QUASI-LOCAL STRUCTURE OF p-FORM THEORY

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(Received October 11, 2000)

We show that the Hamiltonian dynamics of the self-interacting, Abelian p-form theory in $D = 2p+2$ dimensional space-time gives rise to the *quasi-local* structure. Roughly speaking, it means that the field energy is localized but on closed $2p$-dimensional surfaces (quasi-localised). From the mathematical point of view this approach is implied by the boundary value problem for the corresponding field equations. Various boundary problems, *e.g.* Dirichlet or Neumann, lead to different Hamiltonian dynamics. Physics seems to prefer gauge-invariant, positively defined Hamiltonians which turn out to be quasi-local. Our approach is closely related with the standard two-potential formulation and enables one to generate *e.g.* duality transformations in a perfectly local way (but with respect to a new set of nonlocal variables). Moreover, the form of the quantization condition displays very similar structure to that of the symplectic form of the underlying p-form theory expressed in the quasi-local language.

PACS numbers: 11.15-q, 11.10.Kk, 10.10.Lm

1. Introduction

One of the most important idea of modern physics is *locality*. It is strongly related with relativity and quantum mechanics and plays a central role in relativistic (classical and quantum) field theories. Let us cite only two very influential books: *physics is simple when analyzed locally* [1] and *the role of fields is to implement the principle of locality* [2]. It should be stressed, therefore, from the very beginning that we are not going to discuss nonlocal theories. The Abelian p-form theory is a simple generalization of an ordinary electrodynamics in 4-dimensional Minkowski space-time $\mathcal{M}^4$ where the electromagnetic field potential 1-form A_μ is replaced by a p-form in D-dimensional space-time [3, 4]. This theory is perfectly local, *i.e.* it is defined via the local Lagrangian.

The motivations to study p-form theory are already discussed in [3]. Recently the new input comes with electric-magnetic duality [5–7]. It was observed long ago [8] that the duality symmetry for the standard Maxwell electrodynamics in four dimensional Minkowski space-time (*i.e.* $p = 1$ theory) is generated by the nonlocal generator (its physical interpretation as a chirality operator was discussed in [9]), *i.e.* it is nonlocal functional of the electromagnetic field. Therefore, the nonlocality enters into the game in a very natural way. We shall see that the above mentioned nonlocality is closely related with the Hamiltonian description of the field dynamics.

To define the Hamiltonian evolution one splits the entire space-time into space and time and then formulates the initial value problem. But in field theory one has to specify also the boundary condition for the fields. Very often one assumes that all fields do vanish at spatial infinity and simply forgets about this problem. It should be stressed, however, that even if the boundary values vanish numerically they do not vanish functionally, *i.e.* they are necessary in the proper definition of the functional phase space of the dynamical problem. This is typical for the problems with infinitely many degrees of freedom. Boundary value problem is not only a mathematical problem. It also does belong to physics. Different boundary problems lead to different Hamiltonians, *i.e.* different definitions of the field energy, *e.g.* energies defined via canonical and symmetric energy-momentum tensors. Now, in the standard (*i.e.* $p = 1$) electrodynamics the "canonical" energy, which is neither gauge-invariant nor positively defined, is related to the boundary value problem for the scalar potential A_0. On the other hand the "symmetric energy" (defined by the symmetric energy-momentum tensor), which is perfectly gauge-invariant and positively defined, is related to the control of the electric and magnetic fluxes on the boundary [10–13]. Therefore, it distinguishes a new set of electromagnetical variables Q^1 and Q^2 consistent with the boundary problem. Together with the canonically conjugated momenta Π_1 and Π_2 they encode the entire gauge-invariant information about the electromagnetic field $F = dA$, *i.e.* knowing Q's and Π's one may uniquely reconstruct F [10]. Actually, it was shown long ago by Debye [14] that Maxwell theory could be described in terms of two complex functions (so called Debey potentials). It turns out that this formulation is very well suited to describe *e.g.* radiative phenomena [15]. Our Q's and Π's (they may be rearranged into complex Q and Π) are closely related to Debey potentials. They solve the Gauss constraint and, therefore, they reduce the symplectic form in the space of Cauchy data for the field dynamics. However, they are nonlocal functions of the electromagnetic fields $\boldsymbol{D}$ and $\boldsymbol{B}$. The nonlocality is of the very special structure and the Hamiltonian generating the dynamics defines a *quasi-local* functional, *i.e.* performing an integration over angle variables one obtains perfectly local functional.

Now, in the Abelian self-interacting p-form theory in $D = 2p+2$ dimensional space-time one may perform the similar analysis [16]: instead of two complex functions Q and Π, the dynamical information about a p-form electromagnetic fields D and B is now encoded into two complex $(p-1)$-forms. In the present paper we relate the quasi-local picture implied by these $(p-1)$-forms with the proper definition of the Hamiltonian dynamics for a p-form theory. Moreover, we show that this formulation is perfectly suited for the description of the duality symmetry, *i.e.* the duality rotations (for odd p) are generated locally in terms of Q and Π. We show that the canonical generator has the following form:

$$\int Q^1 \Pi_2 - Q^2 \Pi_1 \,. \tag{1.1}$$

It is evident that this approach is closely related to the two-potential formulation [7,17] (see Appendix D).

It is well known that there is a crucial difference between theories with different parities of p, *e.g.* for even p a theory can not be duality invariant. Now, it was observed only recently [7] that the quantization condition for $(p-1)$–brane dyons crucially depends upon p, namely

$$e_1 g_2 + (-1)^p e_2 g_1 = nh \,, \tag{1.2}$$

with integer n (h is the Planck constant). It turns out that the symplectic form of a p-form theory written in terms of Q and Π has very similar structure

$$\Omega_p = \int \delta\Pi_1 \wedge \delta Q^1 + (-1)^{p+1} \delta\Pi_2 \wedge \delta Q^2 \,, \tag{1.3}$$

therefore, there is a striking correspondence between the form of the quantization condition (1.2) and the structure of symplectic form (1.3). This correspondence is universal, *i.e.* it holds for any gauge-invariant, self-interacting theory.

The paper is organized as follows: we remind the quasi-local structure of standard (1-form) electrodynamics in Section 2. This is the prototype of the p-form theory for odd p. Then in Section 3 we make the generalization for $p = 2$ which is the prototype for even p. The general case (*i.e.* an arbitrary p) is discussed in Appendices B and C. In Section 4 we describe the gauge-invariant coupling of p-form electrodynamics to the charged matter and the Hamiltonian structure of the interacting theory. The details of notation are clarified in Appendix A.

2. 1-form theory in $D = 4$

2.1. Generating formula

Let us consider a 1-form theory defined by the Lagrangian $\mathcal{L} = \mathcal{L}(A, \partial A)$. Field dynamics of this theory may be written in terms of the following generating formula (see Appendix A for details of notation):

$$-\delta\mathcal{L} = \partial_\nu(\mathcal{G}^{\nu\mu}\delta A_\mu) = (\partial_\nu\mathcal{G}^{\nu\mu})\delta A_\mu + \mathcal{G}^{\nu\mu}\delta(\partial_\nu A_\mu) \,. \tag{2.1}$$

The formula (2.1) implies the following definition of "momenta":

$$\mathcal{G}^{\mu\nu} = -2\,\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} \,. \tag{2.2}$$

Moreover, (2.1) generates dynamical (in general nonlinear) field equations

$$\partial_\nu\mathcal{G}^{\nu\mu} = -\mathcal{J}^\mu \,, \tag{2.3}$$

where the external 1-form current reads:

$$\mathcal{J}^\mu = \frac{\partial\mathcal{L}}{\partial A_\mu} \,. \tag{2.4}$$

Let us start with a source free theory, *i.e.* $\mathcal{J} = 0$. We shall study the p-form electromagnetism coupled to a charged matter in Sec. 4. To obtain the Hamiltonian description of the field dynamics let us integrate Eq. (2.1) over a 3-dimensional volume V contained in the constant-time hyperplane Σ:

$$-\delta \int_V \mathcal{L} = \int_V \partial_0(\mathcal{G}^{0i}\delta A_i) + \int_{\partial V} \mathcal{G}^{\perp\mu}\delta A_\mu \,, \tag{2.5}$$

where $\perp$ denotes the component orthogonal to the 2-dimensional boundary ∂V. To simplify our notation let us introduce the spherical coordinates on Σ:

$$x^3 = r \,, \quad x^A = \varphi_A \,; \quad A = 1, 2 \,, \tag{2.6}$$

where φ_1, φ_2 denote spherical angles (usually one writes $\varphi_1 = \varphi$ and $\varphi_2 = \Theta$). To enumerate angles we shall use capital letters $A, B, C, \ldots$. The Euclidean metric tensor is diagonal

$$g_{11} = r^2, \quad g_{22} = r^2\sin\varphi_2, \quad g_{rr} = 1 \,, \tag{2.7}$$

and the volume form $\Lambda_1 = \sqrt{\det(g_{kl})} = r^2 \sin\varphi_2$. Let V be a 3-ball with a finite radius R. In such a coordinate system the formula (2.5) takes the following form:

$$\delta \int_V \mathcal{L} = -\int_V \partial_0(\mathcal{D}^i \delta A_i) + \int_{\partial V} \mathcal{D}^r \delta A_0 - \int_{\partial V} \mathcal{G}^{rB} \delta A_B \,, \tag{2.8}$$

where

$$\mathcal{D}_i = \mathcal{G}_{i0} \tag{2.9}$$

denotes the 1-form electric induction density on Σ. Now, performing the Legendre transformation between induction 1-form $\mathcal{D}^i$ and $\dot{A}_i$ one obtains the following Hamiltonian formula:

$$-\delta \mathcal{H}_{\mathrm{can}} = -\int_V \left(\dot{\mathcal{D}}^i \delta A_i - \dot{A}_i \delta \mathcal{D}^i\right) + \int_{\partial V} \mathcal{D}^r \delta A_0 - \int_{\partial V} \mathcal{G}^{rB} \delta A_B \,, \tag{2.10}$$

where the canonical Hamiltonian

$$\mathcal{H}_{\mathrm{can}} = \int_V \left(-\mathcal{D}^i \dot{A}_i - \mathcal{L}\right) . \tag{2.11}$$

Equation (2.10) generates an infinite-dimensional Hamiltonian system in the phase space $\mathcal{P}_p = (\mathcal{D}^i, A_i)$ fulfilling Dirichlet boundary conditions for the 1-form potential A_i: $A_0|\partial V$ and $A_A|\partial V$. From the mathematical point of view this is the missing part of the definition of the functional space. The Hamiltonian structure of a general nonlinear 1-form electrodynamics described above is mathematically well defined, *i.e.* a mixed Cauchy problem (Cauchy data given on Σ and Dirichlet data given on $\partial V \times \boldsymbol{R}$) has a unique solution (modulo gauge transformations which reduce to the identity on $\partial V \times \boldsymbol{R}$).

There is, however, another way to describe the Hamiltonian evolution of fields in the region V. Let us perform the Legendre transformation between $\mathcal{D}^r$ and A_0 at the boundary ∂V. One obtains:

$$-\delta \mathcal{H}_{\mathrm{sym}} = -\int_V \left(\dot{\mathcal{D}}^i \delta A_i - \dot{A}_i \delta \mathcal{D}^i\right) - \int_{\partial V} A_0 \delta \mathcal{D}^r - \int_{\partial V} \mathcal{G}^{rB} \delta A_B \,, \tag{2.12}$$

where the new "symmetric" Hamiltonian

$$\mathcal{H}_{\mathrm{sym}} = \mathcal{H}_{\mathrm{can}} + \int_{\partial V} \mathcal{D}^r A_0 \,. \tag{2.13}$$

Observe, that formula (2.12) defines the Hamiltonian evolution but on a different phase space. In order to kill boundary terms in (2.12) one has to control on ∂V: $\mathcal{D}^r$ (instead of A_0) and A_B. We stress that from the mathematical point of view both descriptions are equally good and an additional physical argument has to be given if we want to choose one of them as more fundamental.

2.2. Canonical vs symmetric energy

Now, let us discuss the relation between $\mathcal{H}_{\rm can}$ and $\mathcal{H}_{\rm sym}$ defined by (2.11) and (2.13), respectively. One has:

$$
\begin{aligned}
\mathcal{H}_{\rm sym} &= \mathcal{H}_{\rm can} + \int\limits_{\partial V} \mathcal{D}^r A_0 = \mathcal{H} + \int\limits_{V} \partial_k \left(\mathcal{D}^k A_0\right) \\
&= \int\limits_{V} \left\{ -\mathcal{D}^i \dot{A}_i - \mathcal{L} + \left(A_0 \partial_k \mathcal{D}^k + \mathcal{D}^k \partial_k A_0 \right) \right\} \\
&= \int\limits_{V} \left(\mathcal{D}^i E_i - \mathcal{L} \right) \,, \qquad (2.14)
\end{aligned}
$$

where the 1-form electric field is defined by

$$
E_i = F_{i0} = \partial_{[i} A_{0]} \,. \qquad (2.15)
$$

Therefore, $\mathcal{H}_{\rm sym}$ is related to $\mathcal{L}$ via different Legendre transformation (compare (2.11) with (2.14)). Contrary to $\mathcal{H}_{\rm can}$, $\mathcal{H}_{\rm sym}$ is perfectly gauge-invariant. It is evident that $\mathcal{H}_{\rm sym}$ is defined via the symmetric energy-momentum tensor:

$$
T^{\mu\nu}_{\rm sym} = F^{\mu\lambda} \mathcal{G}^{\nu}{}_{\lambda} + g^{\mu\nu} \mathcal{L} \,, \qquad (2.16)
$$

whereas $\mathcal{H}_{\rm can}$ via the canonical one:

$$
T^{\mu\nu}_{\rm can} = (\partial^\mu A^\lambda) G^{\nu}{}_{\lambda} + g^{\mu\nu} \mathcal{L} \,, \qquad (2.17)
$$

i.e. $\mathcal{H}_{\rm sym} = \int_V T^{00}_{\rm sym}$ and $\mathcal{H}_{\rm can} = \int_V T^{00}_{\rm can}$. Therefore, the "symmetric energy" $\mathcal{H}_{\rm sym}$ is gauge-invariant and positively defined, *e.g.* for the 1-form Maxwell theory one has

$$
\mathcal{H}^{\rm Maxwell}_{\rm sym} = \frac{1}{2} \int (\mathcal{D}^i D_i + \mathcal{B}^i B_i) \,.
$$

On the other hand, the "canonical energy" $\mathcal{H}_{\rm can}$ is neither positively defined nor gauge-invariant. These properties show that the Hamiltonian evolution based on $\mathcal{H}_{\rm sym}$ is more natural from the physical point of view than the one based on $\mathcal{H}_{\rm can}$ (see also discussion in [12]).

2.3. Reduction of the generating formula

Now, it turns out that the formula (2.12) may be considerably simplified. Any geometrical object on a 3-dimensional hyperplane Σ may be decomposed into the radial and tangential (*i.e.* tangential to any sphere $S^2(r)$) components, *e.g.* a 1-form gauge potential A_i decomposes into the radial A_r and tangential A_A. Now, any 1-form on $S^2(r)$ may be further decomposed into "longitudinal" and "transversal" parts:

$$A_A = \nabla_A u + \varepsilon_{AB}\nabla^B v \,, \tag{2.18}$$

where both u and v are scalar functions on $S^2(r)$. Now, using (2.18) and integrating by parts one gets:

$$\begin{aligned}\int_V \left(\dot{\mathcal{D}}^i \delta A_i - \dot{A}_i \delta \mathcal{D}^i\right) &= \int_V \Big\{\left(\dot{\mathcal{D}}^r \delta A_r - \dot{A}_r \delta \mathcal{D}^r\right) \\ &+ \left[(\partial_r \dot{\mathcal{D}}^r)\delta u - \dot{u}\delta(\partial_r \mathcal{D}^r)\right] - \varepsilon_{AB}\left[\left(\nabla^B \dot{\mathcal{D}}^A\right)\delta v - \dot{v}\delta\left(\nabla^B \mathcal{D}^A\right)\right]\Big\}\,,\end{aligned} \tag{2.19}$$

where we have used the Gauss law

$$\nabla_A \mathcal{D}^A = -\partial_r \mathcal{D}^r \,. \tag{2.20}$$

Moreover, due to (2.18)

$$\int_{\partial V} \mathcal{G}^{rA}\delta A_A = -\int_{\partial V}\left\{-\dot{\mathcal{D}}^r \delta u + \left(\varepsilon_{AB}\nabla^B \mathcal{G}^{rA}\right)\delta v\right\} . \tag{2.21}$$

In deriving (2.21) we have used

$$\nabla_A \mathcal{G}^{Ar} = -\dot{\mathcal{D}}^r \,, \tag{2.22}$$

which follows from the field equations $\nabla_A \mathcal{G}^{Ar} + \partial_0 \mathcal{G}^{0r} = 0$. Now, taking into account (2.19) and (2.21) the generating formula (2.12) may be rewritten in the following way:

$$\begin{aligned}-\delta\mathcal{H}_{\rm sym} \;=\; &-\int_V \Big\{\left[\dot{\mathcal{D}}^r \delta\left(A_r - \partial_r u\right) - \left(\dot{A}_r - \partial_r \dot{u}\right)\delta\mathcal{D}^r\right] \\ &- \left[\left(\varepsilon_{AB}\nabla^B \dot{\mathcal{D}}^A\right)\delta v - \dot{v}\delta\left(\varepsilon_{AB}\nabla^B \mathcal{D}^A\right)\right]\Big\} \\ &- \int_{\partial V}\left\{\left(A_0 - \dot{u}\right)\delta\mathcal{D}^r - \left(\varepsilon_{AB}\nabla^B \mathcal{G}^{rA}\right)\delta v\right\} .\end{aligned} \tag{2.23}$$

Note, that although A_r, A_0 and u are manifestly gauge-dependent, the combinations $A_r - \partial_r u$ and $A_0 - \partial_0 u$ are gauge-invariant. To simplify our consideration we choose the special gauge $u \equiv 0$, *i.e.* a 1-form A_A on $S^2(r)$ is purely transversal. This condition, due to (2.18), may be equivalently rewritten as

$$\nabla_A A^A = 0 \,. \tag{2.24}$$

Assuming (2.24) one may show [16]

$$\varDelta_0 \, A^r = r^2 \, \varepsilon^{AB} \, \nabla_B B_A \,, \tag{2.25}$$

where

$$\varDelta_0 = r^2 \, \nabla_A \nabla^A \tag{2.26}$$

denotes the 2-dimensional Laplacian on $S^2(1)$, *i.e.* the 2-dim. Laplace–Beltrami operator on scalar functions (0-forms). Moreover,

$$B^r = \varepsilon^{AB} \, \nabla_A A_B = -r^{-2} \varDelta_0 v \,. \tag{2.27}$$

Since $\varDelta_0$ is invertible in the source free theory [10] the formula (2.23) may be rewritten as follows:

$$\begin{aligned} -\delta\mathcal{H}_{\rm sym} \;=\; & -\int\limits_V \Big\{ \Big[(r\dot{\mathcal{D}}^r)\delta\left(r\varDelta_0^{-1}\varepsilon_{AB}\,\nabla^B B^A\right) - \left(r\varDelta_0^{-1}\varepsilon_{AB}\,\nabla^B \dot{B}^A\right)\delta(r\mathcal{D}^r)\Big] \\ & + \Big[\left(r\varDelta_0^{-1}\varepsilon_{AB}\nabla^B\dot{\mathcal{D}}^A\right)\delta(rB^r) - (r\dot{B}^r)\delta\left(r\varDelta_0^{-1}\varepsilon_{AB}\nabla^B\mathcal{D}^A\right)\Big]\Big\} \\ & - \int\limits_{\partial V} \left\{ (r^{-1}A_0)\delta(r\mathcal{D}^r) + \left(\varDelta_0^{-1}\varepsilon_{AB}\nabla^B\mathcal{G}^{rA}\right)\delta(rB^r)\right\} \,. \end{aligned} \tag{2.28}$$

Now, introducing the following set of variables

$$Q^1 \;=\; rD^r \,, \tag{2.29}$$
$$Q^2 \;=\; rB^r \,, \tag{2.30}$$
$$\varPi_1 \;=\; r\varDelta_0^{-1}\varepsilon^{AB}\nabla_B B_A \,, \tag{2.31}$$
$$\varPi_2 \;=\; -r\varDelta_0^{-1}\varepsilon^{AB}\nabla_B D_A \,, \tag{2.32}$$

Eq. (2.28) simplifies to

$$\begin{aligned} -\delta\mathcal{H}_{\rm sym} \;=\; & \int\limits_V \varLambda_1 \left\{ \left(\dot{\varPi}^1\delta Q_1 - \dot{Q}_1\delta\varPi^1\right) + \left(\dot{\varPi}^2\delta Q_2 - \dot{Q}_2\delta\varPi^2\right)\right\} \\ & + \int\limits_{\partial V} \varLambda_1 \left(\chi^1\delta Q_1 + \chi^2\delta Q_2\right) \,, \end{aligned} \tag{2.33}$$

where we introduced the boundary momenta:

$$\chi_1 = -\frac{1}{r}\Lambda_0 \,, \tag{2.34}$$
$$\chi_2 = -r\Delta_0^{-1}\varepsilon_{AB}\nabla^B G^{rA} \,. \tag{2.35}$$

Tensor $G^{\mu\nu}$ is defined by $\mathcal{G}^{\mu\nu} = \Lambda_1 G^{\mu\nu}$, and, therefore, $\mathcal{D}^i = \Lambda_1 D^i$. Note, that

$$\chi^l = \frac{\delta \mathcal{H}_{\rm sym}}{\delta(\partial_r Q_l)} \,, \qquad l = 1,2 \,. \tag{2.36}$$

For a Maxwell theory one obtains

$$\mathcal{H}_{\rm sym}^{\rm Maxwell} = \frac{1}{2}\int_V \Lambda_1 \sum_{l=1}^{2} \left\{ \frac{1}{r^2} Q_l Q_l - \frac{1}{r^2}\partial_r (rQ_l)\Delta_0^{-1}\partial_r(rQ_l) - \Pi^l \Delta_0 \Pi^l \right\} \,, \tag{2.37}$$

and, therefore

$$\chi^l = \frac{1}{r}\,\Delta_0^{-1}\partial_r(rQ_l) \,, \qquad l = 1,2 \,, \tag{2.38}$$

have perfectly symmetric form.

2.4. Canonical symmetries

The symplectic form $\int \delta\mathcal{D}^k \wedge \delta A_k$ rewritten in terms of Q's and Π's have the following form [10, 16]:

$$\Omega = \mathrm{Im} \int \Lambda_1\, \delta\Pi \wedge \delta\overline{Q} \,, \tag{2.39}$$

where we introduced a complex notation

$$Q = Q^1 + iQ^2 \,, \tag{2.40}$$
$$\Pi = i(\Pi_1 + i\Pi_2) \,. \tag{2.41}$$

The form (2.39) is invariant under the following set of $\boldsymbol{R}$-linear transformations:

$$Q \to \mathrm{e}^{i\alpha} Q \,, \tag{2.42}$$
$$Q \to \cosh\alpha\, Q + i\sinh\alpha\, \overline{Q} \,, \tag{2.43}$$
$$Q \to \cosh\lambda\, Q + \sinh\lambda\, \overline{Q} \,, \tag{2.44}$$

and the same rules for Π. It is easy to see that these transformations form the group SO(2,1). In terms of $\boldsymbol{D}$ and $\boldsymbol{B}$, (2.42)–(2.44) have more familiar form:
(2.42) corresponds to orthogonal SO(2) duality rotations:

$$\begin{aligned} \boldsymbol{D} &\to \boldsymbol{D}\cos\alpha - \boldsymbol{B}\sin\alpha\,, \\ \boldsymbol{B} &\to \boldsymbol{D}\sin\alpha + \boldsymbol{B}\cos\alpha\,, \end{aligned} \tag{2.45}$$

(2.43) corresponds to hyperbolic SO(1,1) rotations:

$$\begin{aligned} \boldsymbol{D} &\to \boldsymbol{D}\cosh\alpha + \boldsymbol{B}\sinh\alpha\,, \\ \boldsymbol{B} &\to \boldsymbol{D}\sinh\alpha + \boldsymbol{B}\cosh\alpha\,, \end{aligned} \tag{2.46}$$

(2.44) corresponds to scaling transformations:

$$\begin{aligned} \boldsymbol{D} &\to \mathrm{e}^{\lambda}\,\boldsymbol{D}\,, \\ \boldsymbol{B} &\to \mathrm{e}^{-\lambda}\,\boldsymbol{B}\,. \end{aligned} \tag{2.47}$$

The canonical generators corresponding to (2.42)–(2.44) have the following form:

$$G_1 = \int \Lambda_1\,(Q^2\Pi_1 - Q^1\Pi_2) = \mathrm{Re}\int \Lambda_1\,(\Pi\overline{Q})\,, \tag{2.48}$$

$$G_2 = -\int \Lambda_1\,(Q^2\Pi_1 + Q^1\Pi_2) = \mathrm{Re}\int \Lambda_1\,(\Pi Q)\,, \tag{2.49}$$

$$G_3 = \int \Lambda_1\,(Q^1\Pi_1 - Q^2\Pi_2) = \mathrm{Im}\int \Lambda_1\,(\Pi Q)\,. \tag{2.50}$$

Note, that for the duality invariant theory G_1 defined in (2.48) is constant in time. Its physical interpretation was clarified in [9]. Obviously, G_1, G_2 and G_3 rewritten in terms of $\boldsymbol{D}$ and $\boldsymbol{B}$ are highly nonlocal functionals of the fields [8, 9].

2.5. Summary

The reduced variables (Q_l, Π^l) play the role of generalized positions and momenta for an electromagnetic field. They are perfectly gauge-invariant and contain the entire (gauge-invariant) information about $\boldsymbol{D}$ and $\boldsymbol{B}$. Let us note that Q's and Π's are nonlocal functions of $\boldsymbol{D}$ and $\boldsymbol{B}$. The nonlocality enters via the operations on each sphere $S^2(r)$, *i.e.* via the operator Δ_0^{-1}. On the other hand the operations in the radial direction do not produce any nonlocality.

The Hamiltonian generating the dynamics is perfectly local in $\boldsymbol{D}$ and $\boldsymbol{B}$ but is nonlocal in Q's and Π's. The field functional with the above described nonlocality we shall call *quasi-local*. Note, that generators G_i are perfectly local in reduced variables.

The "symmetric" Hamiltonian dynamics is defined by the Dirichlet boundary conditions for positions Q_l. On the other hand the "canonical" formula (2.12) is defined by the Dirichlet boundary condition for χ^1 and Q_2. Note, however, that in the Maxwell case

$$\int_{\partial V} \Lambda_1 Q_1 \delta\chi^1 = \int_{\partial V} \Lambda_1 \frac{1}{r} \left(\Delta_0^{-1} Q_1\right) \delta\ \partial_r(r^2 D^r) = \int_{\partial V} r \left(\Delta_0^{-1} Q_1\right) \delta\ (\partial_r \mathcal{D}^r)\,, \tag{2.51}$$

i.e. a Dirichlet condition $\chi^1|\partial V$ is equivalent to the Neumann condition $\partial_r \mathcal{D}^r|\partial V$.

3. 2-form theory in $D = 6$

3.1. Generating formula

Now, consider a 2-form theory defined by the Lagrangian $\mathcal{L} = \mathcal{L}(A, \partial A)$. Field dynamics of this theory may be written in terms of the following generating formula:

$$-\delta\mathcal{L} = \partial_\nu(\mathcal{G}^{\nu\mu\lambda}\delta A_{\mu\lambda}) = (\partial_\nu \mathcal{G}^{\nu\mu\lambda})\delta A_{\mu\lambda} + \mathcal{G}^{\nu\mu\lambda}\delta(\partial_\nu A_{\mu\lambda})\,. \tag{3.1}$$

The formula (3.1) implies the following definition of "momenta":

$$\mathcal{G}^{\mu\nu\lambda} = -3!\,\frac{\partial\mathcal{L}}{\partial F_{\mu\nu\lambda}}\,. \tag{3.2}$$

Moreover, (3.1) generates dynamical (in general nonlinear) field equations

$$\partial_\nu \mathcal{G}^{\nu\mu\lambda} = -\mathcal{J}^{\mu\lambda}\,, \tag{3.3}$$

where the external 2-form current reads:

$$\mathcal{J}^{\mu\lambda} = 2\,\frac{\partial\mathcal{L}}{\partial A_{\mu\lambda}}\,. \tag{3.4}$$

In the present section we consider only $\mathcal{J} = 0$ (for $\mathcal{J} \neq 0$ see Section 4.) To obtain the Hamiltonian description of the field dynamics let us integrate equation (3.1) over a 5-dimensional volume V contained in the constant-time hyperplane Σ:

$$-\delta \int_V \mathcal{L} \;=\; \int_V \partial_0(\mathcal{G}^{0ij}\delta A_{ij}) + \int_{\partial V} \mathcal{G}^{\perp\mu\nu}\delta A_{\mu\nu}\,, \tag{3.5}$$

where $\perp$ denotes the component orthogonal to the 4-dimensional boundary ∂V. To simplify our notation let us introduce the spherical coordinates on Σ:

$$x^5 \; = \; r \, , \qquad x^A = \varphi_A \; ; \qquad A = 1,2,3,4 \, , \tag{3.6}$$

where $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ denote spherical angles (to enumerate angles we shall use capital letters $A, B, C, ...$). The Euclidean metric on Σ reads:

$$\begin{aligned} g_{11} \; &= \; r^2 \sin^2 \varphi_2 \sin^2 \varphi_3 \sin^2 \varphi_4 \, , \qquad g_{22} = r^2 \sin^2 \varphi_3 \sin^2 \varphi_4 , \\ g_{33} \; &= \; r^2 \sin^2 \varphi_4 \, , \qquad g_{44} = r^2 \, , \qquad g_{55} \equiv g_{rr} = 1 \, , \end{aligned} \tag{3.7}$$

and the corresponding volume form

$$\Lambda_2 = \sqrt{\det(g_{ij})} = r^4 \sin \varphi_2 \sin^2 \varphi_3 \sin^3 \varphi_4 \, . \tag{3.8}$$

Let V be a 5-dim. ball with a finite radius R. In such a coordinate system the formula (3.5) takes the following form:

$$\delta \int\limits_V \mathcal{L} \; = \; \int\limits_V \partial_0 (\mathcal{D}^{ij} \delta A_{ij}) - \int\limits_{\partial V} 2 \, \mathcal{D}^{rA} \delta A_{0A} - \int\limits_{\partial V} \mathcal{G}^{rAB} \delta A_{AB} \, , \tag{3.9}$$

where

$$\mathcal{D}_{ij} = \mathcal{G}_{ij0} \tag{3.10}$$

denotes the 2-form electric induction density. Now, performing the Legendre transformation between induction 2-form $\mathcal{D}^{ij}$ and $\dot{A}_{ij}$ one obtains the following Hamiltonian formula:

$$-\delta \mathcal{H}_{\mathrm{can}} = \int\limits_V \left(\dot{\mathcal{D}}^{ij} \delta A_{ij} - \dot{A}_{ij} \delta \mathcal{D}^{ij} \right) - \int\limits_{\partial V} 2 \, \mathcal{D}^{rA} \delta A_{0A} - \int\limits_{\partial V} \mathcal{G}^{rAB} \delta A_{AB} \, , \tag{3.11}$$

where the canonical Hamiltonian

$$\mathcal{H}_{\mathrm{can}} = \int\limits_V \left(\mathcal{D}^{ij} \dot{A}_{ij} - \mathcal{L} \right) \, . \tag{3.12}$$

Equation (3.11) generates an infinite-dimensional Hamiltonian system in the phase space $\mathcal{P}_2 = (\mathcal{D}^{ij}, A_{ij})$ fulfilling Dirichlet boundary conditions for the 2-form potential A_{ij}: $A_{0A}|\partial V$ and $A_{AB}|\partial V$. From the mathematical point of view this is the missing part of the definition of the functional space. The Hamiltonian structure of a general nonlinear 2-form electrodynamics described above is mathematically well defined, *i.e.* a mixed Cauchy problem (Cauchy data given on Σ and Dirichlet data given on $\partial V \times \boldsymbol{R}$) has a unique solution (modulo gauge transformations which reduce to the identity on $\partial V \times \boldsymbol{R}$).

Note the difference in signs between corresponding formulae of the present section and that of Section 2. This difference follows from the difference between corresponding symplectic structures [16]. For 1-form theory one has

$$\Omega_1 = \int_V \delta\mathcal{G}^{0i} \wedge \delta A_i = +\int_V \delta\mathcal{D}^i \wedge \delta A_i \,, \tag{3.13}$$

whereas for 2-form theory

$$\Omega_2 = \int_V \delta\mathcal{G}^{0ij} \wedge \delta A_{ij} = -\int_V \delta\mathcal{D}^{ij} \wedge \delta A_{ij} \,, \tag{3.14}$$

Now, in analogy to (2.12) we pass to another Hamiltonian description of the field evolution in the finite region V. Let us perform the Legendre transformation between $\mathcal{D}^{rA}$ and A_{0A} at the boundary ∂V. One obtains:

$$-\delta\mathcal{H}_{\rm sym} = \int_V \left(\dot{\mathcal{D}}^{ij}\delta A_{ij} - \dot{A}_{ij}\delta\mathcal{D}^{ij}\right) + \int_{\partial V} 2\, A_{0A}\delta\mathcal{D}^{rA} - \int_{\partial V} \mathcal{G}^{rAB}\delta A_{AB} \,, \tag{3.15}$$

where the new "symmetric" Hamiltonian

$$\mathcal{H}_{\rm sym} = \mathcal{H}_{\rm can} - \int_{\partial V} 2\,\mathcal{D}^{rA} A_{0A} \,. \tag{3.16}$$

Observe, that formula (3.15) defines the Hamiltonian evolution but on a different phase space. In order to kill boundary terms in (3.15) one has to control on ∂V: $\mathcal{D}^{rA}$ (instead of A_{0A}) and A_{AB}. We stress that from the mathematical point of view both descriptions are equally good and an additional physical argument has to be given if we want to choose one of them as more fundamental.

3.2. Canonical vs symmetric energy

The relation between $\mathcal{H}_{\rm can}$ and $\mathcal{H}_{\rm sym}$ is exactly the same as in $p = 1$ case:

$$\begin{aligned} \mathcal{H}_{\rm sym} &= \mathcal{H}_{\rm can} - \int_{\partial V} 2\,\mathcal{D}^{rA} A_{0A} = \mathcal{H}_{\rm can} - \int_V 2\,\partial_k \left(\mathcal{D}^{ki} A_{0i}\right) \\ &= \int_V \left\{\mathcal{D}^{ij}\dot{A}_{ij} - \mathcal{L} + 2\left(A_{0i}\partial_k\mathcal{D}^{ki} + \mathcal{D}^{ki}\partial_k A_{0i}\right)\right\} \\ &= \int_V \left(\tfrac{1}{2}\,\mathcal{D}^{ij} E_{ij} - \mathcal{L}\right) \,, \end{aligned} \tag{3.17}$$

where the 2-form electric field is defined by

$$E_{ij} = F_{ij0} = \partial_{[i} A_{j0]} \,. \tag{3.18}$$

Therefore, $\mathcal{H}_{\rm sym} = \int T^{00}_{\rm sym}$ and $\mathcal{H}_{\rm can} = \int T^{00}_{\rm can}$ with

$$T^{\mu\nu}_{\rm sym} = \tfrac{1}{2} F^{\mu\lambda\sigma} G^{\nu}{}_{\lambda\sigma} + g^{\mu\nu}\mathcal{L} \,, \tag{3.19}$$

and

$$T^{\mu\nu}_{\rm can} = (\partial^\mu A^{\lambda\sigma}) G^{\nu}{}_{\lambda\sigma} + g^{\mu\nu}\mathcal{L} \,. \tag{3.20}$$

In the 2-form Maxwell theory the "symmetric energy" (gauge-invariant and positively defined) reads:

$$\mathcal{H}^{\rm Maxwell}_{\rm sym} = \tfrac{1}{4} \int (\mathcal{D}^{ij} D_{ij} + \mathcal{B}^{ij} B_{ij}) \,.$$

3.3. Reduction of the generating formula

Now, in analogy to (2.18) let as make the following decomposition:

$$A_{AB} = \nabla_{[A} u_{B]} + \varepsilon_{ABCD} \nabla^C v^D \,, \tag{3.21}$$

where ∇_A denotes a covariant derivative on each $S^4(r)$ defined by the induced metric g_{AB} and ε_{ABCD} stands for the Lévi–Civita tensor density such that $\varepsilon_{1234} = \Lambda_2$. Both u_A and v^A are 1-forms on $S^4(r)$. Using (3.21) and integrating by parts one gets:

$$\begin{aligned}
\int_V \left(\dot{\mathcal{D}}^{ij} \delta A_{ij} - \dot{A}_{ij} \delta \mathcal{D}^{ij} \right) &= \int_V \Big\{ 2 \left(\dot{\mathcal{D}}^{rA} \delta A_{rA} - \dot{A}_{rA} \delta \mathcal{D}^{rA} \right) \\
&+2 \left[\left(\partial_r \dot{\mathcal{D}}^{rA} \right) \delta u_A - \dot{u}_A \delta \left(\partial_r \mathcal{D}^{rA} \right) \right] \\
&- \varepsilon_{ABCD} \left[\left(\nabla^C \dot{\mathcal{D}}^{AB} \right) \delta v^D - \dot{v}^D \delta \left(\nabla^C \mathcal{D}^{AB} \right) \right] \Big\} \,,
\end{aligned} \tag{3.22}$$

where we have used the Gauss law

$$\nabla_A \mathcal{D}^{AB} = -\partial_r \mathcal{D}^{rB} \,. \tag{3.23}$$

Moreover, due to (3.21)

$$\int_{\partial V} \mathcal{G}^{rAB} \delta A_{AB} = - \int_{\partial V} \left\{ -2 \dot{\mathcal{D}}^{rA} \delta u_A + \left(\varepsilon_{ABCD} \nabla^C \mathcal{G}^{rAB} \right) \delta v^D \right\} \,. \tag{3.24}$$

In deriving (3.24) we have used

$$\nabla_A \mathcal{G}^{ABr} = -\dot{\mathcal{D}}^{rB} , \tag{3.25}$$

which follows from the field equations $\nabla_A \mathcal{G}^{ABr} + \partial_0 \mathcal{G}^{0Br} = 0$. Now, taking into account (3.22) and (3.24) the generating formula (3.15) may be rewritten in the following way:

$$\begin{aligned} -\delta\mathcal{H}_{\rm sym} \;=\; & \int_V \Big\{ \Big[\dot{\mathcal{D}}^{rA} \delta(2\, A_{rA} - 2\partial_r u_A) - \Big(2\dot{A}_{rA} - 2\partial_r \dot{u}_A \Big) \delta\mathcal{D}^{rA} \Big] \\ & - \; \Big[\Big(\varepsilon_{ABCD} \nabla^C \dot{\mathcal{D}}^{AB} \Big) \delta v^D - \dot{v}^D \delta \left(\varepsilon_{ABCD} \nabla^C \mathcal{D}^{AB} \right) \Big] \Big\} \\ & + \int_{\partial V} \left\{ (2A_{0A} - 2\dot{u}_A) \delta\mathcal{D}^{rA} + \int_{\partial V} \left(\varepsilon_{ABCD} \nabla^C \mathcal{G}^{rAB} \right) \delta v^D \right\} . \end{aligned} \tag{3.26}$$

Note, that although A_{rA}, A_{0A} and u_A are manifestly gauge-dependent, the combinations $A_{rA} - \partial_r u_A$ and $A_{0A} - \partial_0 u_A$ are gauge-invariant. To simplify our consideration we choose the special gauge $u \equiv 0$, *i.e.* a 2-form A_{AB} on $S^4(r)$ is purely transversal. This condition, due to (3.21), may be equivalently rewritten as

$$\nabla_A A^{AB} = 0 . \tag{3.27}$$

But now, contrary to the $p = 1$ case, we have an additional covector field on $S^4(r)$, namely A_{rA}. For this covector we choose an analogous gauge condition, *i.e.*

$$\nabla_A A^{rA} = 0 . \tag{3.28}$$

Assuming (3.27) and (3.28) one may show [16]

$$\varDelta_1 \, A^{rD} = -\frac{r^2}{4} \, \varepsilon^{ABCD} \, \nabla_C B_{AB} , \tag{3.29}$$

where

$$\varDelta_1 = r^2 \nabla_A \nabla^A - 3 , \tag{3.30}$$

equals to the Laplace–Beltrami operator on co-exact 1-forms on $S^4(1)$ [16]. Moreover, in analogy to (2.27) one has [16]

$$B^{rA} = -2r^{-2} \varDelta_1 v^A , \tag{3.31}$$

and, therefore, the formula (3.26) simplifies to

$$
\begin{aligned}
-\delta\mathcal{H}_{\rm sym} = \frac{1}{2}\int\limits_V \Big\{ - \Big[\Big(r\dot{\mathcal{D}}^{rD} \Big)\, \delta\left(r\Delta_1^{-1}\varepsilon_{ABCD}\, \nabla^C B^{AB}\right) \\
- \Big(r\Delta_1^{-1}\varepsilon_{ABCD}\nabla^C \dot{B}^{AB}\Big)\, \delta\left(r\mathcal{D}^{rD}\right) \Big] \\
+ \Big[\Big(r\Delta_1^{-1}\varepsilon_{ABCD}\nabla^C\dot{\mathcal{D}}^{AB}\Big)\, \delta\left(rB^{rD}\right) - \Big(r\dot{B}^{rD}\Big)\delta\Big(r\Delta_1^{-1}\varepsilon_{ABCD}\nabla^C\mathcal{D}^{AB}\Big)\Big]\Big\} \\
+ \int\limits_{\partial V} \Big\{ (2r^{-1}A_{0A})\delta\left(r\mathcal{D}^{rA}\right) - \Big(\frac{1}{2} r\Delta_1^{-1}\varepsilon_{ABCD}\nabla^C\mathcal{G}^{rAB}\Big)\, \delta\left(rB^{rD}\right) \Big\} \, . \quad (3.32)
\end{aligned}
$$

Now, introducing the following set of variables

$$
\begin{aligned}
{Q_1}^A &= rD^{rA}\, , && (3.33)\\
{Q_2}^A &= rB^{rA}\, , && (3.34)\\
{\mathit{\Pi}^1}_D &= \frac{r}{2}\Delta_1^{-1}\varepsilon_{ABCD}\nabla^C B^{AB}\, , && (3.35)\\
{\mathit{\Pi}^2}_D &= -\frac{r}{2}\Delta_1^{-1}\varepsilon_{ABCD}\nabla^C D^{AB}\, , && (3.36)
\end{aligned}
$$

Eq. (3.32) simplifies to

$$
\begin{aligned}
-\delta\mathcal{H}_{\rm sym} &= \int\limits_V \Lambda_2 \left\{ \left({\dot{\mathit{\Pi}}^1}_A \delta {Q_1}^A - {\dot{Q}_1}{}^A \delta {\mathit{\Pi}^1}_A \right) - \left({\dot{\mathit{\Pi}}^2}_A \delta {Q_2}^A - {\dot{Q}_2}{}^A \delta {\mathit{\Pi}^2}_A \right) \right\} \\
&+ \int\limits_{\partial V} \Lambda_2 \left({\chi^1}_A \delta {Q_1}^A + {\chi^2}_A \delta {Q_2}^A \right)\, , \qquad (3.37)
\end{aligned}
$$

where we introduced the boundary momenta:

$$
\begin{aligned}
{\chi^1}_A &= \frac{2}{r} A_{0A}\, , && (3.38)\\
{\chi^2}_D &= -\frac{r}{2}\Delta_1^{-1}\varepsilon_{ABCD}\nabla^C G^{rAB}\, . && (3.39)
\end{aligned}
$$

In (3.37) we defined

$$
Q_{l\,A} := g_{AB}\; {Q_l}^B\, , \qquad \mathit{\Pi}^{l\,A} := g^{AB}\; {\mathit{\Pi}^l}_B\, . \qquad (3.40)
$$

Note the crucial difference between (3.37) and (2.33): the sign "+" in (2.33) is replaced by "−" in (3.37).

For a Maxwell theory one obtains

$$\mathcal{H}^{\mathrm{Maxwell}}_{\mathrm{sym}} = \frac{1}{4}\int_V \Lambda_2 \sum_{l=1}^{2}\left\{\frac{1}{r^2}Q_l{}^A Q_{lA} - \frac{1}{r^4}\partial_r(r^3 Q_{lA})\Delta_1^{-1}\partial_r(rQ_l^A) - \Pi^{lA}\Delta_1\Pi^l_A\right\} \tag{3.41}$$

and, therefore

$$\chi^l{}_A = \frac{1}{r^3}\Delta_1^{-1}\partial_r(r^3 Q_{lA})\,, \qquad l = 1,2\,. \tag{3.42}$$

3.4. Canonical symmetries

The symplectic form $-\int \delta\mathcal{D}^{ij}\wedge\delta A_{ij}$ rewritten in terms of Q's and Π's have the following form [16]:

$$\Omega = \mathrm{Im}\int \Lambda_2\,\delta\Pi^A \wedge \delta Q_A\,, \tag{3.43}$$

where we introduced a complex notation

$$Q_A \;=\; Q^1_A + iQ^2_A\,, \tag{3.44}$$
$$\Pi^A \;=\; i\left(\Pi^A_1 + i\Pi^A_2\right)\,. \tag{3.45}$$

The form (3.43) contrary to (2.39) is invariant only under the following transformations:

$$Q_A \;\to\; \cosh\lambda\, Q_A + \sinh\lambda\,\overline{Q}_A\,, \tag{3.46}$$

and the same rule for Π^A. It is easy to see that these transformations form the group SO(1,1). In terms of D^{ij} and B^{ij}, (3.46) reads:

$$\begin{aligned} D^{ij} &\to \mathrm{e}^{\lambda}\, D^{ij}\,, \\ B^{ij} &\to \mathrm{e}^{-\lambda}\, B^{ij}\,. \end{aligned} \tag{3.47}$$

The canonical generator corresponding to (3.46) has the following form:

$$G_4 \;=\; -\int \Lambda_2\,\left(Q^1{}_A\Pi^A_1 + Q^2_A\Pi^A_2\right) = \mathrm{Im}\int \Lambda_2\,\left(\Pi^A\overline{Q}_A\right)\,. \tag{3.48}$$

3.5. Summary

Contrary to the $p = 1$ case the reduced variables (Q_l^A, Π^l_A) do not solve completely the Gauss constraints $\partial_i D^{ij} = \partial_i B^{ij} = 0$. They fulfill the following additional conditions [16]:

$$\nabla_A Q_l^A = \nabla^A \Pi^l_A = 0\,, \qquad l = 1, 2. \tag{3.49}$$

In the geometric language it means that $\star Q_l$ and $\star \Pi^l$ are closed 3-forms on $S^4(r)$ ($\star$ denotes the Hodge dual defined via ε^{ABCD}). They are gauge-invariant and contain the entire information about 2-forms D^{ij} and B^{ij}. The dynamics is generated by the *quasi-local* functional of Q's and Π's.

The "symmetric" dynamics defined by (3.37) corresponds to the Dirichlet boundary condition for positions Q_l whereas the "canonical" dynamics corresponds to the Dirichlet conditions for χ^1 and Q_2. But Dirichlet condition for ${\chi^1}_A$ is equivalent to the Neumann condition for $\partial_r {\mathcal{D}^r}_A$

$$\int\limits_{\partial V} \Lambda_2 Q_1^A \delta \chi^1_A = \int\limits_{\partial V} r \Delta_1^{-1} Q_1^A \delta(\partial_r \mathcal{D}^r_A)\,. \tag{3.50}$$

4. Coupling to the charged matter

In the present section we study the coupling of p-form electrodynamics to the charged matter. We present parallel discussion for $p = 1$ and $p = 2$. The general case is presented in Appendix C.

4.1. $p = 1$

Consider a 1-form electromagnetism interacting with the charged matter field Φ (for simplicity let Φ be a complex scalar field). In the presence of charged matter the Lagrangian generating formula (2.1) has to be replaced by:

$$-\delta \mathcal{L} = \partial_\nu (G^{\nu\mu} \delta A_\mu + \mathcal{P}^\nu \delta \Phi)\,, \tag{4.1}$$

where the matter "momentum"

$$\mathcal{P}^\nu = -\frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)}\,. \tag{4.2}$$

Because $\mathcal{L}$ should define a gauge-invariant theory let us assume that there is a group of gauge transformations U_Λ parameterized by a a function (0-form) Λ acting in the following way: $A_\mu \to A_\mu + \partial_\mu \Lambda$ and $\Phi \to U_\Lambda(\Phi)$.

Now, the target space of the matter field Φ may be reparameterized $\Phi = (\varphi, U)$ in such a way that, a parameter U is gauge invariant and φ is the phase undergoing the following gauge transformation: $\varphi \to \varphi + \Lambda$. For the scalar (complex) field one has: $U := |\Phi|$ and the $\varphi = \operatorname{Arg} \Phi$. Therefore, the matter part in (4.1) may be rewritten as follows:

$$\mathcal{P}^\nu \delta\Phi = J^\nu \delta\varphi + p^\nu \delta U \, . \tag{4.3}$$

Gauge invariance of the theory means that the gauge dependent quantities, *i.e.* A_μ and φ, enter into $\mathcal{L}$ via the gauge-invariant combinations only:

$$\mathcal{L} = \mathcal{L}(F_{\mu\nu}, D_\mu\varphi, U, \partial_\mu U) \, , \tag{4.4}$$

where

$$D_\mu\varphi := \partial_\mu\varphi - A_\mu \tag{4.5}$$

denotes a covariant derivative of φ. This implies, that the momentum J^μ canonically conjugated to φ is equal to the electric current

$$J^\mu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \frac{\partial\mathcal{L}}{\partial A_\mu} = -\partial_\nu G^{\nu\mu} \, . \tag{4.6}$$

Now, instead of (2.8) one has

$$\begin{aligned} -\delta \int_V \mathcal{L} \; = \; & \int_V \partial_0 \left(\mathcal{D}^i \delta A_i + \rho\delta\varphi + p^0 \delta U \right) \\ & + \int_{\partial V} \left(-\mathcal{D}^r \delta A_0 + \mathcal{G}^{rB} \delta A_B + J^r \delta\varphi + p^r \delta U \right) \, , \end{aligned} \tag{4.7}$$

with $\rho := J^0$. Performing the set of Lagrange transformations between: (1) $\mathcal{D}^k$ and $\dot{A}_k$, (2) ρ and $\dot{\varphi}$, (3) $\pi := p^0$ and $\dot{U}$ in the volume V, and between $\mathcal{D}^r$ and A_0 at the boundary ∂V, one obtains the following generalization of (2.12):

$$\begin{aligned} -\delta\mathcal{H}_{\rm sym} = \; & - \int_V \left\{ \left(\dot{\mathcal{D}}^i \delta A_i - \dot{A}_i \delta\mathcal{D}^i \right) + (\dot{\rho}\delta\varphi - \dot{\varphi}\delta\rho) + \left(\dot{\pi}\delta U - \dot{U}\delta\pi \right) \right\} \\ & - \int_{\partial V} \left\{ A_0 \delta\mathcal{D}^r + \mathcal{G}^{rB} \delta A_B + J^r \delta\varphi + p^r \delta U \right\} \, , \end{aligned} \tag{4.8}$$

where the "symmetric" Hamiltonian of the interacting electromagnetic field and the charged matter represented by Φ reads:

$$\mathcal{H}_{\rm sym} = \int_V \left(-\mathcal{D}^i \dot{A}_i - \rho\dot{\varphi} - \pi\dot{U} - \mathcal{L} + \partial_k \left(A_0 \mathcal{D}^k \right) \right) \, . \tag{4.9}$$

Now, using
$$\partial_k \mathcal{D}^k = \rho\,, \tag{4.10}$$
implied by (4.6), one gets the following formula for $\mathcal{H}_{\rm sym}$:
$$\mathcal{H}_{\rm sym} = \int\limits_V \left(\mathcal{D}^i E_i - \rho D_0\varphi - \pi\dot{U} - \mathcal{L} \right)\,. \tag{4.11}$$

Note, that the gauge-dependent phase φ enters into $\mathcal{H}_{\rm sym}$ via the gauge-invariant combination $D_0\varphi$ only. Moreover, due to (4.10), we may rewrite the dynamical part for φ in (4.8) as follows:
$$\int\limits_V (\dot{\rho}\delta\varphi - \dot{\varphi}\delta\rho) = \int\limits_V \left(-\dot{\mathcal{D}}^k\delta(\partial_k\varphi) + (\partial_k\dot{\varphi})\delta\mathcal{D}^k \right) + \int\limits_{\partial V} \left(\dot{\mathcal{D}}^r\delta\varphi - \dot{\varphi}\delta\mathcal{D}^r \right)\,. \tag{4.12}$$
Now, the $\dot{\mathcal{D}}^r$ at the boundary may be easily eliminated by the field equations (4.6)
$$\dot{\mathcal{D}}^r = -\partial_0\mathcal{G}^{r0} = \partial_\mu\mathcal{G}^{\mu r} - \partial_A\mathcal{G}^{Ar} = -J^r + \partial_A\mathcal{G}^{rA}\,. \tag{4.13}$$

Introducing a hydrodynamical variables:
$$V_\mu := -D_\mu\varphi\,, \tag{4.14}$$

we may rewrite finally (4.8) as follows:
$$\begin{aligned} -\delta\mathcal{H}_{\rm sym} \;&=\; -\int\limits_V \left\{ \left(\dot{\mathcal{D}}^i\delta V_i - \dot{V}_i\delta\mathcal{D}^i \right) + \left(\dot{\pi}\delta U - \dot{U}\delta\pi \right) \right\} \\ &\quad - \int\limits_{\partial V} \left\{ V_0\delta\mathcal{D}^r + \mathcal{G}^{rB}\delta V_B + p^r\delta U \right\}\,, \end{aligned} \tag{4.15}$$

i.e. (4.15) has exactly the same form as (2.12) with A_μ replaced by the gauge-invariant V_μ and supplemented by the gauge-invariant canonical pair (U,π) together with the boundary momentum p^r.

4.2. $p = 2$

Now, consider a 2-form electromagnetism interacting with the charged matter field Φ_μ (for simplicity let Φ_μ be a complex vector field). In the presence of charged matter the Lagrangian generating formula (3.1) has to be replaced by:
$$-\delta\mathcal{L} = \partial_\nu\left(G^{\nu\mu\lambda}\delta A_{\mu\lambda} + \mathcal{P}^{\nu\mu}\delta\Phi_\mu \right)\,, \tag{4.16}$$

where the matter "momentum"

$$\mathcal{P}^{\nu\mu} = -2\frac{\partial \mathcal{L}}{\partial(\partial_{[\nu}\Phi_{\mu]})} \,. \tag{4.17}$$

Because $\mathcal{L}$ should define a gauge-invariant theory let us assume that there is a group of gauge transformations U_Λ parameterized by a a 1-form Λ acting in the following way: $A \to A + d\Lambda$ and $\Phi \to U_\Lambda(\Phi)$.

Now, the target space of the matter field Φ_μ may be reparameterized $\Phi_\mu = (\varphi_\mu, U_\mu)$ in such a way that a 1-form U_μ is gauge invariant and a 1-form φ_μ is the phase undergoing the following gauge transformation: $\varphi \to \varphi + \Lambda$. For the vector (complex) field one has: $U_\mu := |\Phi_\mu|$ and $\varphi_\mu = \operatorname{Arg}\Phi_\mu$. Therefore, the matter part in (4.16) may be rewritten as follows:

$$\mathcal{P}^{\nu\mu}\delta\Phi_\mu = J^{\nu\mu}\delta\varphi_\mu + p^{\nu\mu}\delta U_\mu \,. \tag{4.18}$$

Gauge invariance of the theory means that the gauge dependent quantities, *i.e.* $A_{\mu\nu}$ and φ_μ, enter into $\mathcal{L}$ via the gauge-invariant combinations only:

$$\mathcal{L} = \mathcal{L}(F_{\mu\nu\lambda}, D_\mu\varphi_\nu, U_\mu, \partial_\mu U_\nu) \,, \tag{4.19}$$

where

$$D_\mu\varphi_\nu := \frac{1}{2}\,\partial_{[\mu}\varphi_{\nu]} - A_{\mu\nu} \tag{4.20}$$

denotes a "covariant derivative" of φ_ν. This implies, that the momentum $J^{\mu\lambda}$ canonically conjugated to φ_λ is equal to the electric current

$$J^{\mu\lambda} = -2\,\frac{\partial\mathcal{L}}{\partial(\partial_{[\mu}\varphi_{\lambda]})} = 2\,\frac{\partial\mathcal{L}}{\partial A_{\mu\lambda}} = -\partial_\nu G^{\nu\mu\lambda} \,. \tag{4.21}$$

Now, instead of (3.9) one has

$$\begin{aligned} -\delta \int\limits_V \mathcal{L} &= \int\limits_V \partial_0 \left(-\mathcal{D}^{ij}\delta A_{ij} - \rho^k\delta\varphi_k + \pi^k\delta U_k \right) \\ &+ \int\limits_{\partial V} \left(2\mathcal{D}^{rA}\delta A_{0A} + \mathcal{G}^{rAB}\delta A_{AB} + \rho^r\delta\varphi_0 + J^{rA}\delta\varphi_A - \pi^r\delta U_0 + p^{rA}\delta U_A \right) \,, \end{aligned} \tag{4.22}$$

with $\rho^k := J^{k0}$ (it defines a 1-form charge density on 5-dim. hyperplane Σ) and $\pi^k := p^{0k}$. Now, to pass to the Hamiltonian picture one has to perform the following Legendre transformations between: (1) $\mathcal{D}$ and $\dot{A}$, (2) ρ and $\dot{\varphi}$,

(3) π and $\dot{U}$ in the volume V, and between (4) $\mathcal{D}^r$ and A_0, (5) ρ^r and φ_0 and (6) π^r and U_0 at the boundary ∂V. One obtains the following generalization of (3.15):

$$-\delta\mathcal{H}_{\rm sym} = \int\limits_V \left\{\left(\dot{\mathcal{D}}^{ij}\delta A_{ij} - \dot{A}_{ij}\delta\mathcal{D}^{ij}\right) + \left(\dot{\rho}^k\delta\varphi_k - \dot{\varphi}_k\delta\rho^k\right) - \left(\dot{\pi}^k\delta U_k - \dot{U}_k\delta\pi^k\right)\right\}$$
$$- \int\limits_{\partial V}\left\{-2A_{0A}\delta\mathcal{D}^{rA} + \mathcal{G}^{rAB}\delta A_{AB} - \varphi_0\delta\rho^r + J^{rA}\delta\varphi_A + U_0\delta\pi^r + p^{rA}\delta U_A\right\}, \tag{4.23}$$

where the "symmetric" Hamiltonian of the interacting electromagnetic field and the charged matter represented by Φ_μ reads:

$$\mathcal{H}_{\rm sym} = \int\limits_V \left\{\mathcal{D}^{ij}\dot{A}_{ij} + \rho^k\left(\dot{\varphi}_k - \partial_k\varphi_0\right) - \pi^k\dot{U}_k - \partial_k\left(2A_{0i}\mathcal{D}^{ki} - U_0\pi^k\right) - \mathcal{L}\right\}, \tag{4.24}$$

where we have used $\partial_k\rho^k = 0$. Now, using

$$\partial_i\mathcal{D}^{ik} = \rho^k\,, \tag{4.25}$$

one gets the following formula for $\mathcal{H}_{\rm sym}$:

$$\mathcal{H}_{\rm sym} = \int\limits_V \left(\frac{1}{2}\mathcal{D}^{ij}E_{ij} + 2\rho^k D_0\varphi_k - \pi^k\dot{U}_k - \mathcal{L} + \partial_k(\pi^k U_0)\right). \tag{4.26}$$

Note, that the gauge-dependent phase φ_μ enters into $\mathcal{H}_{\rm sym}$ *via* the gauge-invariant combination $D_0\varphi_\mu$. Moreover, due to (4.25), we may rewrite the dynamical part for φ_μ in (4.23) as follows:

$$\begin{aligned}\int\limits_V\left(\dot{\rho}^k\delta\varphi_k - \dot{\varphi}_k\delta\rho^k\right) &= \int\limits_V\left(-\dot{\mathcal{D}}^{ik}\delta(\partial_i\varphi_k) + (\partial_i\dot{\varphi}_k)\delta\mathcal{D}^{ik}\right)\\ &\quad + \int\limits_{\partial V}\left(\dot{\mathcal{D}}^{rA}\delta\varphi_A - \dot{\varphi}_A\delta\mathcal{D}^{rA}\right).\end{aligned} \tag{4.27}$$

Now, the term $\dot{\mathcal{D}}^{rA}$ at the boundary may be easily eliminated by the field equations (4.21)

$$\dot{\mathcal{D}}^{rA} = J^{rA} + \partial_B\mathcal{G}^{rAB}\,. \tag{4.28}$$

Introducing hydrodynamical variables:

$$V_{\mu\nu} := -D_\mu\varphi_\nu\,, \tag{4.29}$$

we may rewrite finally (4.23) as follows:

$$
\begin{aligned}
-\delta\mathcal{H}_{\rm sym} \;=\; & \int\limits_V \left\{ \left(\dot{\mathcal{D}}^{ij}\delta V_{ij} - \dot{V}_{ij}\delta\mathcal{D}^{ij} \right) - \left(\dot{\pi}^k \delta U_k - \dot{U}_k \delta\pi^k \right) \right\} \\
& - \int\limits_{\partial V} \left\{ -2V_{0A}\delta\mathcal{D}^{rA} + \mathcal{G}^{rAB}\delta V_{AB} - U_0\delta\pi^r + p^{rA}\delta U_A \right\} \;, \qquad (4.30)
\end{aligned}
$$

i.e. (4.15) has exactly the same form as (3.15) with $A_{\mu\nu}$ replaced by the gauge-invariant 2-form $V_{\mu\nu}$ and supplemented by the gauge-invariant canonical pair (U_k, π^k) together with the boundary momenta U_0 and p^{rA}. All gauge-dependent terms dropped out.

Appendix A

Notation

Consider a p-form potential A defined in the $D = 2p+2$ dimensional Minkowski space-time $\mathcal{M}^{2p+2}$ with the signature of the metric tensor $(-,+,\ldots,+)$. The corresponding field tensor is defined as a $(p+1)$-form by $F = dA$:

$$F_{\mu_1\ldots\mu_{p+1}} = \partial_{[\mu_1}A_{\mu_2\ldots\mu_{p+1}]} \,, \qquad (A.1)$$

where the antisymmetrization is defined by $X_{[kl]} := X_{kl} - X_{lk}$. Having a Lagrangian $\mathcal{L}$ of the theory one defines another $(p+1)$-form G as follows:

$$\mathcal{G}^{\mu_1\ldots\mu_{p+1}} = -(p+1)!\frac{\partial\mathcal{L}}{\partial F_{\mu_1\ldots\mu_{p+1}}} \,. \qquad (A.2)$$

Now one may define the electric and magnetic intensities and inductions in the obvious way:

$$
\begin{aligned}
E_{i_1\ldots i_p} &= F_{i_1\ldots i_p 0} \,, & (A.3)\\
B_{i_1\ldots i_p} &= \frac{1}{(p+1)!}\,\varepsilon_{i_1\ldots i_p j_1\ldots j_{p+1}} F^{j_1\ldots j_{p+1}} \,, & (A.4)\\
\mathcal{D}_{i_1\ldots i_p} &= \mathcal{G}_{i_1\ldots i_p 0} \,, & (A.5)\\
\mathcal{H}_{i_1\ldots i_p} &= \frac{1}{(p+1)!}\,\varepsilon_{i_1\ldots i_p j_1\ldots j_{p+1}} \mathcal{G}^{j_1\ldots j_{p+1}} \,, & (A.6)
\end{aligned}
$$

where the indices $i_1, i_2, \ldots, j_1, j_2, \ldots$ run from 1 up to $2p+1$ and $\varepsilon_{i_1 i_2\ldots i_{2p+1}}$ is the Lévi–Civita tensor in $2p+1$ dimensional Euclidean space, *i.e.* a space-like hyperplane Σ in the Minkowski space-time. The field equations are given by the Bianchi identities $dF = 0$, or in components

$$\partial_{[\lambda}F_{\mu_1\ldots\mu_{p+1}]} = 0 \,, \qquad (A.7)$$

and the *true* dynamical equations $d \star \mathcal{G} = 0$, or equivalently

$$\partial_{[\lambda} \star \mathcal{G}_{\mu_1 \dots \mu_{p+1}]} = 0\,, \tag{A.8}$$

where the Hodge star operation in $\mathcal{M}^{2p+2}$ is defined by:

$$\star X^{\mu_1 \dots \mu_{p+1}} = \frac{1}{(p+1)!}\, \eta^{\mu_1 \dots \mu_{p+1} \nu_1 \dots \nu_{p+1}}\; X_{\nu_1 \dots \nu_{p+1}} \tag{A.9}$$

and $\eta^{\mu_1 \mu_2 \dots \mu_{2p+2}}$ is the covariantly constant volume form in the Minkowski space-time. Note, that $\varepsilon^{i_1 \dots i_{2p+1}} := \eta^{0 i_1 \dots i_{2p+1}}$. In terms of electric and magnetic fields defined in (A.3)–(A.6) the field equations (A.7)–(A.8) have the following form:

$$\begin{aligned}
\partial_0 B^{i_1 \dots i_p} &= (-1)^p \frac{1}{p!}\, \varepsilon^{i_1 \dots i_p k j_1 \dots j_p}\; \nabla_k E_{j_1 \dots j_p}\,, && \text{(A.10)}\\
\nabla_{i_1} B^{i_1 \dots i_p} &= 0\,, && \text{(A.11)}\\
\partial_0 \mathcal{D}^{i_1 \dots i_p} &= \frac{1}{p!}\, \varepsilon^{i_1 \dots i_p k j_1 \dots j_p}\; \nabla_k \mathcal{H}_{j_1 \dots j_p}\,, && \text{(A.12)}\\
\nabla_{i_1} \mathcal{D}^{i_1 \dots i_p} &= 0\,, && \text{(A.13)}
\end{aligned}$$

where ∇_k denotes the covariant derivative on Σ compatible with the metric g_{kl} induced from $\mathcal{M}^{2p+2}$. The Lévi–Civita tensor density satisfies $\varepsilon_{12 \dots 2p+1} = \sqrt{g}$, with $g = \det(g_{kl})$.

Appendix B

General p-form theory without matter

B.1 Generating formula

For an arbitrary p the formulae (2.1) and (3.1) generalize to:

$$-\delta \mathcal{L} = (\partial_\nu \mathcal{G}^{\nu \mu_1 \dots \mu_p} \delta A_{\mu_1 \dots \mu_p}) = (\partial_\nu \mathcal{G}^{\nu \mu_1 \dots \mu_p}) \delta A_{\mu_1 \dots \mu_p} + \mathcal{G}^{\nu \mu_1 \dots \mu_p} \delta(\partial_\nu A_{\mu_1 \dots \mu_p})\,. \tag{B.1}$$

The formula (B.1) implies the following definition of "momenta":

$$\mathcal{G}^{\mu_1 \dots \mu_{p+1}} = -(p+1)!\, \frac{\partial \mathcal{L}}{\partial F_{\mu_1 \dots \mu_{p+1}}}\,. \tag{B.2}$$

Moreover, (B.1) generates dynamical (in general nonlinear) field equations

$$\partial_\nu \mathcal{G}^{\nu \mu_1 \dots \mu_p} = -\mathcal{J}^{\mu_1 \dots \mu_p}\,, \tag{B.3}$$

where the external p-form current reads:

$$\mathcal{J}^{\mu_1\ldots\mu_p} = p!\,\frac{\partial\mathcal{L}}{\partial A_{\mu_1\ldots\mu_p}}\,. \tag{B.4}$$

Let us start with $\mathcal{J} = 0$ and discuss a general p-form charged matter in Appendix C. To obtain the Hamiltonian description of the field dynamics let us integrate equation (B.1) over a $(2p{+}1)$-dimensional volume V contained in the constant-time hyperplane Σ:

$$-\delta\int\limits_V \mathcal{L} = \int\limits_V \partial_0\left(\mathcal{G}^{0i_1\ldots i_p}\delta A_{i_1\ldots i_p}\right) + \int\limits_{\partial V} \mathcal{G}^{\perp\mu_1\ldots\mu_p}\delta A_{\mu_1\ldots\mu_p}\,, \tag{B.5}$$

where $\perp$ denotes the component orthogonal to the $2p$-dimensional boundary ∂V. To simplify our notation let us introduce the spherical coordinates on Σ:

$$x^{2p+1} \;=\; r\,, \qquad x^A = \varphi_A\;; \qquad A = 1,2,\ldots,2p\,, \tag{B.6}$$

where $\varphi_1,\varphi_2,\ldots,\varphi_{2p}$ denote spherical angles (to enumerate angles we shall use capital letters $A,B,C,\ldots$). The metric tensor g_{ij} is diagonal and has the following form:

$$\begin{aligned}
g_{11} &= r^2\sin^2\varphi_2\sin^2\varphi_3\ldots\sin^2\varphi_{2p}\,,\\
g_{22} &= r^2\sin^2\varphi_3\sin^2\varphi_4\ldots\sin^2\varphi_{2p}\,,\\
&\;\vdots\\
g_{2p-1,2p-1} &= r^2\sin^2\varphi_{2p-1}\sin^2\varphi_{2p}\\
g_{2p,2p} &= r^2\sin^2\sin_{2p}\,,\\
g_{rr} &= r^2\,.
\end{aligned} \tag{B.7}$$

Therefore, the volume form

$$\Lambda_p = \sqrt{\det(g_{ij})} = r^{2p}\sin\varphi_2\sin^2\varphi_3\ldots\sin^{2p-2}\varphi_{2p-1}\sin^{2p-1}\varphi_{2p}\,. \tag{B.8}$$

Let V be a $(2p{+}1)$–dim. ball with a finite radius R. In such a coordinate system the formula (B.5) takes the following form:

$$\begin{aligned}
\delta\int\limits_V \mathcal{L} \;=\; &(-1)^p\int\limits_V \partial_0\left(\mathcal{D}^{i_1\ldots i_p}\delta A_{i_1\ldots i_p}\right) - (-1)^p\int\limits_{\partial V} p\,\mathcal{D}^{rA_2\ldots A_p}\delta A_{0A_2\ldots A_p}\\
&-\int\limits_{\partial V}\mathcal{G}^{rB_1\ldots B_p}\delta A_{B_1\ldots B_p}\,,
\end{aligned} \tag{B.9}$$

where

$$\mathcal{D}_{i_1\ldots i_p} = \mathcal{G}_{i_1\ldots i_p 0} \tag{B.10}$$

denotes the p-form electric induction density. Now, performing the Legendre transformation between induction p-form $\mathcal{D}^{i_1\ldots i_p}$ and $\dot{A}_{i_1\ldots i_p}$ one obtains the following Hamiltonian formula:

$$\begin{aligned}-\delta\mathcal{H}_{\mathrm{can}} &= (-1)^p \int\limits_V \left(\dot{\mathcal{D}}^{i_1\ldots i_p}\delta A_{i_1\ldots i_p} - \dot{A}_{i_1\ldots i_p}\delta\mathcal{D}^{i_1\ldots i_p}\right) \\ &-(-1)^p \int\limits_{\partial V} p\,\mathcal{D}^{rA_2\ldots A_p}\delta A_{0A_2\ldots Ap} - \int\limits_{\partial V} \mathcal{G}^{rB_1\ldots B_p}\delta A_{B_1\ldots B_p}\,,\end{aligned} \tag{B.11}$$

where the canonical Hamiltonian

$$\mathcal{H}_{\mathrm{can}} = \int\limits_V \left((-1)^p\mathcal{D}^{i_1\ldots i_p}\dot{A}_{i_1\ldots i_p} - \mathcal{L}\right)\,. \tag{B.12}$$

Equation (B.11) generates an infinite-dimensional Hamiltonian system in the phase space $\mathcal{P}_p = (\mathcal{D}^{i_1\ldots i_p}, A_{i_1\ldots i_p})$ fulfilling Dirichlet boundary conditions for the p-form potential $A_{i_1\ldots i_p}$: $A_{0A_2\ldots A_p}|\partial V$ and $A_{A_1A_2\ldots A_p}|\partial V$. From the mathematical point of view this is the missing part of the definition of the functional space. The Hamiltonian structure of a general nonlinear p-form electrodynamics described above is mathematically well defined, *i.e.* a mixed Cauchy problem (Cauchy data given on Σ and Dirichlet data given on $\partial V\times\boldsymbol{R}$) has a unique solution (modulo gauge transformations which reduce to the identity on $\partial V\times\boldsymbol{R}$).

The presence of a p-dependent sign $(-1)^p$ follows from the p-dependence of the corresponding symplectic form:

$$\Omega_p = \int\limits_V \delta\mathcal{G}^{0i_1\ldots i_p}\wedge\delta A_{i_1\ldots i_p} = (-1)^{p+1}\int\limits_V \delta\mathcal{D}^{i_1\ldots i_p}\wedge\delta A_{i_1\ldots i_p}\,. \tag{B.13}$$

There is, however, another way to describe the Hamiltonian evolution of fields in the region V. Let us perform the Legendre transformation between $\mathcal{D}^{rA_2\ldots A_p}$ and $A_{0A_2\ldots A_p}$ at the boundary ∂V. One obtains:

$$\begin{aligned}-\delta\mathcal{H}_{\mathrm{sym}} &= (-1)^p \int\limits_V \left(\dot{\mathcal{D}}^{i_1\ldots i_p}\delta A_{i_1\ldots i_p} - \dot{A}_{i_1\ldots i_p}\delta\mathcal{D}^{i_1\ldots i_p}\right) \\ &+(-1)^p \int\limits_{\partial V} pA_{0A_2\ldots A_p}\delta\mathcal{D}^{rA_2\ldots A_p} - \int\limits_{\partial V} \mathcal{G}^{rB_1\ldots B_p}\delta A_{B_1\ldots B_p}\,,\end{aligned} \tag{B.14}$$

where the new "symmetric" Hamiltonian

$$\mathcal{H}_{\text{sym}} = \mathcal{H}_{\text{can}} - (-1)^p \int_{\partial V} p\, \mathcal{D}^{rA_2 \ldots A_p} A_{0A_2 \ldots A_p} \,. \tag{B.15}$$

Observe, that formula (B.14) defines the Hamiltonian evolution but on a different phase space. In order to kill boundary terms in (B.14) one has to control on ∂V: $\mathcal{D}^{rA_2 \ldots A_p}$ (instead of $A_{0A_2 \ldots A_p}$) and $A_{B_1 \ldots B_p}$. We stress that from the mathematical point of view both descriptions are equally good and an additional physical argument has to be given if we want to choose one of them as more fundamental.

B.2 Canonical vs symmetric energy

Now, let us discuss the relation between $\mathcal{H}_{\text{can}}$ and $\mathcal{H}_{\text{sym}}$ defined by (B.12) and (B.15) respectively. One has:

$$\begin{aligned}
\mathcal{H}_{\text{sym}} &= \mathcal{H}_{\text{can}} - (-1)^p \int_{\partial V} p\, \mathcal{D}^{rA_2 \ldots A_p} A_{0A_2 \ldots A_p} \\
&= \mathcal{H} - (-1)^p \int_V p\, \partial_k \left(\mathcal{D}^{ki_2 \ldots i_p} A_{0i_2 \ldots i_p} \right) \\
&= \int_V \Big\{ (-1)^p \mathcal{D}^{i_1 \ldots i_p} \dot{A}_{i_1 \ldots i_p} - \mathcal{L} + (-1)^p p \left(A_{0i_2 \ldots i_p} \partial_k \mathcal{D}^{ki_2 \ldots i_p} + \mathcal{D}^{ki_2 \ldots i_p} \partial_k A_{0i_2 \ldots i_p} \right) \Big\} \\
&= \int_V \left(\frac{1}{p!} \mathcal{D}^{i_1 \ldots i_p} E_{i_1 \ldots i_p} - \mathcal{L} \right) ,
\end{aligned} \tag{B.16}$$

where the p-form electric field is defined by

$$E_{i_1 \ldots i_p} = F_{i_1 \ldots i_p 0} = \partial_{[i_1} A_{i_2 \ldots i_p 0]} \,. \tag{B.17}$$

Therefore, $\mathcal{H}_{\text{sym}} = \int_V T^{00}_{\text{sym}}$ and $\mathcal{H}_p = \int_V T^{00}_{\text{can}}$, where

$$T^{\mu\nu}_{\text{sym}} = \frac{1}{p!} F^{\mu\nu_1 \ldots \nu_p} G^{\nu}{}_{\nu_1 \ldots \nu_p} + g^{\mu\nu} \mathcal{L} \,, \tag{B.18}$$

$$T^{\mu\nu}_{\text{can}} = \partial^{\mu} A^{\nu_1 \ldots \nu_p} G^{\nu}{}_{\nu_1 \ldots \nu_p} + g^{\mu\nu} \mathcal{L} \,. \tag{B.19}$$

Obviously, for the Maxwell theory one has:

$$\mathcal{H}^{\text{Maxwell}}_{\text{sym}} = \frac{1}{2p!} \int \left(\mathcal{D}^{i_1 \ldots i_p} D_{i_1 \ldots i_p} + \mathcal{B}^{i_1 \ldots i_p} B_{i_1 \ldots i_p} \right) . \tag{B.20}$$

B.3 Reduction of the generating formula

Any geometrical object on $(2p+1)$-dimensional hyperplane Σ may be decomposed into the radial and tangential components, *e.g.* a p-form gauge potential $A_{i_1\ldots i_p}$ decomposes into the radial $A_{rA_2\ldots A_p}$ and tangential $A_{A_1\ldots A_p}$. On each sphere $2p$-dimensional sphere $S^{2p}(r)$, $A_{rA_2\ldots A_p}$ defines a $(p-1)$-form whereas $A_{A_1\ldots A_p}$ a p-form. Now, any p-form on $S^{2p}(r)$ may be further decomposed into "longitudinal" and "transversal" parts:

$$A_{A_1\ldots A_p} = \nabla_{[A_1} u_{A_2\ldots A_p]} + \varepsilon_{A_1\ldots A_p B_1\ldots B_p} \nabla^{B_1} v^{B_2\ldots B_p} \, , \tag{B.21}$$

where $\varepsilon_{A_1\ldots A_p B_1\ldots B_p}$ denotes the Lévi–Civita tensor density on $S^{2p}(r)$ such that $\varepsilon_{12\ldots 2p} = \Lambda_p$. Both u and v are $(p-1)$-forms on $S^{2p}(r)$. Now, using (B.21) and integrating by parts one gets:

$$\begin{aligned}
&\int_V \left(\dot{\mathcal{D}}^{i_1\ldots i_p} \delta A_{i_1\ldots i_p} - \dot{A}_{i_1\ldots i_p} \delta \mathcal{D}^{i_1\ldots i_p} \right) \\
&= \int_V p \left\{ \left(\dot{\mathcal{D}}^{rA_2\ldots A_p} \delta A_{rA_2\ldots A_p} - \dot{A}_{rA_2\ldots A_p} \delta \mathcal{D}^{rA_2\ldots A_p} \right) \right. \\
&+ p! \left[\left(\partial_r \dot{\mathcal{D}}^{rA_2\ldots A_p} \right) \delta u_{A_2\ldots A_p} - \dot{u}_{A_2\ldots A_p} \delta \left(\partial_r \mathcal{D}^{rA_2\ldots A_p} \right) \right] \\
&\left. - \varepsilon_{A_1\ldots A_p B_1\ldots B_p} \left[\left(\nabla^{B_1} \dot{\mathcal{D}}^{A_1\ldots A_p} \right) \delta v^{B_2\ldots B_p} - \dot{v}^{B_2\ldots B_p} \delta \left(\nabla^{B_1} \mathcal{D}^{A_1\ldots A_p} \right) \right] \right\} ,
\end{aligned} \tag{B.22}$$

where we have used the Gauss law

$$\nabla_{A_1} \mathcal{D}^{A_1\ldots A_p} = -\partial_r \mathcal{D}^{rA_2\ldots A_p} \, . \tag{B.23}$$

Moreover, due to (B.21)

$$\begin{aligned}
&\int_{\partial V} \mathcal{G}^{rA_1\ldots A_p} \delta A_{A_1\ldots A_p} \\
&= \int_{\partial V} \left\{ (-1)^p p! \, \dot{\mathcal{D}}^{rA_2\ldots A_p} \delta u_{A_2\ldots A_p} - \left(\varepsilon_{A_1\ldots A_p B_1\ldots B_p} \nabla^{B_1} \mathcal{G}^{rA_1\ldots A_p} \right) \delta v^{B_2\ldots B_p} \right\} .
\end{aligned} \tag{B.24}$$

In deriving (B.24) we have used

$$\nabla_{A_1} \mathcal{G}^{A_1\ldots A_p r} = -\dot{\mathcal{D}}^{rA_2\ldots A_p} \, , \tag{B.25}$$

which follows from the field equations $\nabla_{A_1}\mathcal{G}^{A_1\ldots A_p r}+\partial_0\mathcal{G}^{0A_2\ldots A_p r}=0$. Now, taking into account (B.22) and (B.24) the generating formula (B.14) may be rewritten in the following way:

$$\begin{aligned}
-\delta\mathcal{H}_{\mathrm{sym}} &= (-1)^p\int\limits_V\Big\{\Big[\dot{\mathcal{D}}^{rA_2\ldots A_p}\delta\left(pA_{rA_2\ldots A_p}-p!\partial_r u_{A_2\ldots A_p}\right)\\
&\quad-\left(p\dot{A}_{rA_2\ldots A_p}-p!\partial_r\dot{u}_{A_2\ldots A_p}\right)\delta\mathcal{D}^{rA_2\ldots A_p}\Big]\\
&\quad-\Big[\left(\varepsilon_{A_1\ldots A_p B_1\ldots B_p}\nabla^{B_1}\dot{\mathcal{D}}^{A_1\ldots A_p}\right)\delta v^{B_2\ldots B_p}\\
&\quad-\dot{v}^{B_2\ldots B_p}\delta\left(\varepsilon_{A_1\ldots A_p B_1\ldots B_p}\nabla^{B_1}\mathcal{D}^{A_1\ldots A_p}\right)\Big]\Big\}\\
&\quad+(-1)^p\int\limits_{\partial V}\left(p\,A_{0A_2\ldots A_p}-p!\,\dot{u}_{A_2\ldots A_p}\right)\delta\mathcal{D}^{rA_2\ldots A_p}\\
&\quad+\int\limits_{\partial V}\left(\varepsilon_{A_1\ldots A_p B_1\ldots B_p}\nabla^{B_1}\mathcal{G}^{rA_1\ldots A_p}\right)\delta v^{B_2\ldots B_p}\,.
\end{aligned}\tag{B.26}$$

Note, that although $A_{rA_2\ldots A_p}$, $A_{0A_2\ldots A_p}$ and $u_{A_2\ldots A_p}$ are manifestly gauge-dependent, the combinations $p\,A_{rA_2\ldots A_p}-p!\,\partial_r u_{A_2\ldots A_p}$ and $p\,A_{0A_2\ldots A_p}-p!\,\partial_0 u_{A_2\ldots A_p}$ are gauge-invariant. To simplify our consideration we choose the special gauge $u\equiv 0$, *i.e.* a p-form $A_{A_1\ldots A_p}$ on $S^{2p}(r)$ is purely transversal. This condition, due to (B.21), may be equivalently rewritten as

$$\nabla_{A_1}A^{A_1\ldots A_p}=0\,.\tag{B.27}$$

Let us choose the same condition for the radial part

$$\nabla_{A_2}A^{rA_2\ldots A_p}=0\,.\tag{B.28}$$

Assuming (B.27) and (B.28) one may show [16]

$$\varDelta_{p-1}A^{rB_2\ldots B_p}=(-1)^{p+1}\frac{r^2}{p\,p!}\,\varepsilon^{A_1\ldots A_p B_1\ldots B_p}\nabla_{B_1}B_{A_1\ldots A_p}\,,\tag{B.29}$$

where

$$\varDelta_{p-1}=(p-1)!\left[r^2\nabla_A\nabla^A-(p^2-1)\right]\tag{B.30}$$

equals the Laplace–Beltrami operator on co-exact $(p-1)$-forms on $S^{2p}(1)$ [16]. In the same way

$$B^{rA_2\ldots A_p}=-\frac{p!}{r^2}\,\varDelta_{p-1}v^{A_2\ldots A_p}\,.\tag{B.31}$$

Finally, introducing

$$
\begin{aligned}
Q_1{}^{A_2\ldots A_p} &= D^{rA_2\ldots A_p}\,, && \text{(B.32)}\\
Q_2{}^{A_2\ldots A_p} &= B^{rA_2\ldots A_p}\,, && \text{(B.33)}\\
\Pi^1{}_{B_2\ldots B_p} &= \frac{r}{p!}\,\Delta_{p-1}^{-1}\left(\varepsilon_{A_1\ldots A_p B_1\ldots B_p}\nabla^{B_1}B^{A_1\ldots A_p}\right)\,, && \text{(B.34)}\\
\Pi^2{}_{B_2\ldots B_p} &= -\frac{r}{p!}\,\Delta_{p-1}^{-1}\left(\varepsilon_{A_1\ldots A_p B_1\ldots B_p}\nabla^{B_1}D^{A_1\ldots A_p}\right)\,, && \text{(B.35)}
\end{aligned}
$$

the formula (B.26) simplifies to

$$
\begin{aligned}
-\delta\mathcal{H}_{\mathrm{sym}} = \int_V \Lambda_p \Big\{ &\left(\dot{\Pi}^1{}_{A_2\ldots A_p}\delta Q_1{}^{A_2\ldots A_p} - \dot{Q}_1{}^{A_2\ldots A_p}\delta\Pi^1{}_{A_2\ldots A_p}\right)\\
&+ (-1)^{p+1}\left(\dot{\Pi}^2{}_{A_2\ldots A_p}\delta Q_2{}^{A_2\ldots A_p} - \dot{Q}_2{}^{A_2\ldots A_p}\delta\Pi^2{}_{A_2\ldots A_p}\right)\Big\}\\
&+ \int_{\partial V}\Lambda_p\left(\chi^1{}_{A_2\ldots A_p}\delta Q_1{}^{A_2\ldots A_p} + \chi^1{}_{A_2\ldots A_p}\delta Q_1{}^{A_2\ldots A_p}\right)\,, \qquad \text{(B.36)}
\end{aligned}
$$

where we introduced the boundary momenta:

$$
\begin{aligned}
\chi^1_{A_2\ldots A_p} &= (-1)^p\frac{p}{r}\,A_{0A_2\ldots A_p}\,, && \text{(B.37)}\\
\chi^2_{B_2\ldots B_p} &= -\frac{r}{p!}\,\Delta_1^{-1}\varepsilon_{A_1\ldots A_p B_1\ldots B_p}\nabla^{B_1}G^{rA_1\ldots A_p}\,. && \text{(B.38)}
\end{aligned}
$$

In the formula (B.36) we have introduced:

$$
\begin{aligned}
Q_{l\,A_2\ldots A_p} &:= g_{A_2B_2}\ldots g_{A_pB_p}\,Q_l{}^{B_2\ldots B_p}\,, && \text{(B.39)}\\
\Pi^{l\,A_2\ldots A_p} &:= g^{A_2B_2}\ldots g^{A_pB_p}\,\Pi^l{}_{B_2\ldots B_p}\,, && \text{(B.40)}
\end{aligned}
$$

for $l = 1, 2$. For the Maxwell theory

$$
\begin{aligned}
\mathcal{H}^{\mathrm{Maxwell}}_{\mathrm{sym}} = \frac{1}{2(p-1)!}\int_V \Lambda_p\sum_{l=1}^{2}\Big\{&\frac{1}{r^2}Q_l{}^{A_2\ldots A_p}Q_{l\,A_2\ldots A_p}\\
&-\Pi^{l\,A_2\ldots A_p}\Delta_{p-1}\Pi^l{}_{A_2\ldots A_p}\\
&-\frac{1}{r^{2p}}\partial_r(r^{2p-1}Q_{l\,A_2\ldots A_p})\Delta_{p-1}^{-1}\partial_r\left(rQ_l{}^{A_2\ldots A_p}\right)\Big\}\,, \qquad \text{(B.41)}
\end{aligned}
$$

and, therefore, the boundary momenta read:

$$
\chi^l{}_{A_2\ldots A_p} = \frac{1}{r^{2p-1}}\Delta_1^{p-1}\partial_r\left(r^{2p-1}Q_{l\,A_2\ldots A_p}\right)\,, \qquad l = 1, 2. \qquad \text{(B.42)}
$$

B.4 Summary

The *quasi-local* reduced variables $\left(Q_l{}^{A_2\ldots A_p}, \Pi^l{}_{A_2\ldots A_p}\right)$ fulfill the following conditions [16]:

$$\nabla_{A_2} Q_l{}^{A_2\ldots A_p} = \nabla^{A_2} \Pi^l{}_{A_2\ldots A_p} = 0\,, \qquad l = 1, 2, \tag{B.43}$$

which follow from the Gauss laws. In the geometric language it means that $\star Q_l$ and $\star \Pi^l$ are closed $(p+1)$-forms on $S^{2p}(r)$ ($\star$ denotes the Hodge dual defined via $\varepsilon^{A_1\ldots A_p B_1\ldots B_p}$). They are gauge-invariant and contain the entire information about p-forms D and B.

The "symmetric" dynamics defined by (B.36) corresponds to the Dirichlet boundary condition for positions Q_l whereas the "canonical" dynamics corresponds to the Dirichlet conditions for $\chi^1{}_{A_2\ldots A_P}$ and Q_2. But Dirichlet condition for $\chi^1{}_A$ is equivalent to the Neumann condition for $\partial_r \mathcal{D}^r{}_{A_2\ldots A_p}$

$$\int_{\partial V} \Lambda_p Q_1{}^{A_2\ldots A_p} \delta \chi^1{}_{A_2\ldots A_p} = \int_{\partial V} r \Delta^{-1}_{p-1} Q_1{}^{A_2\ldots A_p} \delta \left(\partial_r \mathcal{D}^r{}_{A_2\ldots A_p}\right)\,. \tag{B.44}$$

Appendix C

General p-form theory with matter

Now, consider a p-form electromagnetism interacting with the charged matter field Φ (for simplicity let Φ be a complex $(p-1)$-form). In the presence of charged matter the Lagrangian generating formula (B.1) has to be replaced by:

$$-\delta \mathcal{L} = \partial_\nu \left(G^{\nu\mu_1\ldots\mu_p} \delta A_{\mu_1 \mu_p} + \mathcal{P}^{\nu\mu_2\ldots\mu_p} \delta \Phi_{\mu_2\ldots\mu_p} \right)\,, \tag{C.1}$$

where the matter "momentum"

$$\mathcal{P}^{\mu_1\mu_2\ldots\mu_p} = -p! \frac{\partial \mathcal{L}}{\partial(\partial_{[\mu_1} \Phi_{\mu_2\ldots\mu_p]})}\,. \tag{C.2}$$

Because $\mathcal{L}$ should define a gauge-invariant theory let us assume that there is a group of gauge transformations U_Λ parameterized by a a p-form Λ acting in the following way: $A \to A + d\Lambda$ and $\Phi \to U_\Lambda(\Phi)$.

Now, the target space of the matter field Φ may be reparameterized $\Phi = (\varphi, U)$ in such a way that a $(p-1)$-form U is gauge invariant and a $(p-1)$-form φ is the phase undergoing the following gauge transformation: $\varphi \to \varphi + \Lambda$. For the (complex) $(p-1)$-form one has: $U_{\mu_1\ldots\mu_{p-1}} := |\Phi_{\mu_1\ldots\mu_{p-1}}|$

and $\varphi_{\mu_1\ldots\mu_{p-1}} = \mathrm{Arg}\,\Phi_{\mu_1\ldots\mu_{p-1}}$. Therefore, the matter part in (C.1) may be rewritten as follows:

$$\mathcal{P}^{\mu_1\ldots\mu_p}\delta\Phi_{\mu_2\ldots\mu_p} = J^{\mu_1\ldots\mu_p}\delta\varphi_{\mu_2\ldots\mu_p} + p^{\mu_1\ldots\mu_p}\delta U_{\mu_2\ldots\mu_p}\,. \tag{C.3}$$

Gauge invariance of the theory means that the gauge dependent quantities, *i.e.* A and φ, enter into $\mathcal{L}$ via the gauge-invariant combinations only:

$$\mathcal{L} = \mathcal{L}\left(F_{\mu_1\ldots\mu_{p+1}}, D_\nu\varphi_{\mu_1\ldots\mu_{p-1}}, U_{\mu_1\ldots\mu_{p-1}}, \partial_\nu U_{\mu_1\ldots\mu_{p-1}}\right)\,, \tag{C.4}$$

where

$$D_\nu\varphi_{\mu_1\ldots\mu_{p-1}} := \frac{1}{p}\,\partial_{[\nu}\varphi_{\mu_1\ldots\mu_{p-1}]} - A_{\nu\mu_1\ldots\mu_{p-1}} \tag{C.5}$$

denotes a covariant derivative of $\varphi_{\mu_1\ldots\mu_{p-1}}$. This implies, that the momentum $J^{\mu_1\ldots\mu_p}$ canonically conjugated to $\varphi_{\mu_1\ldots\mu_p}$ is equal to the electric current

$$J^{\mu_1\ldots\mu_p} = -p!\,\frac{\partial\mathcal{L}}{\partial(\partial_{[\mu_1}\varphi_{\mu_2\ldots\mu_p]})} = p!\,\frac{\partial\mathcal{L}}{\partial A_{\mu_1\ldots\mu_p}} = -\partial_\nu G^{\nu\mu_1\ldots\mu_p}\,. \tag{C.6}$$

Now, instead of (B.9) one has

$$\begin{aligned}
\delta\int_V \mathcal{L} = \int_V \partial_0\Big\{&(-1)^p\mathcal{D}^{i_1\ldots i_p}\delta A_{i_1\ldots i_p}\\
&+(-1)^p\rho^{i_1\ldots i_{p-1}}\delta\varphi_{i_1\ldots i_{p-1}} - \pi^{i_1\ldots i_{p-1}}\delta U_{i_1\ldots i_{p-1}}\Big\}\\
&-\int_{\partial V}\Big\{(-1)^p p\,\mathcal{D}^{rA_2\ldots A_p}\delta A_{0A_2\ldots A_p}\\
&+\mathcal{G}^{rA_1\ldots A_p}\delta A_{A_1\ldots A_p} + (-1)^p(p-1)\rho^{rA_3\ldots A_p}\delta\varphi_{0A_3\ldots A_p}\\
&+J^{rA_2\ldots A_p}\delta\varphi_{A_2\ldots A_p} - (p-1)\pi^{rA_3\ldots A_p}\delta U_{0A_3\ldots A_p} + p^{rA_2\ldots A_p}\delta U_{A_2\ldots A_p}\Big\}\,,
\end{aligned} \tag{C.7}$$

with $\rho^{i_1\ldots i_{p-1}} := J^{i_1\ldots i_{p-1}0}$ (it defines a $(p-1)$-form charge density on $(2p+1)$-dim. hyperplane Σ) and $\pi^{i_1\ldots i_{p-1}} := p^{0i_1\ldots i_{p-1}}$. Now, to pass to the Hamiltonian picture one has to perform the following Legendre transformations between: (1) $\mathcal{D}$ and $\dot{A}$, (2) ρ and $\dot{\varphi}$, (3) π and $\dot{U}$ in the volume V, and between (4) $\mathcal{D}^r$ and A_0, (5) ρ^r and φ_0 and (6) π^r and U_0 at the boundary ∂V. One obtains the following generalization of (B.14):

$$
\begin{aligned}
-\delta\mathcal{H}_{\mathrm{sym}} = & \int\limits_V \Big\{(-1)^p \left(\dot{\mathcal{D}}^{i_1\dots i_p}\delta A_{i_1\dots i_p} - \dot{A}_{i_1\dots i_p}\delta\mathcal{D}^{i_1\dots i_p}\right) \\
& +(-1)^p \left(\dot{\rho}^{i_1\dots i_{p-1}}\delta\varphi_{i_1\dots i_{p-1}} - \dot{\varphi}_{i_1\dots i_{p-1}}\delta\rho^{i_1\dots i_{p-1}}\right) \\
& - \left(\dot{\pi}^{i_1\dots i_{p-1}}\delta U_{i_1\dots i_{p-1}} - \dot{U}_{i_1\dots i_{p-1}}\delta\pi^{i_1\dots i_{p-1}}\right)\Big\} \\
& - \int\limits_{\partial V} \Big\{(-1)^p A_{0A_2\dots A_p}\delta\mathcal{D}^{rA_2\dots A_p} \\
& +\mathcal{G}^{rA_1\dots A_p}\delta A_{A_1\dots A_p} + (-1)^p(p-1)\varphi_{0A_3\dots A_p}\delta\rho^{rA_3\dots A_p} \\
& -J^{rA_2\dots A_p}\delta\varphi_{A_2\dots A_p} - (p-1)U_{0A_3\dots A_p}\delta\pi^{rA_3\dots A_p} - p^{rA_2\dots A_p}\delta U_{A_2\dots A_p}\Big\} ,
\end{aligned}
\tag{C.8}
$$

where the "symmetric" Hamiltonian of the interacting electromagnetic field and the charged matter represented by Φ reads:

$$
\begin{aligned}
\mathcal{H}_{\mathrm{sym}} = & \int\limits_V \Big\{(-1)^p\mathcal{D}^{i_1\dots i_p}\dot{A}_{i_1\dots i_p} \\
& +(-1)^p\rho^{i_1\dots i_{p-1}}\dot{\varphi}_{i_1\dots i_{p-1}} - \pi^{i_1\dots i_{p-1}}\dot{U}_{i_1\dots i_{p-1}} - \mathcal{L} \\
& -\partial_k\Big[(-1)^p p\,\mathcal{D}^{ki_2\dots i_p}A_{0i_2\dots i_p} \\
& +(-1)^p(p-1)\rho^{ki_3\dots i_p}\varphi_{0i_3\dots i_p} - (p-1)\pi^{ki_3\dots i_p}U_{0i_3\dots i_p}\Big]\Big\} .
\end{aligned}
\tag{C.9}
$$

Now, using

$$
\partial_k\mathcal{D}^{ki_2\dots i_p} = \rho^{i_2\dots i_p} , \tag{C.10}
$$

one gets the following formula for $\mathcal{H}_{\mathrm{sym}}$:

$$
\begin{aligned}
\mathcal{H}_{\mathrm{sym}} = & \int\limits_V \Big\{\frac{1}{p!}\mathcal{D}^{i_1\dots i_p}E_{i_1\dots i_p} \\
& +(-1)^p p\,\rho^{i_1\dots i_{p-1}}D_0\varphi_{i_1\dots i_{p-1}} - \pi^{i_1\dots i_{p-1}}\dot{U}_{i_1\dots i_{p-1}} - \mathcal{L} \\
& +\partial_k\Big[(p-1)\pi^{ki_3\dots i_p}U_{0i_3\dots i_p}\Big]\Big\} .
\end{aligned}
\tag{C.11}
$$

Moreover, due to (C.10), we may rewrite the dynamical part for φ in (C.8) as follows:

$$\begin{aligned}
&\int_V \left(\dot{\rho}^{i_2\ldots i_p}\delta\varphi_{i_2\ldots i_p} - \dot{\varphi}_{i_2\ldots i_p}\delta\rho^{i_2\ldots i_p}\right) \\
&= \int_V \left(-\dot{\mathcal{D}}^{ki_2\ldots i_p}\delta(\partial_k\varphi_{i_2\ldots i_p}) + (\partial_k\dot{\varphi}_{i_2\ldots i_p})\delta\mathcal{D}^{ki_2\ldots i_p}\right) \\
&+ \int_{\partial V} \left(\dot{\mathcal{D}}^{rA_2\ldots A_p}\delta\varphi_{A_2\ldots A_p} - \dot{\varphi}_{A_2\ldots A_p}\delta\mathcal{D}^{rA_2\ldots A_p}\right) . \qquad \text{(C.12)}
\end{aligned}$$

Now, the term $\dot{\mathcal{D}}^{rA_2\ldots A_p}$ at the boundary may be easily eliminated by the field equations (C.6)

$$\dot{\mathcal{D}}^{rA_2\ldots A_p} = (-1)^p\left(J^{rA_2\ldots A_p} - \partial_{A_1}\mathcal{G}^{rA_1A_2\ldots A_p}\right) . \qquad \text{(C.13)}$$

Introducing hydrodynamical variables:

$$V_{\mu_1\mu_2\ldots\mu_p} := -D_{\mu_1}\varphi_{\nu_2\ldots\mu_p} , \qquad \text{(C.14)}$$

we may rewrite finally (C.8) as follows:

$$\begin{aligned}
&-\delta\mathcal{H}_{\rm sym} = \int_V \Big\{(-1)^p\left(\dot{\mathcal{D}}^{i_1\ldots i_p}\delta V_{i_1\ldots i_p} - \dot{V}_{i_1\ldots i_p}\delta\mathcal{D}^{i_1\ldots i_p}\right) \\
&- \left(\dot{\pi}^{i_1\ldots i_{p-1}}\delta U_{i_1\ldots i_{p-1}} - \dot{U}_{i_1\ldots i_{p-1}}\delta\pi^{i_1\ldots i_{p-1}}\right)\Big\} \\
&- \int_{\partial V} \Big\{(-1)^p V_{0A_2\ldots A_p}\delta\mathcal{D}^{rA_2\ldots A_p} + \mathcal{G}^{rA_1\ldots A_p}\delta V_{A_1\ldots A_p} \\
&-(p-1)U_{0A_3\ldots A_p}\delta\pi^{rA_3\ldots A_p} - p^{rA_2\ldots A_p}\delta U_{A_2\ldots A_p}\Big\} , \qquad \text{(C.15)}
\end{aligned}$$

i.e. (C.15) has exactly the same form as (B.14) with A replaced by the gauge-invariant p-form V and supplemented by the gauge-invariant canonical pair of $(p-1)$-forms (U,π) together with the boundary momenta: $(p-2)$-form U_0 and $(p-1)$-form p^r on ∂V. All gauge-dependent terms dropped out.

Appendix D

2 potentials vs reduced variables

Let us introduce a second p-form gauge potential Z on Σ such that

$$D^{i_1\ldots i_p} = \varepsilon^{i_1\ldots i_p k j_1\ldots j_p}\partial_k Z_{j_1\ldots j_p} \,. \tag{D.1}$$

Assuming for Z the same gauge conditions as for A, *i.e.*

$$\nabla_{A_1} Z^{A_1\ldots A_p} = 0\,, \tag{D.2}$$

$$\nabla_{A_2} Z^{rA_2\ldots A_p} = 0\,, \tag{D.3}$$

we have in analogy to (B.29)

$$\Delta_{p-1} Z^{rB_2\ldots B_p} = (-1)^{p+1}\,\frac{r^2}{p\,p!}\,\varepsilon^{A_1\ldots A_p B_1\ldots B_p}\nabla_{B_1} D_{A_1\ldots A_p}\,. \tag{D.4}$$

Therefore, taking into account (B.34)–(B.35) one has:

$$\Pi^1{}_{B_2\ldots B_p} = (-1)^{p+1}\,\frac{r}{p}\,A_{rB_2\ldots B_p}\,, \tag{D.5}$$

$$\Pi^2{}_{B_2\ldots B_p} = (-1)^{p}\,\frac{r}{p}\,Z_{rB_2\ldots B_p}\,, \tag{D.6}$$

i.e. the entire gauge-invariant information about two p-forms Z and A on Σ is encoded into two complex $(p-1)$-forms Q and Π on each $S^{2p}(r)$.

This work was partially supported by the Polish State Committee for Scientific Research (KBN) Grant no 2 P03A 047 15.

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