PHASE-COHERENCE AND AMPLITUDE-COHERENCE

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Weight function $P(\alpha)$ in the diagonal representation of density operator, $\rho = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$, is reduced to define separately the weight functions for phase, $\arg(\alpha)$, and amplitude, $|\alpha|$, which leads to concepts of phase-coherence and amplitude-coherence. For a single mode phase-coherent field, it is shown that *(i)* we can have Hermitian operator of form, $a\rho e^{i\psi}$, where a is annihilation operator and ψ is a constant, and *(ii)* the normally ordered characteristic function, $\chi_N(\xi)$, is a function of only the imaginary part of $\xi e^{i\psi}$. For a single mode amplitude-coherent field, it is shown that $a\rho a^\dagger = ka^\dagger$, where $a^\dagger$ is creation operator and k is a positive real constant. When the weight function for the field is non-classical, each of the reduced weight function may or may not be non-classical irrespective of the nature of the other. Examples of generation of phase-coherent and amplitude-coherent fields are given.

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1. Introduction

The concept of coherence can be traced back to the Young's double slit experiment which gives rise to interference fringes. The superposition of optical beams leads to interference fringes depending upon the coherence of the beams. This phenomenal definition of the coherence leads to the fact that coherence is due to correlation of optical signals at two space-time points separated in space or in time or in both [1]. The early experiments involved the observations of quantities which were quadratic in field strength and demonstrated such correlations, but the milestone experiment of Brown and Twiss [2] and latter experiments in nonlinear optics involved quantities which are higher orders in field strength.

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Optical signals have random variations in both amplitudes and phase. Classical study of coherence, thus involves a probability distribution function containing, for each mode, one amplitude and one phase variable. If the analytic signal for a single mode field is

$$v = v_r + iv_i = x\mathrm{e}^{i\theta}\,, \quad x = |v|\,, \quad \theta = \arg v\,, \tag{1.1}$$

properties of the field are described by coherence functions defined by

$$\begin{aligned} \Gamma^{(m,n)} &= \langle v^{*m} v^n \rangle = \int d^2v f(v) v^{*m} v^n\,, \\ d^2 v &\equiv dv_r dv_i = x\,dx\,d\theta\,, \end{aligned} \tag{1.2}$$

where (n,m) is the order of coherence function, $\langle\ \rangle$ denotes ensemble average and $f(v)d^2v$ is the probability for analytic signal to be in the interval d^2v at v. In quantum theory of coherence, the probability distribution function is replaced by the weight function of the diagonal representation for the density operator. For each mode, coherent states $|\alpha\rangle$ are defined by

$$a|\alpha\rangle = \alpha|\alpha\rangle\,, \quad \alpha = \alpha_r + i\alpha_i = x\mathrm{e}^{i\theta}\,. \tag{1.3}$$

Here a is annihilation operator and α is any complex number. The density operator ρ can be written in the form [3, 4],

$$\rho = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|\,, \tag{1.4}$$

where $P(\alpha)$ is the weight function which has the obvious properties of being a real function and having normalization,

$$\int d^2\alpha P(\alpha) = 1\,, \quad d^2\alpha \equiv d\alpha_r d\alpha_i = x\,dx\,d\theta\,. \tag{1.5}$$

Whenever $P(\alpha)$ is non-negative definite, $P(\alpha)$ plays the role of a probability distribution function, properties of the field can be described by classical optics and $\alpha = x\mathrm{e}^{i\theta}$ corresponds to classical amplitude x and phase variable θ. On the other hand, when $P(\alpha)$ takes negative values also, the field can be described only quantum mechanically. Quantum analogue of Eq. (1.2) is

$$\Gamma^{(m,n)} = \mathrm{Tr}\left[\rho a^{\dagger m} a^n\right] = \int d^2\alpha P(\alpha)\alpha^{*m}\alpha^n\,. \tag{1.6}$$

Kelly and Kleiner [5] discussed separately the distribution functions for amplitude and phase by considering the special case where $P(\alpha)$ can be factorized and written as a product of function $Q(|\alpha|)$ of the amplitude $|\alpha|$

and a function $R(\theta_\alpha)$ of the phase angle θ_α where $\alpha = |\alpha|\mathrm{e}^{i\theta_\alpha}$. Chandra, Prakash and Vachaspati [6, 7] introduced the concept of phase-coherence by considering a Dirac-delta function type distribution of classical phase of a single mode field as revealed by the definition (1.2) of the coherence functions. Eq. (1.2) gives

$$\Gamma^{(m,n)} = B_{m+n}\mathrm{e}^{i(m-n)\psi}\,, \tag{1.7}$$

where $B_{m+n} = \langle x^{m+n}\rangle$, in a special case when the random phase variable θ takes only one fixed value — ψ. Chandra, Prakash and Vachaspati defined phase-coherence with the help of Eq. (1.7) taking B_{m+n} and ψ real. A less stringent case

$$\Gamma^{(m,n)} = B_{m,n}\mathrm{e}^{i(m-n)\psi} \tag{1.8}$$

defines quasi-phase-coherence. These authors introduced S-representation [6–8] for ρ

$$\rho = \int dx\, dy S(x,y) \left|x\mathrm{e}^{-i\psi}\right\rangle \left\langle y\mathrm{e}^{-i\psi}\right| \left(\left\langle x\mathrm{e}^{-i\psi}\middle| y\mathrm{e}^{-i\psi}\right\rangle\right)^{-1} \tag{1.9}$$

and showed that $S(x,y)$ is real for quasi-phase-coherence, if phase is peaked at ψ, and that, for phase-coherence, $S(x,y)$ involves $\delta(x-y)$. The authors also demonstrated conversion of chaotic light into quasi-phase-coherent and phase-coherent light in degenerate parametric amplification [6, 7]. Here it is emphasized that these authors considered only the classical notion of the phase and we also follow the same in this paper [9].

In this paper, we discuss reduced phase- and amplitude-weight functions of optical fields. We also discuss concepts of absolute phase-coherence and amplitude-coherence. Section 2 deals with the definitions. In Section 3, we derive some properties of phase and amplitude coherent optical fields and discuss how such fields can be identified, if ρ or $P(\alpha)$ or the characteristic function is given. In Section 4, non-classical nature of reduced weight functions is discussed. Finally, in Section 5, examples of producing only phase- and amplitude-coherent optical fields are described.

2. Reduced phase and amplitude weight functions, and phase- and amplitude-coherence

Without demanding the factorization of $P\left(x\mathrm{e}^{i\theta}\right)$ into functions of x and θ (*cf.* Kelly and Kleiner [5]), we can obtain such functions by reducing $P\left(x\mathrm{e}^{i\theta}\right)$. The reduced phase and amplitude distribution functions can be defined by

$$P_p^{\mathrm{R}}(\theta) \equiv \int_0^\infty dx\, xP\left(x\mathrm{e}^{i\theta}\right)\,, \quad P_a^{\mathrm{R}}(x) \equiv \int_0^{2\pi} d\theta x P\left(x\mathrm{e}^{i\theta}\right)\,, \tag{2.1}$$

with the normalization properties,

$$\int_0^{2\pi} d\theta P_p^{\rm R}(\theta) = 1\,, \quad \int_0^{\infty} dx\, P_a^{\rm R}(x) = 1\,, \tag{2.2}$$

obtained from Eqs. (1.5). As an example, the coherent state $|x_0 {\rm e}^{i\theta}\rangle$ (with x_0 real and positive) has the weight function,

$$\delta^2\left(x{\rm e}^{i\theta} - x_0{\rm e}^{i\theta_0}\right) = x_0^{-1}\delta(x-x_0)\delta(\theta-\theta_0)\,, \tag{2.3}$$

i.e., it has both a stabilized amplitude and a stabilized phase.

Definition of phase-coherence as given by Chandra, Prakash and Vachaspati, Eq. (1.7) with real B_{m+n} and ψ, demands that $P\left(x{\rm e}^{i\theta}\right)$ must involve $\delta(\theta+\psi)$ and $\delta(\theta+\psi+\pi)$ in the most general case and is of the form,

$$P\left(x{\rm e}^{i\theta}\right) = P_1(x)\delta(\theta+\psi) + P_2(x)\delta(\theta+\psi+\pi) \tag{2.4}$$

giving

$$P_p^{\rm R}(\theta) = A_1\delta(\theta+\psi) + A_2\delta(\theta+\psi+\pi)\,, \quad A_i \equiv \int_0^{\infty} dx\, xP_i(x)\,. \tag{2.5}$$

We can define the field as absolutely phase-coherent, if only one delta function is involved.

In principle, if all coherence functions are given, we can find P_1 and P_2 separately. Eqs. (1.6), (1.7) and (2.4) give

$$B_{m+n} = \int_0^{\infty} dx\, x^{m+n+1}\left[P_1(x) + (-1)^{m+n}P_2(x)\right]\,. \tag{2.6}$$

If we define

$$Q(x) \equiv xP_1(x)U(x) - xP_2(-x)U(-x)\,, \tag{2.7}$$

where $U(x)$ is the unit step function defined by $U(x)=1$ for $x>0$, $U(x)=1/2$ for $x=0$, and $U(x)=0$, for $x<0$, we have

$$B_{m+n} = \int_{-\infty}^{\infty} dx\, Q(x)x^{m+n}\,, \tag{2.8}$$

which shows that constants B's are the moments of function $Q(x)$. Information of all B's determines Q and therefore P_1 and P_2. Formally, one may write

$$Q(x) = \sum_{n=0}^{\infty} (-1)^n (n!)^{-1} B_n \frac{d^n}{dx^n}[\delta(x)] \tag{2.9}$$

and therefore

$$P\left(xe^{i\theta}\right) = x^{-1}[Q(x)\delta(\theta+\psi) + Q(-x)\delta(\theta+\psi+\pi)\,. \tag{2.10}$$

We can similarly discuss the case where reduced amplitude weight function $P_a^{\mathrm{R}}(x)$ is of the form $\delta(x-x_0)$. Here x_0 is a positive constant and we call the field as amplitude-coherent. For such a field, Eqs. (1.6) and (2.1) give

$$\Gamma^{(m,n)} = x_0^{m+n} A_{m-n}\,, \quad A_{m-n} = \int_0^{2\pi} d\theta P_p^{\mathrm{R}}(\theta) e^{-i(m-n)\theta}\,. \tag{2.11}$$

If all coherence function $\Gamma^{(m,n)}$ of such a field are known, obviously,

$$\begin{aligned} P_p^{\mathrm{R}}(\theta) &= (2\pi)^{-1} \sum_{n=-\infty}^{n=\infty} A_n e^{in\theta}\,; \\ P\left(xe^{i\theta}\right) &= P_p^{\mathrm{R}}(\theta) x_0^{-1} \delta(x-x_0)\,. \end{aligned} \tag{2.12}$$

3. Some properties of phase-coherent and amplitude-coherent fields

Eqs. (1.4) and (2.4) give

$$\rho = \int_0^{\infty} dx\, x \left[P_1(x) \left|xe^{-i\psi}\right\rangle \left\langle xe^{-i\psi}\right| + P_2(x) \left|xe^{-i(\psi+\pi)}\right\rangle \left\langle xe^{-i(\psi+\pi)}\right|\right]\,. \tag{3.1}$$

This equation and the definition in Eq. (1.3) of coherent states show that

$$a\rho = \rho a^\dagger e^{-2i\psi} \tag{3.2}$$

or, that $a\rho e^{i\psi}$ is Hermitian. It is easily seen that this statement is equivalent to the definition, Eq. (1.7). Incidentally, Eq. (3.2) also tells why this definition of phase-coherence fails to distinguish between phases ψ and $\psi+\pi$. If we write $\beta = \alpha e^{i\psi}$ and use Eq. (2.7), we can write Eq. (3.1) in the form,

$$\rho = \int\limits_{-\infty}^{\infty} dx\, xQ(x) \left| x e^{-i\psi} \right\rangle \left\langle x e^{-i\psi} \right| , \tag{3.3}$$

or in the form,

$$\rho = \int d^2\beta P'(\beta) \left| \beta e^{-i\psi} \right\rangle \left\langle \beta e^{-i\psi} \right| , \quad P'(\beta) = \beta_r Q(\beta_r)\delta(\beta_i) \, . \tag{3.4}$$

Eq. (3.3) expresses ρ in the S-representation [6–8] with a weight function involving $\delta(x-y)$. A comparison of Eq. (3.4) with Eq. (1.4) shows that the weight function $P(\alpha) = P'(\alpha e^{i\psi})$ involves $\delta(\mathrm{Im}\ \alpha e^{i\psi})$ for phase-coherent fields. The Fourier transform of weight function, the normally ordered characteristic function, $\chi_n(\xi)$, is a function of only the imaginary part of $\xi e^{i\psi}$. This can be seen as follows:

$$\begin{aligned}
\chi_N(\xi) &= \mathrm{Tr}\left[\rho e^{\xi a^\dagger} e^{-\xi^* a}\right] \\
&= \sum_{m,n} (m!n!)^{-1} \xi^m (-\xi^*)^n \Gamma^{(m,n)} \\
&= \sum_{m,n} B_{m+n} (m!n!)^{-1} \left(\xi e^{i\psi}\right)^m \left(-\xi^* e^{-i\psi}\right)^n \\
&= \sum_{p} B_p (p!)^{-1} \left(\xi e^{i\psi} - \xi^* e^{-i\psi}\right)^p \, .
\end{aligned}$$

For amplitude-coherent field Eqs. (1.4) and (2.12) give

$$\rho = \int\limits_{0}^{2\pi} d\theta\, P_p^{\mathrm{R}}(\theta) \left| x_0 e^{i\theta} \right\rangle \left\langle x_0 e^{i\theta} \right| \tag{3.5}$$

and therefore the property,

$$a\rho a^\dagger = x_0^2 \rho \, , \tag{3.6}$$

where x_0^2 is a positive quantity.

4. Non-classical phase and amplitude weight functions

A field is said to be non-classical if $P(\alpha)$ is not non-negative definite. In an interesting paper, Glauber [3] introduced non-classical properties of field by considering a simple case where the weight function is a difference of two functions, one of which is a displaced Gaussian and the other is a Dirac-delta function. We can follow Glauber and construct an example where weight function is given by,

$$P\left(x\mathrm{e}^{i\theta}\right) = A\delta^2\left(x\mathrm{e}^{i\theta} - x_1\mathrm{e}^{i\phi}\right) + B\delta^2\left(x\mathrm{e}^{i\theta} - x_2\mathrm{e}^{i\chi}\right) - C\delta^2\left(x\mathrm{e}^{i\theta} - x_1\mathrm{e}^{i\chi}\right). \tag{4.1}$$

Here A, B and C are real constants with $A + B - C = 1$. The field described by this weight function is non-classical as $P\left(x\mathrm{e}^{i\theta}\right)$ is not non-negative definite. The reduced weight functions are

$$P_p^{\mathrm{R}}(\theta) = A\delta(\theta - \phi) + (B - C)\delta(\theta - \chi) \tag{4.2}$$

and

$$P_a^{\mathrm{R}}(x) = (A - C)\delta(x - x_1) + B\delta(x - x_2)\,, \tag{4.3}$$

each of which may be classical or non-classical depending upon the relative values of A, B and C. This clearly shows that for a non-classical field, we can have all four cases, viz;

1. non-classical $P_p^{\mathrm{R}}(\theta)$ and $P_a^{\mathrm{R}}(x)$,
2. classical $P_p^{\mathrm{R}}(\theta)$ and $P_a^{\mathrm{R}}(x)$,
3. non-classical $P_p^{\mathrm{R}}(\theta)$ and classical $P_a^{\mathrm{R}}(x)$ and
4. classical $P_p^{\mathrm{R}}(\theta)$ and non-classical $P_a^{\mathrm{R}}(x)$.

5. Examples of generation of phase-coherent and amplitude-coherent optical fields

Random phase-modulation of coherent state was discussed by Estes, Kuppenheimer and Narducci [10] and by Picinbono [11]. As coherent state has stabilized amplitude and phase in the sense of Section 2, this should generate amplitude-coherent field. Hamiltonian [10] $H = (\omega + f(t))a^\dagger a$ with a Gaussian random variable f leads to time evolution operator

$$U = \exp\left[-i\left\{\omega t + \int_0^t f(t)dt\right\}a^\dagger a\right]$$

and hence to

$$a(t) = a\exp\left[-i\left\{\omega t + \int_0^t f(t)dt\right\}a^\dagger a\right].$$

For initial state given by $\rho(0) = |\alpha_0\rangle\langle\alpha_0|,\ \alpha = x_0 e^{i\theta_0}$ on taking average over the random variable $f(t)$, using

$$\langle f(t_1) f(t_2)\rangle = \sigma^2 \delta(t_1 - t_2) \tag{5.1}$$

we then have[1]

$$\begin{aligned} \Gamma^{(m,n)} &= \left\langle \alpha_0^{*m} \alpha_0^n \exp\left[i(m-n) \int_0^t f(t')dt' \right] \right\rangle \\ &= \alpha_0^{*m} \alpha_0^n \exp\left[i(m-n)\omega t - \tfrac{1}{2}(m-n)^2 \sigma^2 t \right] . \end{aligned} \tag{5.2}$$

Since (5.1) is of the form[2], Eq. (2.11), the generated field is amplitude--coherent.

A random modulation of amplitude of coherent beam can be accomplished by taking a linearly polarized coherent beam, making optical rotation by a random angle θ and extracting polarized component in the direction of incident polarization. For a given propagation vector $\boldsymbol{k}$ along z-axis, coherent state $|\alpha_0, \beta_0\rangle$ with $a_x|\alpha_0,\beta_0\rangle = \alpha_0|\alpha_0,\beta_0\rangle$, $a_y|\alpha_0,\beta_0\rangle = \beta_0|\alpha_0,\beta_0\rangle$ generates state $|\alpha_0 \cos\theta - \beta_0 \sin\theta,\ \alpha_0 \sin\theta + \beta_0 \cos\theta\rangle$ on optical rotation by angle θ. Incident x-polarized state $|\alpha_0, 0\rangle$ will then generate a field for which the reduced properties of x-components give

$$\Gamma_x^{(m,n)} = \mathrm{Tr}\left[\rho a_x^{\dagger m} a_x^n\right] = \alpha_0^{*m} \alpha_0^n \left\langle \cos^{m+n}\theta \right\rangle . \tag{5.3}$$

This is of the form[3], Eq. (1.7) and hence describes a phase-coherent field.

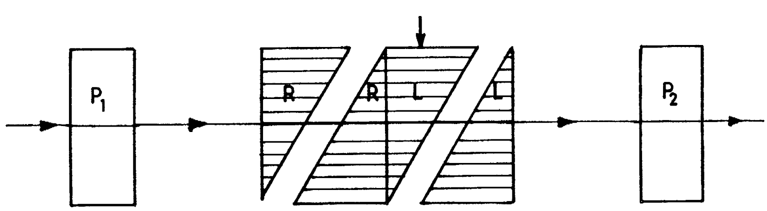

Fig. 1.

[1] Relations $\int_0^t \int_0^t dt_1 dt_2 \langle f(t_1) f(t_2)\rangle = \sigma^2 t$, $\int_0^t dt_1 \ldots \int_0^t dt_{2n} \langle f(t_1) \ldots f(t_{2n})\rangle = (\sigma^2 t)^n (2n-1)(2n-3)\ldots 3.1$, $\int_0^t dt_1 \ldots \int_0^t dt_{2n+1} \langle f(t_1) \ldots f(t_{2n+1})\rangle = 0$ give $\langle \exp[il \int_0^t f(t')dt']\rangle = \exp[-\frac{1}{2} l^2 \sigma^2 t]$; see also Ref. [10].

[2] Obviously $A_l = \exp\left[-il\omega t - \frac{1}{2} l^2 \sigma^2 t\right]$; compare with Eq. (2.1).

[3] Obviously $B_l = |\alpha_0|^l \langle \cos^l \theta\rangle$; compare with Eq. (1.7).

Experimentally, this can be achieved [12] by following the arrangement shown in Fig. 1. Plane polarized light obtained from polariser P_1 may be passed through four geometrically identical quartz prisms, out of which the first and second are made of R quartz and third and fourth are made of L quartz. Equal paths in R and L quartz ensure that the polarization of light is finally unaltered. But, if inner two prisms are joined face to face and this joint prisms are given a random motion perpendicular to the direction of propagation of light, the total paths in quartz and in air do not change but a random variation of difference of paths in R and L quartz is introduced with a zero average. This results in optical rotation by a randomly varying angle θ proportional to the displacement of the prism. If plane polarized component in the direction of incident polarization is now extracted using polariser P_2, random modulation of amplitude by a factor of $\cos\theta$ results.

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