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# ASYMPTOTIC EXPANSION OF THE MAXWELL FIELD IN A NEIGHBOURHOOD OF A MULTIPOLE PARTICLE*

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Electromagnetic field in a neighbourhood of an arbitrarily moving (electric or magnetic) dipole particle is described up to $r^1$-terms, where $r$ is the distance from the particle.

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## 1. Introduction

In the famous paper [1] Dirac proposed a theory of interaction between electromagnetic radiation and a point-like particle, where the latter *is not* treated as a test particle. The theory displays instability (better known as existence of "runaway solutions"). There were many attempts to improve this theory (see [2]), consisting *e.g.* in adding a dipole structure (both electric and magnetic) to the particle.

Recently, a new approach to the interaction problem was proposed (see [3] or [4]). It is not based on any *ad hoc* equations of motion imposed on a particle, but on the conservation principle of the total four-momentum of the composed "particle + field" system. The latter quantity is defined by a certain renormalization procedure which approximates the corresponding four-momentum carried by an extended particle. Equations of motion are then obtained from the above conservation laws in a way, which realizes the Einstein programme of deriving equations of motion from field equations.

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In this approach the source of Dirac's instability is easily understood: the approximation of energy-momentum tensor used in this theory was too rough because it did not take into account any deformation energy of the particle, due to the external field. As a result, the total energy was not bounded from below. To cure this disease a "new generation" theory is being developed by the present authors and will be soon published.

For all these goals a detailed analysis of the Maxwell field in a vicinity of an arbitrarily moving dipole particle is necessary. In the simplest Dirac case, only the monopole particle was considered and only the terms up to $r^0$ in the field expansion around the particle were taken into account.

For our purposes expansion up to $r^1$ terms for a dipole particle is necessary. It was never done in the literature. Even if the idea is simple and consists in an appropriate Taylor expansion of the well known formulae (of the Liénard–Wiechert type), an enormous complexity of the problem makes such an exercise almost infeasible, unless a special algorithm is found which simplifies sufficiently the calculations.

In the present paper we propose such an algorithm, based on the use of the co-moving coordinate system for the particle trajectory (the so called Fermi frame — *cf.* [4] or [5]). This allows us to remove all the unnecessary parameters, related to the special arrangements of measuring instruments, and to express the results in terms of intrinsic characteristics of the trajectory and of the time-dependent dipole moment of the particle. For the actual Taylor expansion the packages of symbolic calculus of the program MAPLE V were used. We stress that the straightforward use of the same packages (*i.e.* without any simplification and standardization proposed in the present paper) does not lead to any manageable result within a finite time.

Whenever a complicated formula is obtained with use of a computer algebra, it is important to have an independent criterion to check its validity. In our case, the entire singular part of the field (more than 70% of the obtained terms) may be calculated by an independent approach, which does not use any power series expansion (see [6]). The method is purely algebraic and consists in solving step by step Maxwell equations in particle's rest frame. The consistence of the two procedures is an important test for the correctness of results presented here.

## 2. The field of a moving dipole in laboratory frame

Suppose, therefore, that an electric dipole is moving along an arbitrary spacetime trajectory (we show in the sequel that the duality transformation between electric and magnetic fields enables us to obtain easily the corresponding result for a magnetic dipole particle). Our starting point is the standard Liénard–Wiechert formula for the (retarded or advanced) field of a *monopole* particle equipped with charge $e$ and moving along the trajectory $q^\mu(t) = (t, \vec{q}(t))$:

$$D_{\mathrm{mon}}(\vec{x},t) = \frac{e}{r^2}\frac{(1-vv)(N+\varepsilon V)}{(1+\varepsilon\, nv)^3} + \frac{e}{r}\frac{na(N+\varepsilon V)-(1+\varepsilon\, nv)A}{(1+\varepsilon\, nv)^3}\,, \quad (1)$$

$$B_{\mathrm{mon}}(\vec{x},t) = -\varepsilon\, N \times D_{\mathrm{mon}}\,, \quad (2)$$

where $\varepsilon$ equals $-1$ for the retarded and 1 for the advanced solution. Capital letters denote three-dimensional (space-like) vectors, any pair of lower case letters denotes their scalar product (*i.e.* $nv = N_i V^i$), $r = |\vec{x} - \vec{q}(\tau)|$, $N = (\vec{x} - \vec{q}(\tau))/r$, where $\vec{x}$ is the observer's position, $\vec{q}(\tau)$ denotes retarded or advanced position of the particle, $V$ stands for the particle's velocity, $A$ for acceleration, both taken at retarded or advanced time $\tau = \tau(t, \vec{x})$. This field satisfies the Maxwell equations with the "delta-like" current:

$$\mathcal{J}^0(\vec{x},t) = e\delta(\vec{x}-\vec{q}(t))\,, \qquad \mathcal{J}^k(\vec{x},t) = eV^k(t)\delta(\vec{x}-\vec{q}(t))\,, \quad (3)$$

where

$$(1, V^k(t)) = \left(1, \frac{d}{dt}\vec{q}(t)\right) = \frac{d}{dt}q^\mu(t) = V^\mu(t)\,.$$

The retarded (advanced) time $\tau$ is an implicit function of the parameters $(x^\mu) = (t, \vec{x})$, defined by the equation:

$$-(x^\mu - q^\mu(\tau))(x^\nu - q^\nu(\tau))g_{\mu\nu} = (t-\tau)^2 - ||\vec{x} - \vec{q}(\tau)||^2 = 0\,. \quad (4)$$

To obtain the retarded (or advanced) solution for a *dipole particle*, we are going to represent it as a derivative of the above monopole field with respect to an auxiliary parameter $l$. Assume, therefore, that we have a 1-parameter family of trajectories $\vec{q}(t,l)$. The derivative of (1) and (2) with respect to the parameter $l$ gives us the retarded and advanced solution, corresponding to the dipole-like current. Indeed, with the following notation for the (time dependent) dipole moment:

$$W(t) := e\,\frac{\partial \vec{q}}{\partial l}(t,0)\,,$$

we obtain the corresponding dipole-like current, equal to the derivative of monopole-like current (3) with respect to the parameter $l$:

$$\mathcal{J}^0 = -W^k \partial_k \delta(\vec{x}-\vec{q}(t))\,, \quad (5)$$

$$\mathcal{J}^k = \left(\frac{d}{dt}W^k\right)\delta(\vec{x}-\vec{q}(t)) - V^k W^i \partial_i \delta(\vec{x}-\vec{q}(t))\,. \quad (6)$$

The above current obviously fulfills continuity equation $\partial_\mu \mathcal{J}^\mu = 0$.

The corresponding (retarded or advanced) field is obtained as a derivative of (1) and (2) with respect to the parameter $l$. For this purpose we must be able to calculate derivatives of the quantities appearing in these formulae. In the Appendix we derive all the necessary ingredients. In particular, we prove the following:

$$e\,\frac{\partial\tau}{\partial l} = -\varepsilon\frac{nw}{1+\varepsilon\, nv}\,, \qquad e\,\frac{\partial N}{\partial l} = -\frac{1}{r}\left(W + (N+\varepsilon\, V)\, e\,\varepsilon\,\frac{\partial\tau}{\partial l}\right)\,. \tag{7}$$

The final formula for the Liénard–Wiechert field of a moving dipole $W$ reads:

$$\begin{aligned} D_{\rm dip} \;=\; & \left(\frac{\alpha}{r^3}+\frac{\beta}{r^2}+\frac{\gamma}{r}\right)(N+\varepsilon V) - \left(\frac{1-vv}{r^3}+\frac{na}{r^2}\right)\frac{W}{(1+\varepsilon\, nv)^3} \\ & + \left(\frac{\psi}{r^2}+\frac{\omega}{r}\right)A + \left(\frac{(1-vv)}{r^2}+\frac{na}{r}\right)\frac{\varepsilon\, W_1}{(1+\varepsilon\, nv)^3} + \frac{\varepsilon\, nw\, A_1}{r(1+\varepsilon\, nv)^3} \\ & - \frac{W_2}{r(1+\varepsilon\, nv)^2}\,, \end{aligned} \tag{8}$$

$$B_{\rm dip} \;=\; -\varepsilon\,\left(\frac{dN}{dl}\times D_{\rm mon} + N\times D_{\rm dip}\right)\,, \tag{9}$$

where:

$$\alpha \;=\; \frac{-3\varepsilon\,(1-vv)}{(1+\varepsilon\, nv)^4}\left(-wv+(1-vv)\, e\frac{\partial\tau}{\partial l}\right)\,, \tag{10}$$

$$\begin{aligned} \beta \;=\; & \frac{-1}{(1+\varepsilon\, nv)^4}\Big((1+\varepsilon\, nv)(2\, w_1 v + wa) - 3\varepsilon\,\left((av)\,(nw)+(wv)\,(na)\right) \\ & + 3\varepsilon\,(1-vv)\left(nw_1 + 2\, na\, e\frac{\partial\tau}{\partial l}\right)\Big)\,, \end{aligned} \tag{11}$$

$$\begin{aligned} \gamma \;=\; & \frac{1}{(1+\varepsilon\, nv)^4}\Big((1+\varepsilon\, nv)\, nw_2 - \varepsilon\,(na_1)\,(nw) \\ & - 3\varepsilon\, na\left(nw_1 + na\, e\frac{\partial\tau}{\partial l}\right)\Big)\,, \end{aligned} \tag{12}$$

$$\psi \;=\; \frac{-1}{(1+\varepsilon\, nv)^3}\left((2-3\, vv-\varepsilon\, nv)\, nw + 2\varepsilon\, wv\right)\,, \tag{13}$$

$$\omega \;=\; \frac{\varepsilon}{(1+\varepsilon\, nv)^3}\left(2\, nw_1 + 3\, na\, e\frac{\partial\tau}{\partial l}\right)\,. \tag{14}$$

To simplify notation we have introduced an additional index related to the rank of the time derivative applied to quantities like dipole moment and acceleration, *i.e.* $V_1 := \frac{\partial}{\partial t}V$, $A_1 := \frac{\partial}{\partial t}A$, $W_1 := \frac{\partial}{\partial t}W$, $W_2 := \frac{\partial^2}{\partial t^2}W$ and $nw_2 := N_i W_2^i = (N|W_2)$. All these quantities have to be taken at the advanced (for $\varepsilon = 1$) or the retarded (for $\varepsilon = -1$) time.

## 3. Expansion in the powers of $r$

### *3.1. Calculation of the advanced (retarded) time*

To obtain useful information about the behaviour of the field in a neighbourhood of the particle we will expand the above results in powers of $r = |\vec{x}|$ on a fixed hyperplane $\{t = t_0\}$. For the sake of simplicity put $t_0 = 0$ and $\vec{q}(0) = \vec{0}$. Now, put $V = V_0$ and use the Taylor series for the particle's position and its dipole moment:

$$\vec{q}(t) = \sum_{k=1}^{5} \frac{V_{k-1}}{k!} t^k + \mathcal{O}(t^5)\,. \tag{15}$$

As will be seen in the sequel, the fifth order expansion is sufficient to obtain the behaviour of the field up to terms proportional to $r$. Equation (4) for the advanced (retarded) time $\tau(\vec{x}) := \tau(0, \vec{x})$ now reads:

$$\tau^2 - \vec{q}(\tau)^2 - r^2 + 2(\vec{q}(\tau)|\vec{x}) = 0\,. \tag{16}$$

Using the following *Ansatz* for $\tau$:

$$\tau(\vec{x}) = r(f_0 + f_1 r + f_2 r^2 + f_3 r^3 + f_4 r^4) + \mathcal{O}(r^5)\,, \tag{17}$$

where $f_i$ depend on the direction $M = \vec{x}/r$ of the vector $\vec{x}$ only (*i.e.* are functions of angles on the sphere $S^2$), and inserting it into (16), together with expansion (15) for $\vec{q}(\tau)$, the coefficients $f_i$ may be finally calculated, but the formulae obtained this way are extremely complicated. Using them in the further expansion of the field produces formulae, which can hardly be useful for our purposes because of their complexity. This problem may be avoided if we decide from the very beginning to describe the field on the particle's rest hyperplane, *i.e.* we put $V_0(0) = 0$. This simplifies considerably our results as the power series expansion becomes feasible and reduces to

$$f_0 = \varepsilon\,, \tag{18}$$

$$f_1 = -\frac{1}{2}\,\varepsilon\, mv_1\,, \tag{19}$$

$$f_2 = -\frac{1}{6}\, mv_2 + \frac{1}{8}\varepsilon\, \left(3\,(mv_1)^2 + v_1 v_1\right)\,, \tag{20}$$

$$\begin{aligned} f_3 = {} & -\frac{1}{24}\,\varepsilon\, mv_3 + \frac{1}{12}\, v_1 v_2 + \frac{1}{3}\,(mv_1)\,(mv_2) \\ & -\frac{5}{16}\varepsilon\, mv_1\, \left(v_1 v_1 + (mv_1)^2\right)\,, \end{aligned} \tag{21}$$

$$
\begin{aligned}
f_4 = & -\frac{1}{120}\,mv_4 - \frac{1}{8}\left((v_1v_1)\,(mv_2) + 2\,(v_1v_2)\,(mv_1)\right) \\
& +\varepsilon\,\frac{1}{8}\left(\frac{1}{9}v_2v_2 + \frac{1}{6}\,v_1v_3 + \frac{7}{16}(v_1v_1)^2\right) \\
& +\frac{5}{8}\,\varepsilon\,\left(\frac{1}{6}\,(mv_1)\,(mv_3) + \frac{1}{9}(mv_2)^2 + \frac{7}{8}\,(v_1v_1)\,(mv_1)^2\right) \\
& -\frac{1}{2}\,mv_2\,(mv_1)^2 + \frac{35}{128}\,\varepsilon\,(mv_1)^4 .
\end{aligned} \tag{22}
$$

### *3.2. The Fermi frame*

The reference system in which the particle moving with an acceleration $a^i$ would remain at rest, can be constructed in many ways. Here we choose the Fermi frame construction (see [5,4]). This means that the time vector $\boldsymbol{e}_0$ of the *tetrad* $(\boldsymbol{e}_\mu)$ defining the system is always equal to the particle's four velocity $u = (u^\mu)$. At each point $q(t)$ of the particle trajectory the space-like hypersurfaces $\Sigma_t$ orthogonal to the trajectory are spanned by the remaining three elements of the tetrad: $(\boldsymbol{e}_k)$, where $k = 1, 2, 3$. The above triad defines uniquely the Cartesian (orthonormal) coordinate system $(x^k)$ on $\Sigma_t$. The Fermi condition imposed on the system means that the covariant derivative $\nabla_{\boldsymbol{e}_0}\boldsymbol{e}_k$ has no space-like component, *i.e.* is proportional to $\boldsymbol{e}_0$. It is easy to show that this condition implies the following relations:

$$
\nabla_{\boldsymbol{e}_0}\boldsymbol{e}_k = a_k\boldsymbol{e}_0 \ , \quad \nabla_{\boldsymbol{e}_0}\boldsymbol{e}_0 = \boldsymbol{a} \ , \tag{23}
$$

where by $\boldsymbol{a} := a^k e_k$ (and $a_k = g_{kl}a^l$) we denote the acceleration (curvature) of the trajectory. It is a vector orthogonal to the trajectory. As the time parameter we take *the proper time* $s$ along the trajectory.

We are going to describe both the particle's motion and its dipole moment with respect to the above, degenerate coordinate system. It is degenerate, because different surfaces $\Sigma_t$ may intersect, but this does not lead to any difficulty. Technically, we may treat our spacetime as an abstract manifold $\Sigma \times R^1$, equipped with time dependent metric (see [5]):

$$
g_{\mu\nu} = \left(\begin{array}{c|c} N_kN^k - N^2 & N_l \\ \hline N_k & g_{kl} \end{array}\right), \quad g^{\mu\nu} = \left(\begin{array}{c|c} -\frac{1}{N^2} & \frac{N^l}{N^2} \\ \hline \frac{N^k}{N^2} & g^{kl} - \frac{N^kN^l}{N^2} \end{array}\right), \tag{24}
$$

where $g_{kl}$ is the flat (euclidean) metric on $\Sigma$. The lapse function $N$ and the shift vector $N^m$ encode information about the particular (3+1) — decomposition of spacetime. The Fermi–Walker transport (23) of the tetrad means that the shift vector vanishes identically: $N^m \equiv 0$. It is easy to check that the lapse function equals $N = 1 + a_ix^i$.

The components of the flat, Minkowski connection, can be obtained as Christoffel symbols of the above metric (*cf.* [3]):

$$\Gamma^0_{00} = \frac{\dot{a}_j x^j}{1 + a_i x^i} \ ; \quad \Gamma^0_{0k} = \frac{a_k}{1 + a_i x^i} \ ; \quad \Gamma^k_{00} = (1 + a_i x^i) a^k \, ; \tag{25}$$

and the remaining components vanish. In particular, if we calculate covariant derivatives of vectors attached at the particle's position $x^i = 0$, the only non-vanishing connection components are:

$$\Gamma^0_{0k} = a_k \ ; \quad \Gamma^k_{00} = a^k \, . \tag{26}$$

We are going to express the electromagnetic field surrounding the particle in terms of the two three-dimensional vectors: the acceleration $\vec{A}$ and the dipole moment $\vec{P}$, together with their time derivatives, calculated with respect to the particle's proper time $s$ in the above frame. Only these derivatives will be denoted by "dots" in the sequel.

#### 3.2.1. Transformation of velocities

In Fermi frame the velocity always vanishes. The entire information about particle's trajectory is encoded in the (time dependent) acceleration $a^i = a^i(s)$, which is a vector orthogonal to the trajectory. As a first step of our standardization procedure, we are going to express all the laboratory-frame derivatives $V_i$ (calculated with respect to the laboratory time $t$) in terms of the above quantity and its Fermi-frame covariant derivatives (calculated with respect to the proper time $s$). We denote the latter derivative by $\nabla_0$. In the next step we express the covariant derivatives in terms of ordinary derivatives ("dots").

To obtain this expression for $V_i$ we introduce the constant laboratory vector field $\frac{\partial}{\partial t} = K^\mu = (1, 0, 0, 0)$ and note that

$$(1, V^k) = V^\mu = \frac{dq^\mu}{dt} = \frac{ds}{dt}\frac{dq^\mu}{ds} = \frac{ds}{dt} u^\mu, \tag{27}$$

and

$$-1 = V^\mu K_\mu = \frac{ds}{dt} u^\mu K_\mu \Rightarrow \frac{ds}{dt} = -\frac{1}{u^\mu K_\mu}, \tag{28}$$

where we used $(-, +, +, +)$ signature for the spacetime metric. This gives us

$$V^\mu = -\frac{1}{u^\mu K_\mu} u^\mu \, . \tag{29}$$

Hence, we have for higher derivatives:

$$V^\mu_{i+1} = -\frac{1}{u^\mu K_\mu} \nabla_0 V^\mu_i \, . \tag{30}$$

The above recurrence can be used to calculate $V_i$ up to required accuracy — for our purposes only terms up to $V_4^\mu$ are necessary. We have: $\nabla_0 u^\mu = a^\mu$. To simplify further our notation we denote $\nabla_0 a^\mu =: a_1^\mu$ and $\nabla_0 a_i^\mu =: a_{i+1}^\mu$. This way we obtain:

$$V^\mu = -\frac{1}{u^\lambda K_\lambda} u^\mu, \tag{31}$$

$$V_1^\mu = -\frac{1}{u^\lambda K_\lambda} \nabla_0 V^\mu = \frac{1}{(u^\lambda K_\lambda)^2} \left( a^\mu + a^\lambda K_\lambda V^\mu \right), \tag{32}$$

$$V_2^\mu = \frac{1}{(u^\lambda K_\lambda)^2} \left( 3\, a^\lambda K_\lambda V_1^\mu - \frac{1}{u^\lambda K_\lambda} \left( a_1^\mu + a_1^\lambda K_\lambda V^\mu \right) \right), \tag{33}$$

$$V_3^\mu = \frac{1}{(u^\lambda K_\lambda)^2} \left( 6\, a^\lambda K_\lambda V_2^\mu - \frac{1}{u^\lambda K_\lambda} \left( 4\, a_1^\lambda K_\lambda V_1^\mu - \frac{1}{u^\lambda K_\lambda} \left( a_2^\mu + a_2^\lambda K_\lambda V^\mu - 3 \left( a^\lambda K_\lambda \right)^2 V_1^\mu \right) \right) \right), \tag{34}$$

$$V_4^\mu = \frac{10\, a_\lambda K^\lambda}{(u^\lambda K_\lambda)^2} \left( V_3^\mu - \frac{V_2^\mu}{u^\lambda K_\lambda} \right) + \frac{1}{(u^\lambda K_\lambda)^4} \left( 5\, a_2^\lambda K_\lambda V_1^\mu - 15 \left( a^\lambda K_\lambda \right)^2 V_2^\mu - \frac{1}{u^\lambda K_\lambda} \left( a_3^\mu + a_3^\lambda K_\lambda V^\mu + 10\, a_1^\lambda K_\lambda\, a^\nu K_\nu\, V_1^\mu \right) \right). \tag{35}$$

Now, we calculate explicitly covariant derivatives in the Fermi frame: $\nabla_0\, a_i^\mu = \dot a_i^\mu + \Gamma^\mu_{0\lambda} a_i^\lambda$, using (26) for the connection coefficients (dot denotes the ordinary proper-time derivative in the Fermi frame). This way we obtain:

$$a^\mu = (0, a^k), \tag{36}$$

$$a_1^\mu = (a_i a^i, \dot a^k), \tag{37}$$

$$a_2^\mu = (3\, a_i \dot a^i, \ddot a^k + a_i a^i\, a^k), \tag{38}$$

$$a_3^k = (\dddot a^k + 5\, a_i \dot a^i\, a^k + a_i a^i\, \dot a^k), \tag{39}$$

$$V^\mu = (1, 0, 0, 0), \tag{40}$$

$$V_1^k = a^k, \tag{41}$$

$$V_2^k = \dot a^k, \tag{42}$$

$$V_3^k = \ddot a^k - 3\, a_i a^i\, a^k, \tag{43}$$

$$V_4^k = \dddot a^k - 10\, a_i \dot a^i\, a^k - 9\, a_i a^i\, \dot a^k\ , \tag{44}$$

and $V_i^0 = 0$ as expected.

**3.2.2. Transformation of dipole momenta**

The four-current density (5), (6) carried by the moving dipole may be rewritten in the following, four dimensional notation

$$\mathcal{J}^\mu = -\partial_\lambda \left( u^\mu W^\lambda \delta_\zeta \right) + \left( u^\lambda \partial_\lambda W^\mu \right) \delta_\zeta \, , \tag{45}$$

where

$$u^\mu = \frac{1}{\sqrt{1 - V^k V_k}} V^\mu \ , \tag{46}$$

and

$$\delta_\zeta = \sqrt{1 - V^k V_k} \, \delta(\vec{x} - \vec{q}(t)) \ . \tag{47}$$

The distribution $\delta_\zeta$ is the Dirac delta assigned in a fully intrinsic (geometric) way to the trajectory $\zeta := \{(t, x^k) | x^k = q^k(t)\}$. Hence, it does not depend upon any choice of a reference system. On the contrary, the description of the dipole moment in terms of the 3-dimensional, laboratory quantity $(W^\mu) = (0, W^k)$ is reference-dependent. To avoid this dependence we are going to replace it by a four-vector $P = (p^\mu)$ *orthogonal to the trajectory* in such a way that the current (45) does not change. This is possible, because of the following identity:

$$\partial_\lambda \left( u^\mu f u^\lambda \delta_\zeta \right) - u^\lambda \left( \partial_\lambda f u^\mu \right) \delta_\zeta = f u^\mu \partial_\lambda \left( u^\lambda \delta_\zeta \right) \equiv 0 \ , \tag{48}$$

where $f$ is any time-dependent scalar. This proves, that the quantity $W^\mu$ in formula (45) may be replaced by $p^\mu := W^\mu + f u^\mu$ and the electric current $\mathcal{J}^\mu$ will not change. To define the vector $P$ uniquely in terms of the current, we impose the orthogonality condition $p^\mu u_\mu = 0$.

The best way to express $W$ in terms of $P$ consists in using formulae: $W^\mu = p^\mu - f u^\mu$ and $W^\mu K_\mu = 0$. This way we obtain:

$$W^\mu = \left( \delta^\mu_\lambda - \frac{1}{u^\lambda K_\lambda} u^\mu K_\lambda \right) p^\lambda \, . \tag{49}$$

In the Fermi frame we have $(u^\mu) = (1, 0, 0, 0,)$ and $P$ becomes a three dimensional quantity: $p^\mu = (0, p^k)$. Hence, the electric current density (45) reduces to:

$$\mathcal{J}^0 = -p^k \partial_k \delta_\zeta \, , \qquad \mathcal{J}^k = \dot{p}^k \delta_\zeta \, . \tag{50}$$

We are going to use the three dimensional vector $\vec{P} = (p^k)$ as the reference-invariant representation of the particle's dipole moment. The quantity $W$ and its derivatives with respect to the laboratory time $t$, which appear in the power series expansion of the field, will be expressed in terms of $\vec{P}$ and its derivatives with respect to the proper time $s$. For this purpose we use formula (49) and its derivatives. Similarly as for velocities $V_i$ we have: $W^\mu_{i+1} = -(1/(u^\mu K_\mu))\nabla_0 W^\mu_i$, where $W^\lambda_0 = W^\lambda$. Putting also $p^\lambda_0 = p^\lambda$ and $p^\lambda_{i+1} = \nabla_0 p^\lambda_i = \dot{p}^\mu_i + \Gamma^\mu_{0\lambda} p^\lambda_i$, we obtain the following recurrence:

$$W^\mu_0 = \left(\delta^\mu_\lambda - \frac{1}{u^\lambda K_\lambda} u^\mu K_\lambda\right) p^\lambda_0 \,, \tag{51}$$

$$W^\mu_1 = -\frac{1}{u^\nu K_\nu} \nabla_0 W^\mu_0 \tag{52}$$

$$= -\frac{1}{u^\lambda K_\lambda}\left(p^\mu_1 - \frac{1}{u^\lambda K_\lambda} K_\lambda \nabla_0 \left(u^\mu p^\lambda_0\right) + \frac{a^\nu K_\nu}{u^\lambda K_\lambda}\left(p^\mu_0 - W^\mu_0\right)\right), \tag{53}$$

$$W^\mu_2 = \frac{1}{(u^\nu K_\nu)^2}\left(3\, a^\nu K_\nu \, W^\mu_1 + p^\mu_2 + \frac{2a^\nu K_\nu}{u^\lambda K_\lambda} p^\mu_1 + \frac{a^\nu_1 K_\nu}{u^\lambda K_\lambda}\left(p^\mu_0 - W^\mu_0\right)\right.$$
$$\left. - \frac{1}{u^\mu K_\mu} K_\lambda \nabla_{00}\left(u^\mu p^\lambda_0\right)\right), \tag{54}$$

$$W^\mu_3 = \frac{1}{(u^\nu K_\nu)^2}\left(6\, a^\nu K_\nu \, W^\mu_2 - \frac{1}{u^\mu K_\mu}\left(4\, a^\nu_1 K_\nu \, W^\mu_1 + p^\mu_3\right)\right.$$
$$- \frac{1}{(u^\mu K_\mu)^2}\left(3\, a^\nu_1 K_\nu \, p^\mu_1 + a^\nu_2 K_\nu \left(p^\mu_0 - W^\mu_0\right) - K_\lambda \nabla_{000}\left(u^\mu p^\lambda_0\right)\right.$$
$$\left.\left. + 3\left(a^\nu K_\nu\right)^2 W^\mu_1 + 3\, a^\nu K_\nu \, p^\mu_2\right)\right). \tag{55}$$

In order to simplify technical obstacles while computing $W^\mu_4$ we can make use of the fact that for initial time $t = 0$ we have: $a^\nu K_\nu = 0$, $(p^\mu_0 - W^\mu_0) = 0$, and $(p^\mu_1 - W^\mu_1) = 0$. Then:

$$W^\mu_4 = \frac{1}{(u^\nu K_\nu)^3}\left(-10\, a^\nu_1 K_\nu \, W^\mu_2 + \frac{1}{u^\nu K_\nu}\left(5\, a^\nu_2 K_\nu \, W^\mu_1 + p^\mu_4\right)\right. \tag{56}$$
$$\left. + \frac{1}{(u^\nu K_\nu)^2}\left(6\, a^\nu_1 K_\nu \, p^\mu_2 + 4\, a^\nu_2 K_\nu \, p^\mu_1 - K_\lambda \nabla_{0000}\left(u^\mu p^\lambda_0\right)\right)\right). \tag{57}$$

The corresponding recurrence for $p_i$ reads:

$$p^\mu_0 = (0, p^k) \,, \tag{58}$$

$$p^\mu_1 = (a_k p^k, \dot{p}^k) \,, \tag{59}$$

$$p_2^\mu = (\dot a_k p^k + 2\, a_k \dot p^k, \ddot p^k + a_i p^i\, a^k)\,, \tag{60}$$

$$p_3^\mu = (3\, a_k \ddot p^k + \ddot a_k p^k + 3\, \dot a_k \dot p^k + a_i a^i a_k p^k, \dddot p^k + a_i p^i\, \dot a_k \tag{61}$$

$$+(2\, \dot a_i p^i, +3\, a_i \dot p^i)\, a^k)\,, \tag{62}$$

$$p_4^k = \ddddot p^k + a_i p^i\, \ddot a^k + (3\, \dot a_i p^i + 4\, a_i \dot p^i)\, \dot a^k \tag{63}$$

$$+(3\, \ddot a_i p^i + 6\, a_i \ddot p^i + 8\, \dot a_i \dot p^i + a_i a^i \dot a_l p^l)\, a^k\,. \tag{64}$$

Comparing these two recurrences we finally obtain:

$$W_0^k = p^k\,, \tag{65}$$

$$W_1^k = \dot p^k\,, \tag{66}$$

$$W_2^k = \ddot p^k - a_i p^i\, a^k, \tag{67}$$

$$W_3^k = \dddot p^k - a_i a^i \dot p^k - 2\, a_i p^i\, \dot a^k - (\dot a_i p^i + 3\, a_i \dot p^i)\, a^k\,, \tag{68}$$

$$W_4^k = \ddddot p^k - 15\, a_i \dot a^i \dot p^k - 10\, a_i a^i \ddot p^k + a_i p^i\, \ddot a^k + (3\, \dot a_i p^i + 4\, a_i \dot p^i)\, \dot a^k \tag{69}$$

$$+(3\, \ddot a_i p^i + 6\, a_i \ddot p^i + 8\, \dot a_i \dot p^i + 11\, a_i a^i\, a_l p^l)\, a^k\,, \tag{70}$$

and $W_i^0 = 0$ as expected.

## 4. The results

The results of this paper consist in calculating the power series expansion of (8) and (9) (together with definitions (10)–(14)) with respect to the distance $r$. For this purpose the trajectory $q(t)$ is taken in the form (15) with $V_0 = 0$. The retarded (advanced) time is given explicitly by formulae (17)–(22). As a second step we translate the results into the standardized form, i. e. we replace all the quantities $V$'s, $W$'s and their derivatives with respect to the laboratory time by the quantities $a^k$, $p^k$ and their proper-time derivatives. For this purpose we use formulae (40)–(44) and (65)–(70).

The expansion was performed with help of the MAPLE V r.5 system for symbolic computation. The algorithm outlined here exists as a program LW'99 and may be downloaded from the internet site of the Department of Mathematical Methods in Physics, University of Warsaw:
`http://info.fuw.edu.pl /~ kmmf/marcin/startlw.html`

Below we give results for both the monopole and the dipole field. We use the following notation. By $M$ we mean vector $M := \vec{x}/|\vec{x}|$, uppercase letters denote three dimensional vectors, any pair of lowercase letters stands for the scalar product of two vectors *i.e.* $pa\, ma = p_i a^i\, m_i a^i$, dots over letters indicate proper time derivatives in the Fermi frame (*i.e.* $\dot a, \ddot a, \dot P, \ddot P, \dddot P, \ddddot P$), $\varepsilon$ equals $+1$ for the advanced and $-1$ for the retarded solution.

The corresponding results for the magnetic dipole may be easily obtained by the duality transformation: $D \to B$ and $B \to -D$. More precisely, this transformation may be used in vacuum case only (*i.e.* outside of the particle). The globally defined dual fields are given by the following formula:

$$B^k_{\text{new}} := D^k_{\text{old}} + P^k \delta^{(3)} , \tag{71}$$

$$D^k_{\text{new}} := -B^k_{\text{old}} . \tag{72}$$

It is easy to check that the new fields fulfill Maxwell equations with the electromagnetic current corresponding to the magnetic dipole: $\mathcal{J}^0 = 0$, $\mathcal{J}^k = \varepsilon^{klm} P_l \partial_m \delta^{(3)}$.

We draw attention of the reader to the fact, that the only terms in the expansion, which are regular ($C^\infty$-derivable) are those containing $\varepsilon$. They depend upon a specific (advanced or retarded) solution. The remaining, singular terms are universal and depend only upon the sources (encoded by $a$, $p$ and their time derivatives). These singular terms may be calculated using a different, purely algebraic procedure (see [6]). The agreement of these terms with the results of [6] is an important validity test for the expansion given below.

### *4.1. The field of a monopole*

$$\underset{-2}{D} = e \frac{M}{r^2} , \tag{73}$$

$$\underset{-1}{D} = -e \frac{1}{2} \frac{ma\, M + A}{r} , \tag{74}$$

$$\underset{0}{D} = e \left[ \left( -\frac{3}{8} aa + \frac{3}{8} ma^2 \right) M + \frac{3}{4} ma\, A - \frac{2}{3} \varepsilon \dot{A} \right] , \tag{75}$$

$$\underset{1}{D} = e \left[ \left( \frac{9}{16} aa\, ma + \frac{1}{8} m\ddot{a} - \frac{5}{16} ma^3 \right) M + \left( \frac{3}{16} aa - \frac{15}{16} ma^2 \right) A \right.$$
$$\left. -\frac{3}{8} \ddot{A} + \frac{2}{3} \left( -a\dot{a}\, M + m\dot{a}\, A + 2\, ma\, \dot{A} \right) \varepsilon \right] r , \tag{76}$$

$$\underset{-1}{B} = e \frac{1}{2} \frac{M \times \dot{A}}{r} , \tag{77}$$

$$\underset{0}{B} = e \left[ -\frac{1}{2} m\dot{a}\, M \times A - \frac{3}{4} ma\, M \times \dot{A} - \frac{1}{4} A \times \dot{A} + \frac{1}{3} \varepsilon M \times \ddot{A} \right] , \tag{78}$$

$$\underset{1}{B} = e\left[\frac{5}{4}\,ma\,m\dot{a}\,M\times A + \left(-\frac{3}{16}\,aa + \frac{15}{16}\,ma^2\right) M\times\dot{A} + \frac{1}{8}M\times\dddot{A} + \frac{5}{8}\,ma\,A\times\dot{A} + \frac{1}{3}\left(-m\ddot{a}\,M\times A - 2\,m\dot{a}\,M\times\dot{A} - 2\,ma\,M\times\ddot{A} - \frac{1}{2}A\times\ddot{A}\right)\varepsilon\right] r\,. \tag{79}$$

### *4.2. The field of an electric dipole*

$$\underset{-3}{D} = \frac{-P + 3\,mp\,M}{r^3}\,, \tag{80}$$

$$\underset{-2}{D} = \frac{\left(-\frac{3}{2}\,ma\,mp + \frac{1}{2}\,pa\right)M + \frac{1}{2}\,ma\,P - \frac{1}{2}\,mp\,A}{r^2}\,, \tag{81}$$

$$\underset{-1}{D} = \left[\left(-\frac{1}{4}\,pa\,ma + \frac{9}{8}\,ma^2\,mp - \frac{3}{8}\,aa\,mp - \frac{1}{2}\,m\ddot{p}\right)M + \left(\frac{3}{4}\,ma\,mp - \frac{1}{4}\,pa\right)A + \left(-\frac{3}{8}\,ma^2 + \frac{3}{8}\,aa\right)P - \frac{1}{2}\,\ddot{P}\right] r^{-1}\,, \tag{82}$$

$$\underset{0}{D} = \left(\frac{3}{16}\,pa\,aa + \frac{9}{16}\,ma\,aa\,mp - \frac{3}{4}\,\ddot{p}a + \frac{1}{2}\,m\dot{a}\,m\dot{p} - \frac{15}{16}\,ma^3\,mp - \frac{1}{2}\,\dot{p}\dot{a} + \frac{3}{16}\,ma^2\,pa - \frac{1}{8}\,p\ddot{a} + \frac{3}{4}\,ma\,m\ddot{p} + \frac{1}{8}\,m\ddot{a}\,mp\right)M + \left(-\frac{15}{16}\,ma^2\,mp + \frac{3}{8}\,pa\,ma + \frac{3}{4}\,m\ddot{p} - \frac{3}{16}\,aa\,mp\right)A + m\dot{p}\,\dot{A} + \frac{3}{8}\,mp\,\ddot{A} + \left(\frac{5}{16}\,ma^3 - \frac{1}{8}\,m\ddot{a} - \frac{9}{16}\,aa\,ma\right)P + \frac{3}{4}\,ma\,\ddot{P} + \frac{2}{3}\left(a\dot{a}\,P + aa\,\dot{P} - \dddot{P}\right)\varepsilon\,, \tag{83}$$

$$\underset{1}{D} = \left[\left(\frac{1}{8}\,m\dddot{p} + \frac{1}{4}\,m\dot{a}\,\dot{p}a + \frac{9}{8}\,ma\,\ddot{p}a + \frac{5}{24}\,mp\,\dot{a}\dot{a} - \frac{5}{16}\,ma\,m\ddot{a}\,mp + \frac{3}{16}\,ma\,p\ddot{a} + \frac{3}{4}\,ma\,\dot{p}\dot{a} - \frac{1}{16}\,m\ddot{a}\,pa + \frac{5}{16}\,mp\,a\ddot{a} - \frac{15}{128}\,aa^2\,mp + \frac{7}{8}\,m\dot{p}\,a\dot{a} - \frac{45}{64}\,ma^2\,mp\,aa - \frac{5}{4}\,m\dot{a}\,ma\,m\dot{p} - \frac{5}{24}\,m\dot{a}^2\,mp + \frac{7}{16}\,m\ddot{p}\,aa - \frac{15}{16}\,ma^2\,m\ddot{p} + \frac{1}{24}\,m\dot{a}\,p\dot{a} - \frac{5}{32}\,ma^3\,pa\right.$$

$$
\begin{aligned}
&+\frac{105}{128}\,ma^4\,mp-\frac{9}{32}\,pa\,aa\,ma\Bigg)M+\left(\frac{1}{16}\,p\ddot{a}+\frac{1}{4}\,\dot{p}\dot{a}\right.\\
&-\frac{15}{8}\,ma\,m\ddot{p}+\frac{3}{8}\,\ddot{p}a-\frac{5}{4}\,m\dot{a}\,m\dot{p}-\frac{15}{32}\,ma^2\,pa+\frac{15}{32}\,ma\,aa\,mp\\
&\left.-\frac{5}{16}\,m\ddot{a}\,mp-\frac{3}{32}\,pa\,aa+\frac{35}{32}\,ma^3\,mp\right)A+\Bigg(-\frac{5}{2}\,ma\,m\dot{p}\\
&-\frac{5}{6}\,m\dot{a}\,mp+\frac{1}{2}\,\dot{p}a+\frac{7}{24}\,p\dot{a}\Bigg)\dot{A}+\left(\frac{3}{16}\,pa-\frac{15}{16}\,ma\,mp\right)\ddot{A}\\
&+\Bigg(\frac{5}{24}\,m\dot{a}^2+\frac{5}{16}\,ma\,m\ddot{a}+\frac{5}{24}\,\dot{a}\dot{a}-\frac{15}{128}\,aa^2+\frac{5}{16}\,a\ddot{a}\\
&-\frac{35}{128}\,ma^4+\frac{45}{64}\,aa\,ma^2\Bigg)P+\frac{9}{8}\,a\dot{a}\,\dot{P}\left(-\frac{15}{16}\,ma^2+\frac{9}{16}\,aa\right)\ddot{P}\\
&-\frac{3}{8}\dddot{P}+\Bigg(\Bigg(-\frac{1}{2}\,\dot{p}\dot{a}+\frac{9}{16}\,mp\,aa\,ma+\frac{1}{2}\,m\dot{a}\,m\dot{p}+\frac{3}{16}\,aa\,pa\\
&+\frac{15}{16}\,mp\,ma^3-\frac{3}{4}\,\ddot{p}a+\frac{1}{8}\,mp\,m\ddot{a}-\frac{1}{8}\,p\ddot{a}+\frac{3}{16}\,pa\,ma^2\\
&+\frac{3}{4}\,ma\,m\ddot{p}\Bigg)M+\Bigg(-\frac{3}{16}\,mp\,aa+\frac{3}{4}\,m\ddot{p}-\frac{15}{16}\,mp\,ma^2\\
&+\frac{3}{8}\,pa\,ma\Bigg)A+\left(-\frac{2}{5}\,mp\,aa+\frac{4}{3}\,m\ddot{p}\right)\dot{A}+\frac{4}{15}\,mp\,\dddot{A}\\
&+\left(-\frac{4}{3}\,a\dot{a}\,ma-\frac{2}{5}\,aa\,m\dot{a}-\frac{1}{15}\,m\dddot{a}\right)P-\frac{4}{3}\,aa\,ma\,\dot{P}\\
&\left.+\frac{2}{3}\,m\dot{a}\,\ddot{P}+\frac{4}{3}\,ma\,\dddot{P}\Bigg)\varepsilon\right]r\,, \qquad (84)
\end{aligned}
$$

$$
\underset{-2}{B} = -\frac{M\times\dot{P}}{r^2}, \qquad (85)
$$

$$
\begin{aligned}
\underset{-1}{B} &= \left[\frac{1}{2}\,ma\,M\times\dot{P}+m\dot{p}\,M\times A+\frac{1}{2}\,mp\,M\times\dot{A}-\frac{1}{2}\,P\times\dot{A}\right.\\
&\left.-\frac{1}{2}\,\dot{P}\times A\right]r^{-1}, \qquad (86)
\end{aligned}
$$

$$
\begin{aligned}
\underset{0}{B} = {} & \left(-\frac{3}{2}\,ma\,m\dot{p}-\frac{1}{2}\,m\dot{a}\,mp\right)M\times A+\Bigg(-\frac{3}{4}\,ma\,mp \\
& -\frac{1}{4}\,pa\Bigg)M\times\dot{A}+\left(-\frac{1}{8}\,aa-\frac{3}{8}\,ma^2\right)M\times\dot{P}+\frac{1}{2}\,M\times\dddot{P} \\
& +\frac{3}{4}\,ma\,P\times\dot{A}+\frac{3}{4}\,ma\,\dot{P}\times A+\frac{1}{4}\,mp\,A\times\dot{A}+\frac{1}{2}\,m\dot{a}\,P\times A \\
& -\frac{1}{3}\left(P\times\ddot{A}+2\dot{P}\times\dot{A}\right)\varepsilon, \qquad (87)
\end{aligned}
$$

$$
\begin{aligned}
\underset{1}{B} = {} & \Bigg[\Bigg(\frac{1}{8}\,aa\,m\dot{p}+\frac{5}{4}\,ma\,mp\,m\dot{a}+\frac{1}{18}\,pa\,m\dot{a}-\frac{1}{2}\,m\dddot{p}+\frac{7}{36}\,a\dot{a}\,mp \\
& +\frac{15}{8}\,ma^2\,m\dot{p}\Bigg)M\times A+\Bigg(\frac{15}{16}\,ma^2\,mp+\frac{41}{72}\,pa\,ma-\frac{1}{144}\,aa\,mp \\
& -\frac{3}{4}\,m\ddot{p}\Bigg)M\times\dot{A}-\frac{1}{2}\,m\dot{p}\,M\times\ddot{A}-\frac{1}{8}\,mp\,M\times\dddot{A}+\Bigg(\frac{7}{36}\,aa\,m\dot{a} \\
& -\frac{7}{36}\,a\dot{a}\,ma\Bigg)M\times P+\left(\frac{3}{16}\,aa\,ma-\frac{1}{8}\,m\ddot{a}+\frac{5}{16}\,ma^3\right)M\times\dot{P} \\
& -\frac{1}{2}\,m\dot{a}\,M\times\ddot{P}-\frac{3}{4}\,ma\,M\times\dddot{P}+\left(-\frac{19}{18}\,ma\,m\dot{a}-\frac{7}{36}\,a\dot{a}\right)P\times A \\
& +\left(-\frac{15}{16}\,ma^2-\frac{1}{16}\,aa\right)\dot{P}\times A+\frac{1}{4}\,\dddot{P}\times A+\Bigg(\frac{55}{144}\,aa \\
& -\frac{163}{144}\,ma^2\Bigg)P\times\dot{A}-\frac{1}{4}\,\ddot{P}\times\dot{A}-\frac{3}{8}\,\dot{P}\times\ddot{A}-\frac{1}{8}\,P\times\dddot{A} \\
& +\left(-\frac{31}{72}\,ma\,mp-\frac{5}{72}\,pa\right)A\times\dot{A}+\frac{1}{3}\Big(-(p\dot{a}+2\,\dot{p}a)\,M\times\dot{A} \\
& -pa\,M\times\ddot{A}-a\dot{a}\,M\times\dot{P}-aa\,M\times\ddot{P}+M\times\ddddot{P}+m\ddot{a}\,P\times A \\
& +2\,m\dot{a}\,P\times\dot{A}+2\,ma\,P\times\ddot{A}+2\,m\dot{a}\,\dot{P}\times A+4\,ma\,\dot{P}\times\dot{A} \\
& +2\,m\dot{p}\,A\times\dot{A}+mp\,A\times\ddot{A}\Big)\varepsilon\Bigg]r\,. \qquad (88)
\end{aligned}
$$

## Appendix

All the technicalities related to differentiation with respect to the parameter $l$ are somewhat similar to the standard procedure used to obtain classical Liénard–Wiechert fields from potentials. In both cases we have to differentiate objects that depend on a variable in both explicit and implicit way (via the variable $\tau$). First we should derive formula for $\frac{\partial \tau}{\partial l}$. Let us expand relation (4):

$$(\tau^l_\pm - t)^2 - ||\vec{q}_l(\tau^l_\pm)||^2 + 2\left(\vec{q}_l(\tau^l_\pm)|\vec{x}\right) - ||\vec{x}||^2 = 0, \tag{89}$$

where by $\pm$ we denote advanced or retarded solution. The superscript "$l$" reminds us that our variables depend additionally on the parameter $l$. Note that:

$$\frac{d\vec{q}_l(\tau^l_\pm)}{dl} = \frac{\partial \vec{q}_l(\tau^l_\pm)}{\partial l} + \frac{\partial \vec{q}_l(\tau^l_\pm)}{\partial \tau}\frac{\partial \tau}{\partial l}\,. \tag{90}$$

We can differentiate both sides of (89) with respect to $l$:

$$\frac{\partial \tau^l_\pm}{\partial l}\left(1 + \left(\frac{\vec{x}-\vec{q}}{\pm|\vec{x}-\vec{q}|}\,\middle|\,\frac{\partial \vec{q}_l(\tau^l_\pm)}{\partial \tau}\right)\right) + \left(\frac{\vec{x}-\vec{q}}{\pm|\vec{x}-\vec{q}|}\,\middle|\,\frac{\partial \vec{q}_l(\tau^l_\pm)}{\partial l}\right) = 0\ , \tag{91}$$

and make use of defined previously (we put $t = 0$):

$$W := e\ \left.\frac{\partial \vec{q}}{\partial l}\right|_{l=0}\ .$$

Finally:

$$e\left.\frac{\partial \tau^l_\pm}{\partial l}\right|_{l=0} = -\varepsilon\frac{nw}{1+\varepsilon\, nv} \quad \text{thus :} \quad e\frac{d\vec{q}_l(\tau^l_\pm)}{dl} = W + Ve\frac{\partial \tau}{\partial l}\,. \tag{92}$$

The procedure outlined here should be then applied to all variables present in formula for monopole's field. Derivatives of velocity and acceleration can be obtained in the same way as $e\frac{d\vec{q}_l(\tau^l_\pm)}{dl}$ and all necessary scalars constructed out of vectors:

$$e\ \frac{\partial \tau}{\partial l} = -\varepsilon\frac{nw}{1+\varepsilon\, nv}, \tag{93}$$

$$e\ \frac{d}{dl}(nv) = -\frac{1}{r}\left(wv + (vv+\varepsilon\, nv)e\frac{\partial \tau}{\partial l}\right) + nw_1 + na\, e\frac{\partial \tau}{\partial l}, \tag{94}$$

$$e\ \frac{d}{dl}V = W_1 + A_0\, e\frac{\partial \tau}{\partial l}, \tag{95}$$

$$e \ \frac{d}{dl}(na) = -\frac{1}{r}\left(wa + (av + \varepsilon\, na)e\frac{\partial\tau}{\partial l}\right) + nw_2 + na_1\, e\frac{\partial\tau}{\partial l}, \tag{96}$$

$$e \ \frac{d}{dl}A = W_2 + A_1\, e\frac{\partial\tau}{\partial l}, \tag{97}$$

$$e \ \frac{\partial N}{\partial l} = -\frac{1}{r}\left(W + (N + \varepsilon\, V)e\,\varepsilon\frac{\partial\tau}{\partial l}\right). \tag{98}$$

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