with RQM to check the behaviour for smaller values of the angular momentum.

M_{nl}^2/m^2	linear potential	harmonic potential
Schr. eq.	$ag^{4/3}l^{4/3} + bg^{4/3}l^{1/3}n$ (38)	$g^2(l^2 + 4ln + \dots)$ (7)
Rel. Schr. eq.	$agl + bgn$ (55)	$ag^{4/3}l^{4/3} + bg^{4/3}l^{1/3}n$ (44)
Klein-Gordon	$agl + bgn$ (63)	$ag^{4/3}l^{4/3} + bg^{4/3}l^{1/3}n$ (69)
Dirac	$agl + bgn$ (89)	$ag^{4/3}l^{4/3} + bg^{4/3}l^{1/3}n$ (95)
RQM $m_1 = m_2$	$ag^{1/2}l + bgn$ (165)	$ag^{4/5}l^{4/5} + bg^{8/3}l^{-2/5}n$ (137)
RQM $m_2 \rightarrow \infty$	$ag^{2/3}l^{2/3} + bg^{2/3}l^{-1/3}n$ (150, 152)	$agl + bgn$ (132)

2. In [3] it was pointed out that an incorrect value of the slope of the Regge trajectory is obtained when the rotational degree of freedom of the string is not taken into account. This deviation was then corrected by adding the extra degree of freedom. Its effect could again be described by an effective Hamiltonian, in which the quark potential is nonlocal and depends on the relative angular momentum. In RQM a similar effect exists, because the (nonlocal) interaction potential carries not only energy, but also momentum — see Eq. (99) — and hence angular momentum. This is most clearly seen in Eqs. (161) and (162), where the tangential derivative and the radial derivative have different coefficients. The same effect is seen for the harmonic oscillator in Eqs. (121) and (123). The suggestion is that the rotational motion of the potential will always be present, whether or not the interaction is described by a gluon string.

3. Without comparing Eq. (140) with the nonrelativistic- and the Relativistic Schrödinger equation for the linear potential, it would have been very difficult to guess the correct asymptotic behaviour of its solution. The success of the method can be understood from the fact that for high energies, where the semi-classical quantisation rule of Bohr-Sommerfeld becomes exact, the relative variation of the energy between the classical turning points is very small.

I want to thank Professor Simonov for explaining his ideas about rotating strings.

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