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THE GEOMETRY OF NONCOMMUTATIVE SYMMETRIES

Mario Paschke

Institut für Physik, Johannes-Gutenberg Universität,
55130 Mainz, Germany
e-mail: paschke@thep.physik.uni-mainz.de

and Andrzej Sitarz†

Laboratoire de Physique Théorique, Batiment 210, Université Paris XI
91405 Orsay Cedex, France
and Institute of Physics, Jagellonian University
Reymonta 4, 30059 Kraków, Poland
e-mail: Andrzej.Sitarz@th.u-psud.fr

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We discuss the notion of noncommutative symmetries based on Hopf algebras in the geometric models constructed within the framework of noncommutative geometry. We introduce and discuss several notions of *noncommutative symmetries* and outline the construction specific examples, for instance, finite algebras and the application of symmetries in the derivation of the Dirac operator for the noncommutative torus.

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1. Aspects of symmetries

It is generally believed that the description of physical systems should have built-in symmetries, which reflect the symmetries of space and associated geometrical structures. In particular, the action functional is required to be invariant under these symmetries, whereas the equations of motions are covariant. The *Noether theorem* relates symmetries of the action with conserved quantities (integrals of motion).

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In quantum field theory the representation of symmetries on the Hilbert space of particles is generated by the conserved charges. The Hilbert space decomposes into irreducible representations and only charged operators, *i.e.* operators which do not commute with the charges, can interpolate between inequivalent representations. The latter form the so-called *superselection sectors*. For instance, one cannot prepare a system in a state, which is a superposition of spin-$\frac{1}{2}$ and spin-1 states (at least, such a state has never been observed). Moreover, by virtue of the symmetries one derives certain identities of amplitudes, the *Ward–Takahashi-* and *Slavnov-Taylor-identities*. They are essential to prove both *renormalisability* and *unitarity*, which are important physical consistency requirements ensuring the "predictive power" of the theory.

The most important examples of symmetries in field theory come from *gauge symmetries* and *Poincaré invariance*. The attempts to extend them have led to concepts that generalized the notion of a group. For instance, in supersymmetric theories, which provide some of the most studied (both theoretically and experimentally) extensions of the standard model, the gauge groups and the Poincaré group are unified in a supergroup.

Other possibilities such as Hopf algebra symmetries might also play an important role in Quantum Field Theories, as has been suggested recently. For instance, they can be used to describe the superselection sectors in certain low dimensional theories (see [13] for details and references). There is also some speculation about the Hopf algebra structure underlying the renormalisability of perturbative Quantum Field Theory [11, 12].

In this paper we shall present some new concepts of symmetries built in the field theory based on noncommutative geometry and formulated in the language of spectral triples. We propose the definitions of quantum symmetries of spectral geometry based on the action (or coaction) of the Hopf algebra. The paper is organized as follows: in section 2 we briefly review the ideas of symmetries in noncommutative models as well as the notation of Hopf algebras and spectral triples. In section 3 we present the proposed definitions of invariant spectral triples (3.1), illustrating it with results on the properties of such objects (differential calculi), and results on symmetries of finite geometries (3.2). Finally, we present a derivation of the spectrum of the invariant Dirac operator for the noncommutative torus. This provides an excellent illustration of the introduced definitions and, although the spectrum is well known, to the best of our knowledge such a derivation following from the symmetry principles has not been published elsewhere.

2. Symmetries of noncommutative spaces

The notion of symmetry in geometry is related with groups and Lie algebras. A space X is said to have a certain symmetry if a group G acts on it. In Noncommutative Geometry, where no notion of space is present, this picture can no longer be used. We are left with two possible ways of generalizations, both coming from the classical picture. First of all, we know that the action of a group on some space X induces automorphisms of the algebra of functions on X. Hence, we can take generalized symmetries as automorphisms of the (not necessarily commutative) algebra. A very good example is the noncommutative interpretation of the Standard Model, which changes the interpretation of known physical symmetries. So, for instance, gauge symmetries are now interpreted as "internal diffeomorphisms".

Another option is to dualize the picture of a group action, which means that we represent it as a coaction of the Hopf algebra $C(G)$ (functions on the group) on the algebra $C(X)$ (functions on the space). Here, the key role of the symmetry is played by the Hopf algebra. This suggests that in the noncommutative case we should also consider these objects as generalized symmetries. A broad class of "noncommutative spaces" with symmetries understood in this context comes from quantum groups and quantum homogeneous spaces. One should stress that at present, there are no known spectral triples for such spaces. Their geometry is usually described by bicovariant differential calculi and Haar measures (if they exist). It is therefore an interesting and important task to build a bridge from quantum group theory to noncommutative geometry based on the Dirac operator.

Within the approach of *spectral triples* symmetries can have yet another meaning. With the basic data consisting of an algebra $\mathcal{A}$, its representation on the Hilbert space $\mathcal{H}$ and the Dirac operator D, we can consider as symmetries transformations of unitary equivalence. These include, of course, mainly the automorphisms of $\mathcal{A}$ ($\mathcal{H}$ is some closure of an appropriate module over $\mathcal{A}$), but also unitaries U, which commute with the algebra, and therefore do not represent diffeomorphisms. The action of physical models based on spectral triples — for instance the Connes–Chamseddine action — only depend on the unitary equivalence class of the data $\mathcal{A}, \mathcal{H}, \gamma, D$, and J.

For the quantization of such a model, it is very important to take into account all its symmetries. In particular, the path integral should only be taken over the space of equivalence classes of field configurations. For instance, if the action is a functional of the eigenvalues of the Dirac operator, such as the bosonic action of the Standard Model, the path integral should be formulated as an integral over the allowed range of these eigenvalues. For simple examples of discrete spectral triples such an invariant mesasure can indeed be defined [18].

One should mention, however, that the eigenvalues of the Dirac operator are, in general, not the only degrees of freedom of such models.

For example, let us consider the discrete spectral triple which is used in the noncommutative description of the Standard Model. Recall that the Dirac operator of this discrete spectral triple is physically interpreted as the fermionic mass matrix. Hence, it contains the fermion masses (its eigenvalues), but also the unitary Cabibbo-Kobayashi-Maskawa matrix as additional degrees of freedom. The latter results from the fact that one cannot simultaneously diagonalize the representation of the algebra and the Dirac operator in the space $\mathbb{C}^3$ of quark-families.

The unitary transformations U which leave all physical observables (the fermion masses *and* the entries of the CKM-matrix) invariant, in the sense that two mass matrices, $\mathcal{M}_1$ and $\mathcal{M}_2$, related by $\mathcal{M}_1 = U\mathcal{M}_2U^*$ describe the same physics, can then obviously be characterized by the requirement that they commute with the grading γ, the charge conjugation J and the representation of the algebra.

Note that in the classical case, where the algebra $\mathcal{A}$ is that of functions on a manifold M, the above condition that it commute with everything except D, states that U is a map from M to the representation of the spin group on spinors.

2.1. Hopf algebras and spectral triples

2.1.1. Hopf algebras

Hopf algebras are the natural generalization (from the algebraic point of view) of the concept of groups and lie algebras and therefore the natural candidate for generalized symmetries. For a comprehensive review of Hopf algebras see, for instance, [1–4].

Here, we shall only recall the notation, for the Hopf algebra H the *coproduct* Δ, is an algebra homomorphism $\Delta : H \to H \otimes H$ (we shall often use Sweedler's notation: $\Delta a = \sum a_{(1)} \otimes a_{(2)}$); the *counit*, an homomorphism $\varepsilon : H \to \mathbb{C}$ and the *antipode*, an antihomomorphism $S : H \to H$, *i.e.*, $S(ab) = S(b)S(a)$.

The notion of an action of a group on a space is reflected by a *coaction* of the Hopf algebra H on an algebra $\mathcal{A}$. A right coaction is given as an algebra homomorphism: $\alpha : \mathcal{A} \to \mathcal{A} \otimes H$, such that $(\alpha \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \Delta)\alpha$ and $\mathrm{id} = (\mathrm{id} \otimes \varepsilon)\alpha$. A good model for that is the coaction of the algebra of functions on a group on the algebra of functions on its homogeneous space. Similarly, Hopf algebras can also *act* on algebras. A left action of H on $\mathcal{A}$, is a map $H \otimes \mathcal{A} \ni h \otimes a \to h \triangleright a$, with the following properties: $1 \triangleright a = a$, $h \triangleright 1 = \varepsilon(h)$, $h \triangleright (g \triangleright a) = (hg) \triangleright a$ and $h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright a)$. To see the commutative example one should consider the action of the lie algebra on the space on which the group acts.

Among possible differential structures on an algebra $\mathcal{A}$ one can single out those on which the Hopf algebra H acts (or coacts). For the Hopf algebra itself, one may distinguish the left (respectively, right) invariant differential calculi, defined by the property that there exist a left (right) coaction β of H on the space of one-forms, which commutes with the external derivative:

$$\beta(da) = (d \otimes \mathrm{id})\Delta a \,.$$

Calculi which are left and right covariant are *bicovariant* [6]. In a similar way one introduces left and right covariant calculi on arbitrary algebras on which Hopf algebras coact or act.

2.1.2. Spectral triples

Spectral triples have been introduced [7] in order to provide an algebraic description of compact spin manifolds. The basic data, which defines a spectral geometry consist of a C^* algebra $\mathcal{A}$, its faithfull representation as bounded operators on a Hilbert space $\mathcal{H}$ and an unbounded selfadjoint operator D, which must satisfy certain algebraic (for instance that its commutators with a dense subalgebra of $\mathcal{A}$ are also bounded) and analytic conditions (for details, we refer to [7], simple finite-dimensional examples are discussed in [8]).

Such data provide information both on topology (C^* algebra), as well as as differential and metric structures of the "noncommutative manifold", which are contained in the operator D. Additional information such as the grading operator (for even-dimensional geometries) and reality operator (for real spectral geometries) further restrict the realm of possibilities to the case of compact real spin manifold. In this situation, the algebra $\mathcal{A}$ is the algebra of continuous functions, while D is the Dirac operator acting on the space of square integrable sections of the spinor bundle.

3. Symmetries of the Dirac operator

One of the most important tasks in field theory models is the construction of the Dirac operator for spin manifolds and characterization of its properties. A similar problem is encountered in the noncommutative case, where, in most examples, we need to know the Dirac operator in advance to construct feasible models. However, just as in the classical case, for some special algebras (manifolds) there might exist a natural method, which uses symmetries of the underlying space as the guiding principle.

Suppose we have an algebra $\mathcal{A}$ for which we would like to construct the spectral data and we know that the Hopf algebra H acts on it. We define a spectral triple with the symmetry H:

Definition 3.1. Let $\mathcal{A}$, with its representation on $\mathcal{H}$, D, γ, J be a spectral triple. Then we call it invariant under the action of H if:

- H acts on the algebra $\mathcal{A}$,
- $\mathcal{H}$ is the representation space of the cross-product of $\mathcal{A}$ and H, *i.e.*, H is represented on $\mathcal{H}$ and the representation obeys:

$$h(av) = \big(h_{(1)} \triangleright (a)\big)\,(h_{(2)}v), \quad \forall a \in \mathcal{A}, h \in H, v \in \mathcal{H}\,, \tag{1}$$

- the Dirac operator D commutes with the representation of H:

$$[D,h] = 0, \;\; \forall h \in H\,. \tag{2}$$

In the classical case, with $\mathcal{A}$ commutative and H a cocommutative Hopf algebra (the group algebra of G or the universal envelope of its Lie-algebra) the second condition is equivalent to H being a G-homogeneous bundle, while the third one states that the metric is invariant under the action of the symmetries, *i.e.* the latter act as isometries.

If H is a compact matrix pseudo-group [5] or its dual (the generalization of a compact group respectively its Lie-algebra) $\mathcal{H}$ decomposes into the finite dimensional irreducible representations of H. One can then use techniques from harmonic analysis to work out the Hilbert-space and, with the help of the order-one condition, also the Dirac operator. We shall illustrate this in the example of the noncommutative torus in the last section of this chapter.

As dual notation (*i.e.* the coaction of the Hopf algebra H on the algebra $\mathcal{A}$ we propose:

Definition 3.2. Let $\mathcal{A}$, with its representation on $\mathcal{H}$, D, γ, J be a spectral triple. Then we call it left invariant under the coaction of H, if:

- H coacts from the left on the algebra $\mathcal{A}$, $\alpha : \mathcal{A} \to H \otimes \mathcal{A}$,
- $\mathcal{H}$ is the left corepresentation of H, so there exists a corepresentation map $\tilde{\alpha} : \mathcal{H} \to H \otimes \mathcal{H}$,
- the coaction and corepresentations commute with each other:

$$\tilde{\alpha}(av) = \alpha(a)\tilde{\alpha}(v), \quad \forall a \in \mathcal{A}, v \in \mathcal{H}\,, \tag{3}$$

- the Dirac operator D commutes with the corepresentation of H:

$$\tilde{\alpha}(Dv) = (\mathrm{id} \otimes D)\tilde{\alpha}(v), \;\; \forall v \in H. \tag{4}$$

3.1. The symmetries of the differential calculi

The above discussed notions of symmetries for the Dirac operator of spectral geometries have immediate consequences for the differential structures encoded in the spectral data. We prove the following:

Lemma 3.3. *Let $\mathcal{A}, \mathcal{H}, D$ be the spectral triple invariant under the action of the Hopf algebra H (as defined in (3.1). Then there exists the action of H on the first-order differential structure determined by the spectral data.*

Proof. Let us simply define $(h \triangleright da) = [D, (h \triangleright a)]$ and extend on the entire bimodule of one-forms through:

$$h \triangleright (adb) = (h_{(1)} \triangleright a) d(h_{(2)} \triangleright b)\,. \tag{5}$$

Clearly this is a well-defined operation, in fact we might define the action of H on any operator O through the identity:

$$h \triangleright O = h_{(1)} O S h_{(2)}. \tag{6}$$

Indeed, we verify that then for every operator O, $h \in H$ and $v \in \mathcal{H}$ we have:

$$\begin{aligned} hOv &= h_{(1)} O \varepsilon(h_{(2)}) v = h_{(1)} O (S h_{(2)}) h_{(3)} v \\ &= \big(h_{(1)} O S h_{(2)}\big)\, h_{(3)} v = \big(h_{(1)} \triangleright O\big)\, h_{(2)} v. \end{aligned}$$

Clearly, this action is a proper action of the Hopf algebra on a bimodule over $\mathcal{A}$ (all operators form a natural bimodule over $\mathcal{A}$), we shall check here only the compatibility with the left module structure:

$$\begin{aligned} h \triangleright (aO) &= h_{(1)} a O S h_{(2)} = h_{(1)} a (S h_{(2)}) h_{(3)} O S h_{(4)} \\ &= (h_{(1)} \triangleright a)(h_{(2)} \triangleright O). \end{aligned}$$

In the case of the considered bimodule of one-forms it is enough to verify that the action maps one-forms onto one-forms, however, this is guaranteed by the definition (5) and the invariance of D. $\square$

The invariance of spectral triples under the action and coaction of the Hopf algebra and the resulting properties of the differential calculus, shall be adressed in a forthcoming paper, with attention focused on the finite case, where the relations between the action and coaction approach can be established (see also [8, 18]).

3.2. The case of finite algebras

The simplest examples of spectral triples as noncommutative geometries are given by finite-dimensional semisimple $*$-algebras. The classification and the rules of constructing the Dirac operator were discussed in [8, 9].

Let us recall that every such algebra (over $\mathbb{C}$) is a finite direct sum of simple matrix algebras. Then the full spectral triple over it is defined by the intersection form matrix q_{ij}, that comes from the bilinear map

$$q \quad : \quad K(\mathcal{A}) \times K(\mathcal{A}) \longrightarrow \mathbb{Z}\,,$$

which is induced by the spectral triple. From the matrix q_{ij} one then obtains the dimension of the representation space, grading and reality operators, as well as the structure of the Dirac operator.

More explicitly, for an algebra $\oplus_{i=1}^{k} M_{n_i}(\mathbb{C})$, q_{ij} is a symmetric non-degenerate $(k \times k)$-matrix with integer entries. The representation space decomposes as $H = \oplus_{i,j} H_{ij}$, where

$$H_{ij} = \mathbb{C}^{n_i |q_{ij}| n_j}.$$

The algebra acts on H_{ij} by left multiplication with the i-th component M_{n_i}.

The grading γ is diagonal with its restriction to the space H_{ij} being q_{ij} times the identity. The reality J is an antilinear operator, which maps H_{ij} to H_{ji}. Finally, the Dirac operator can connect spaces H_{ij} and H_{kl} (of different grading) only if $i = k$ or $j = l$. In the case $i = k$ (or $j = l$ respectively) D must commute with the left action (right action respectively) of M_{n_i} (M_{n_j}) on H_{ij} and H_{il} (H_{ij} and H_{kj}). This is a consequence of the order one condition. Pictorially, if one associates the spaces $\mathcal{H}_{ij}$ to the matrix-elements of q_{ij}, then D acts only along the rows and the columns of the matrix q_{ij}.

3.2.1. The S_0-reality example

The first example has a direct relation to Connes' interpretation of the Standard Model geometry. Consider a $\mathbb{Z}_2$-symmetry, which acts trivially on the algebra $\mathcal{A}$ (hence, as a symmetry it is not a symmetry of the space but only of the fibres of the spin-bundle). It can be represented as an operator S such that $S^2 = 1$ and $[S, a] = 0$ for every $a \in \mathcal{A}$, we require also that it commutes with γ and anticommutes with J. Such symmetry is apparently present in the NCG description of the Standard Model, namely the so-called S_0-reality, which assigns the eigenvalue $+1$ to particles, and -1 to antiparticles.

The physical Dirac operator is invariant under the S_0-reality. However, the most general possible Dirac operator for the Standard Model spectral triple is not invariant. A general Dirac operator would then necessarily lead to additional couplings between leptons and antiquarks, violating the conservation of lepton and baryon numbers and enforcing the existence of scalar particles (leptoquarks), which might break SU(3) color symmetry.

One should mention here that although leptoquarks have not been observed yet, they are experimentally not completely excluded. However, without requiring the S_0-reality the noncommutative description of the Standard Model would lead to unrealistic predictions [10].

3.2.2. Dirac operators for finite Hopf algebras

Suppose that a given semisimple finite algebra allows the Hopf algebra structure. (Note that for one algebra there may exist several inequivalent Hopf algebra structures.)

What can be said of restrictions which this Hopf algebra structure imposes on the construction of spectral triples? There are several possible approaches. We shall briefly show the directions and illustrate them with examples.

Adjoint symmetry

The adjoint action of the Hopf algebra on itself is defined by:

$$h \triangleright_{ad} p = h_{(1)} p (S h_{(2)}). \tag{7}$$

Clearly, every representation of the Hopf algebra is adjoint covariant in the sense of (1):

$$hgv = h_{(1)} g \varepsilon(h_{(2)}) v = h_{(1)} g S(h_{(2)}) h_{(3)} v = (h_{(1)} \triangleright_{ad} g) h_{(2)} v.$$

Then, clearly, the invariance of the Dirac operator with respect to the adjoint action implies that D commutes with the representation of the algebra, and the differential algebra is thus trivial.. However, we may relax the condition and require that D is invariant not under the action of the entire Hopf algebra H but only under the action of its sub-Hopf-algebra $H_0 \subset H$.

Bicovariance of differential structures

Another possibility to restrict the freedom of choice of the Dirac operator, much weaker than the one mentioned in the previous section, is the requirement that the differential structure generated by D is (at least in the first order) bicovariant (or left covariant).

Let us take a simple example of a commutative finite algebra (functions on a discrete group G). The generators of the left-covariant differential calculus are one-forms χ^g, which, in the spectral triple representation are the following operators, using the decomposition of the representation space we have $\forall_{i,j\in G}\chi^g_{ij} : H_{(ig^{-1})j} \to H_{ij}$ and it is easy to verify that $\chi^g_{ij} = D_{ij,(ig^{-1})j}$. First of all the left covariance of a calculus in which χ^g is present enforces that for all i there must exist a j such that $D_{ij,(ig^{-1})j}$ does not vanish, otherwise we would have $e_h\chi^g = 0$, which cannot happen.

Now let us verify whether the calculus given by D is bicovariant. It is known that the generators of such a calculus correspond to orbits of the adjoint action in G. Suppose that for a given $g \quad \chi^g$ vanishes, which means that $\forall i,j \in G \quad D_{ij,(ig^{-1})j} = 0$. Then in order to preserve the bicovariance one must have $D_{ij,(ih^{-1})j} = 0$ for all h in the adjoint orbit of g.

Another simple, interesting example comes from studying group algebras and their spectral triples. This is more interesting since the algebras are no longer necessarily commutative. For instance, the smallest nonabelian group S_3 has the group algebra $M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$. The bicovariant (or, in fact, left covariant, because of cocommutativity) calculi on group algebras correspond to the representations of the group algebra. In the case of $\mathbb{C}S_3$ it happens, that the calculus either has a splitting property (so that it is a direct sum of the calculus on $M_2(\mathbb{C})$ and the calculus on $\mathbb{C} \oplus \mathbb{C}$) or has a central element in the bimodule of one forms. Neither of these is possible for the calculi obtained from the (finite) spectral triples. So the result is, that no Dirac operator respects the $\mathbb{C}S_3$ symmetry in that sense.

Invariance under the action of the dual Hopf algebra

Again, as an illustrative example we shall consider spectral triples built on finite Hopf algebras. The notion of symmetry, which we introduce now is stronger than the previous one. We shall use the canonical action of the dual Hopf algebra $\mathcal{H}^*$ on $\mathcal{H}$: $(h \triangleright a) =< h, a_{(2)} > a_{(1)}$, where $<,>$ is the pairing between the Hopf algebra and its dual. For instance, take a commutative algebra of functions on a group $C(G)$ with its basis $e_g, g \in G$ and its dual, the group algebra $\mathcal{G}$ with the basis $g \in G$. Then the action, expressed in the above basis becomes:

$$g \triangleright e_h = e_{hg^{-1}}.$$

To proceed with the notion of invariance as proposed in definition (3.1) we need to construct the representation of the crossproduct of $\mathcal{H}^*$ and $\mathcal{H}$. The set of algebraic rules for the cross product algebra is given by:

$$e_h e_g = e_h \delta_{gh}, \quad g e_h = e_{hg^{-1}} g,$$

for all elements g of G. We construct the spectral triple for the subalgebra generated by e_h, which is a representation space for the entire cross product algebra and we look for a Dirac operator that commutes with all $g \in G$.

Using the known form of the finite spectral triple we establish first that g act as an operator $g : \oplus_j H_{ij} \to \oplus H_{(ig^{-1})j}$. Let us denote its restriction to H_{ij}, which maps it to $H_{(ig^{-1})l}$ by $g_{ij,l}$. (Note, that, since g is invertible, H_{ij} and $H_{(ig^{-1})l}$ must be of the same dimension if $g_{ij,l} \neq 0$.)

Now, since D commutes with every g we have:

$$\sum_m D_{kl,(ig^{-1})m} g_{ij,m} = \sum_m g_{(kg)m,l} D_{(kg)m,ij},$$

which after taking into account the restriction on D becomes:

$$D_{kj,(ig^{-1})j} g_{ij,j} = g_{(kg)j,j} D_{(kg)j,ij},$$

and

$$\sum_m D_{(ig^{-1})l,(ig^{-1})m} g_{ij,m} = \sum_m g_{im,l} D_{im,ij}.$$

Thus, depending on the spectral triple and the chosen representation of the cross product, we obtain a severe restriction on possible Dirac operators. Physically, the symmetry requirement is a constraint for the fermionic content of the theory, especially their masses. In a few simple examples, that we worked out, these masses, in fact, are fixed (up to a scale) by the above equations on D. Note, that, since the algebra here is commutative there cannot be any mixing of fermions.

The trace and the Haar measure

Finally, let us briefly mention the possibility of using the Haar measure on the Hopf algebra. In the general theory of spectral triples the scalar product of forms is defined through the trace of the representation of the algebra (and differential forms). However, one may modify it, for instance by introducing different weight coefficients for each component of our semisimple algebra. If the algebra is the Hopf algebra, there exists a unique choice of such coefficients, which gives the normalized Haar measure.

The physical consequence of such a choice is the change of possible mass relations within the model.

3.3. Quantum torus: noncommutative space with classical symmetries

After presenting our ideas of symmetries for spectral triples we would like to show their application in the most renowned example in noncommutative geometry: the quantum torus and its symmetries.

Consider the group $\mathrm{U}(1)\times\mathrm{U}(1)$. It has two commuting generators δ_1, δ_2:

$$[\delta_1, \delta_2] = 0,$$

and, accordingly, all its its irreducible representations V_{nm} are one-dimensional, characterized by two integers n, m :

$$\begin{aligned} \delta_1|n,m\rangle &= n|n,m\rangle\,, \\ \delta_2|n,m\rangle &= m|n,m\rangle\,, \qquad\qquad |n,m\rangle \in V_{nm}. \end{aligned}$$

Being a Lie-algebra, the coproduct of the generators is, of course,

$$\triangle\delta_i = \mathrm{id}\otimes\delta_i + \delta_i\otimes\mathrm{id}\,.$$

We are now looking for algebras which contain each irreducible representation exactly once.

Suppose we have two unitaries U, V on which the generators δ_1, δ_2 act in the following way:

$$\begin{aligned} \delta_1 U = U, \qquad\qquad & \delta_2 U = 0\,, \\ \delta_1 V = 0, \qquad\qquad & \delta_2 V = V\,. \end{aligned}$$

Now, clearly the element $U^{-1}V^{-1}UV$ is annihilated both by δ_1 and δ_2, therefore the above symmetry-requirement $\mathrm{U}(1)\times\mathrm{U}(1)$ enforces that $U^{-1}V^{-1}UV$ must be proportional to the identity operator, hence we obtain:

$$UV = \lambda VU, \qquad\qquad |\lambda| = 1,$$

which is, of course, the defining relation for the noncommutative torus.

The noncommutative torus is therefor the most general algebra with two generators possessing the required $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry.

The only covariant representation (up to equivalence) of this algebra is then the one corresponding to the free module of rank 1. In the Hilbert space H_0 there exists a orthonormal basis $|n,m\rangle$ and the representation is given by

$$\begin{aligned} U|n,m\rangle &= |n+1,m\rangle, \\ V|n,m\rangle &= \lambda^{-n}|n,m+1\rangle. \end{aligned}$$

In particular $|0,0\rangle$ is a cyclic separating vector, and the Tomita–Takesaki-theorem provides J_0 as

$$J_0|n,m\rangle = \lambda^{-nm}|-n,-m\rangle.$$

In order to obtain γ and the Dirac operator D one doubles the Hilbert-space $H = H_0 \oplus H_0$ and sets

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad\qquad J = \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}.$$

Being selfadjoint and anticommuting with γ, D is clearly of the form

$$D = \begin{pmatrix} 0 & \partial \\ \partial^* & 0 \end{pmatrix}.$$

Since we would like the group $\mathrm{U}(1)\times\mathrm{U}(1)$ to act as isometries, we require that D commutes with the two generators δ_1, δ_2. As we shall see, this assumption fixes ∂ up to a normalization factor.

Let us denote the basis of common eigenvectors of $\delta_1, \delta_2, \gamma$ in H by $|n, m, \pm\rangle$. It follows that

$$\partial|n, m, -\rangle = d_{n,m}|n, m, +\rangle,$$

and the complex numbers $d_{n,m}$ are to be calculated from the order-one condition $\left[[D, a], b^0\right] = 0$, $\forall a, b \in \mathcal{A}$. This directly leads to the recursion relations between the coefficients $d_{n,m}$:

$$\begin{aligned} d_{n+2,m} &= 2d_{n+1,m} - d_{n,m}, \\ d_{n,m+2} &= 2d_{n,m+1} - d_{n,m} \end{aligned}$$

with the solution (up to a normalization and a constant term)

$$d_{n,m} = n + m\tau \qquad\qquad \tau \in \mathbb{C},$$

The obtained result agrees, of course, with the usual Dirac operator on the noncommutative torus. However, one should stress, that as compared to usual constructions, here we have derived it from the order-one condition. Moreover we have used the symmetries to find the representation of $\mathcal{A}$, which corresponds to the spin-bundle. Note that by doing so, one also introduces the norm on the algebra.

Along the same the lines, on can construct spectral triples for other Lie-algebras. For su(2) and the algebra of functions on the two-sphere, this has been done in [18]. In that case, there are infinitely many su(2)-homogeneous line-bundles, from which the spin-bundle could be constructed. The latter is identified by the requirements that γ be a Hochschild-cycle and that there exists a reality structure J.

An interesting question, which arises in this context is whether also non-cocommutative Hopf-algebras can serve as isometries. For instance, there exists a two-parameter family of deformations $S^2_{q,\lambda}$ of the algebra of functions on the sphere, which for $q = 1, \lambda = 0$ agree with the classical case. In the general case, for any λ there exists however an action of $\mathcal{U}_q(\mathrm{su}(2))$ (the q-deformed universal envelope of su(2)) on this algebra. With the help of their symmetries, one can attempt to construct spectral triples for the quantum sphere. Work in this direction is in progress [17].

4. Conclusions

Symmetries are very important and fundamental in physics and it is commonly believed that they are the key to our understanding of particle physics and gravity and many other phenomena. Perhaps one of the most interesting open questions in this respect is whether symmetries — in the extended sense (supersymmetries, Hopf algebra symmetries) — are as fundamental as group symmetries and what role they play (if any) in particle physics.

Both in theoretical and experimental physics one would like to verify whether there are any "new" symmetries , which are broken at low energies, when physics is effectively described by the Standard Model. For instance, it is an exciting question, whether there is any symmetry behind the observed masses and mixing matrices. Another thing that requires further studies are the possible relations between the finite spectral triples, which are the geometrical setup for the Standard Model and finite Hopf algebras originating from quantum groups at roots of unity — for details see [14–16].

The above discussed examples of possibilities for extended types of symmetries and their realisation open new paths, even within the physics of the Standard Model. They could also provide guiding principles to find new extensions of the Standard model and, even more, they could lead to a better understanding of Quantum (Field) Theory.

It seems appropriate to add at this moment that all symmetries in physics must be verified experimentally. At present, it is rather likely that various extensions of the Standard Model (or at least some of them) might be tested within the next years. Whether this will confirm new symmetries (supersymmetry, for instance) or not is hard to speculate.

Having formulated classical (in the sense of the vanishing Planck constant) field theories on noncommutative spaces, the next task will be the quantisation of such models. In view of the importance of symmetries for the consistency of quantum field theories on commutative spaces, we strongly believe that it will be unavoidable to exploit all the symmetries of these models as the first step.

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